Advanced Engineering Mathematics
Prof. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 06
Complex Integration

Hello friends, welcome to my lecture on complex integration, why do we study complex integration; the integration in the complex plane is important because there are many real integrals which can be evaluated by complex integration while the usual methods of real integral calculus failed in those cases, so and further in the complex integration, we are able to prove some basic properties of analytic functions like their distance of higher order derivatives which will be otherwise difficult to establish.

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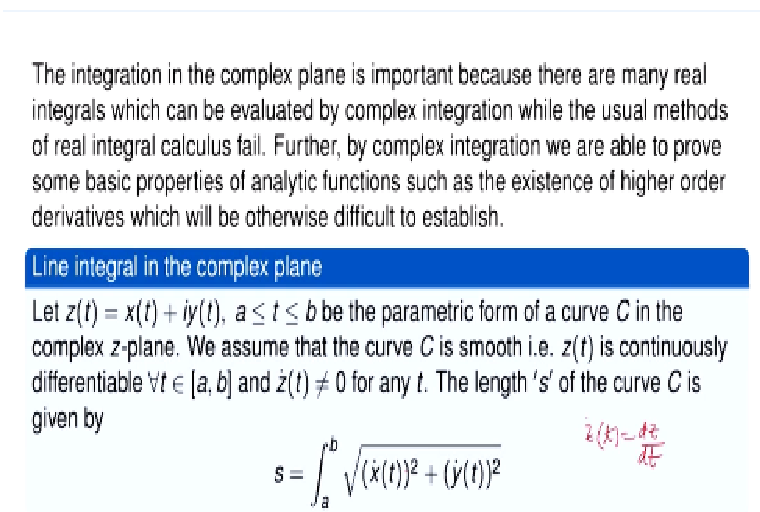

Let us consider first the line integral in the complex line, so let us consider the parametric representation of a curve C in the complex z plane, let jet zt = xt + iy t a < or = t < or = b the; be with the parametric form of a curve C in the complex z plane, let us assume that the curve C is smooth that is the zt function is continuously differentiable for all t belonging to the close interval ab and dz/dt is != 0, z dot t means dz/dt is != 0.

The length of the curve we know; the length of the curve as is then given by S = integral a to b dx/dt whole square + dy/dt whole square dt.

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Then C is a rectifiable curve. Further, the positive direction along C corresponds to the sense of increasing values of t.

Let $f(z)$ be a continuous function at each point of C. We subdivide the interval $a \le t \le b$ by means of points

$$t_0(=a), t_1, t_2, \ldots, t_{n-1}, t_n(=b)$$

where $t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n(=z)$$

where $z_j = z(t_j), j = 0, 1, \ldots, n$.

The curve C is then called the rectifiable curve, now the positive direction along C corresponds to the sense of increasing values of t, the direction in which the values of t increase, okay, a set with the positive sense along the curve C. Let us consider a continuous function fz at each point of C, let us say fz is a continuous function at each point of C, we sub divide the interval ab okay the interval; close interval ab by means of these n +1 points.

T0 = a, t1, t2, tn – 1and tn = b, okay, so this is your suppose interval ab, let us divided it into n parts, okay, so t0 < t1 < t2 < tn -1 < tn, now we have zt = xt + iy t, where a < or = t < or = b, so corresponding to these values of t; t0, t1, t2, t3, tn -1, tn, there will be the values of z, okay, so let us say z ti = zi, where i varies from; i goes from 0 to n, okay so that means z t0 = z0, z t1 = z1, z tn = zn.

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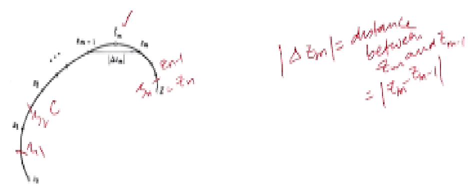

Figure : Fig.1

On each portion of a subdivision of C, let us choose a point, say, ζ_1 between z_0 and z_1, ζ_2 between z_1 and z_2 and so on. Then we form the sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m, \qquad (1)$$

where $\Delta z_m = z_m - z_{m-1}$, $m = 1, 2, \ldots n$

Then, these curve; this curve let us say this curve C okay, z0 is point here, z1 is the point here, z2 here and this is your z, z is; this is zn, okay. Now, on each portion of a subdivision, so this divided the curve into n parts on each portion of a subdivision of C, let us choose a point, say zeta 1 between z0 and z1, so here we choose zeta 1 between z1 and z2, we choose a point zeta 2 and so on which mean, zm – 1 and zm, we choose a point zeta m in a completely arbitrary manner and so on.

So, and then we form S sum, okay Sn, so Sn will be sigma n =; because in the nth interval, zn – 1 zn; zn – 1 zn we choose zeta n okay, so then and since f is a continuous function at each point of C, so we can find the values of f, at zeta 1, zeta 2, zeta m and zeta n and then multiply them by delta zm, delta f zeta m is multiplied by delta zm, where delta zm is zm – zm -1, okay, this is zm – 1, this is zm and the magnitude of delta zm.

Mod of delta zm, this is the distance between zm and zm – 1, okay, so the lines of this curve, okay, mod of delta zm is the length of this curve, so you multiply f zeta m/ delta zm and form the sum from over sigma over m from 1 to m, so Sn you get, sigma n = 1 to n f zeta m delta zm.

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This we do for each $n = 2, 3, \dots$ in a completely independent manner but in such a way that the greatest of $|\Delta z_m| \to 0$ as $n \to \infty$. This gives a sequence of complex numbers $S_2, S_3, S-4, \dots$. The limit of the sequence as $n \to \infty$ is called the line integral of $f(z)$ along the oriented curve C and is denoted by

$$\int_C f(z)\, dz.$$

$\max |\Delta z_m| \to 0$

$\int_C f(z)dz = \lim_{n\to\infty} S_n$, $\max_{1\le m\le n} |\Delta z_m| \to 0$

The curve C is called the path of integration.

And this we do for each value of n, n = 2, 3 and so on, we have divided into n parts, so let us start with n =2, we divide first into 2 parts, okay, form Sn, Sn means from we get S2, okay for n = 2, we get a 2, for n = 3 similarly, we get S3 and so on, okay. So, this we do in a completely independent manner but in such a way that when you sub divide the curve C into n parts, let us and n increase, n goes to infinity.

We do this is such a way that the maximum of mod of delta zm, okay goes to 0, as n goes to infinity that is means the maximum arch length, maximum cord length okay, delta z mod of delta zm is the distance between zm and zm -1, okay, maximum of mod of delta zm tends to 0 as n goes to infinity, then the sequence of complex number S2, S3, and so on that converges to the integral of fz along C, okay.

The limit of the sequence then goes to infinity and it is called the line integral of fz along the oriented curve C, we are considering the orientation of the curve C, okay, we have decided to take the sense along the curve C, okay, so it is called an orientated curve, so oriented curve C and we denoted by integral over C fz dz, the curve C is called the path of the integration. So, integral over C, fz dz here is the limit of Sn as n goes to infinity.

But in such a way that maximum of mod of delta zm, okay, when < or =; m < or = m goes to 0, okay, so this is how be the definition of the line integral of fz, okay is exactly the same as we do it in the case of real integrals.

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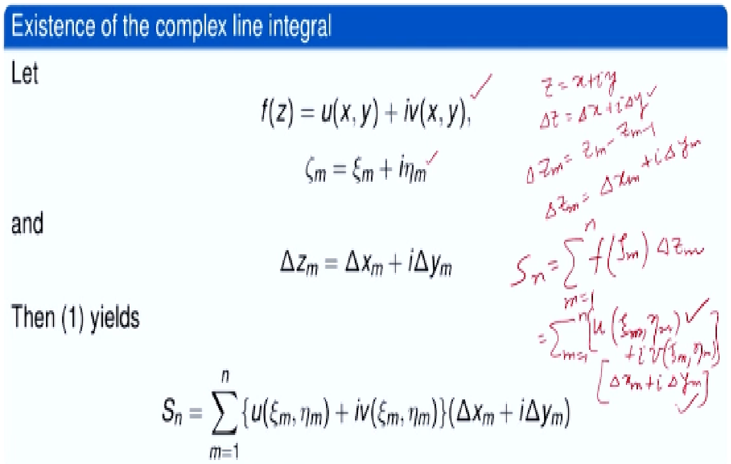

So, now let us discuss the existence of the complex line integral, fz is a complex function, so let us take the real and imaginary parts of fz as u xy and v xy, so fz is u xy + iv xy, let us zeta m is a point which; lie between zm – 1 and zm, so we can take it as Xi m + I eta m and delta zm; z is x + iy, so delta z is delta x + I delta y, so delta zm which is the zm – zm – 1; zm is delta zm is zm – zm – 1, okay.

So, delta zm because of delta z means delta x + I delta y, we write delta zm as delta xm + I delta ym okay, if zm is say xm + I ym and zm -1 is xm – 1 + im – 1, then I ym – 1, so then we can get delta xm delta ym, so delta z is delta xm + I delta ym, then the equation 1 yields, we get the following, equation 1, this equation, Sn =; we have Sn = sigma m = 1to n f zeta m delta zm, okay, this gives you sigma m = 1to n, f zeta m will give you u Xi m eta m.

Because this zeta m is Xi m + i eta m, its real part is Xi m, imaginary part is eta m, so we get u Xi m eta m + iv Xi m eta m, this multiplied by delta zm which is delta xm + I delta ym, okay, so we multiply this expression by delta xm + I delta ym and sum over m = 1 to n.

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$$i^2 = -1$$

$$S_n = \sum_{m=1}^{n} u(\xi_m, \eta_m)\Delta x_m - \sum_{m=1}^{n} v(\xi_m, \eta_m)\Delta y_m + i\left\{ \sum_{m=1}^{n} u(\xi_m, \eta_m)\Delta y_m + \sum_{m=1}^{n} v(\xi_m, \eta_m)\Delta x_m \right\} \quad (2)$$

All these sums are real and since f is continuous, the real functions $u(x, y)$ and $v(x, y)$ are also continuous. As $n \to \infty$, maximum of $|\Delta z_m| \to 0$ and so also maximum of Δx_m and Δy_m will approach zero. Hence each sum in (2) can be replaced by a real line integral.

$$|\Delta z_m| = \sqrt{(\Delta x_m)^2 + (\Delta y_m)^2}$$

$$\max |\Delta z_m| \to 0, \quad \Delta x_m \to 0, \ \Delta y_m \to 0$$

$$\lim_{n \to \infty} S_n = \int_C f(z)dz = \left(\int_C u(x,y)dx - \int_C v(x,y)dy \right) + i\left(\int_C u(x,y)dy + \int_C v(x,y)dx \right)$$

Then if you these 2 vectors; this vector and this vector; this expression and this expression, what you get is this and u iota square = - 1, okay, we get sigma m = 1 to 1, use im eta m, delta xm – sigma m = 1 to 1, these im eta m, delta ym + iota times sigma m = 1 to n, u is im eta m delta ym + m = 1 to n, these im eta m delta xm. Now, all these sums, okay, all these sums are real, you can see, u xy is a real function, delta x is also real.

So, u xy * delta x is the real quantity, so these; all these expressions; this, this, this and this, they are all real sums okay, since f is a continuous function, so its real and imaginary parts are continuous, u xy and v xy are continuous functions and therefore, as n goes to infinity, when maximum of mod of delta zn goes to 0, maximum of mod of delta xm and mod of delta ym also goes to 0, you an see, we have delta zm = this, okay.

So, delta zm goes to 0, mod of delta zm will be what; under root delta xm square delta ym square, okay so when maximum of mod of delta zm goes to 0, when this goes to 0, okay, delta xm and delta ym, they will also tend to 0, okay, so they will also 0 and hence each sum here can be replace by a real integral, so limit n tends to infinity, Sn gives you integral over C fz dz which is =integral over C uxy dx – integral over C v xy dy + iota times integral over C u xy dy.

Because we have delta y here, delta ym, so we get u xy dy + integral over v xy; integral over C v xy, okay, so the evaluation of a complex function, okay, a complex integral reduces to a evaluation of 4 real integrals, this integral, this integral, this integral and this integral.

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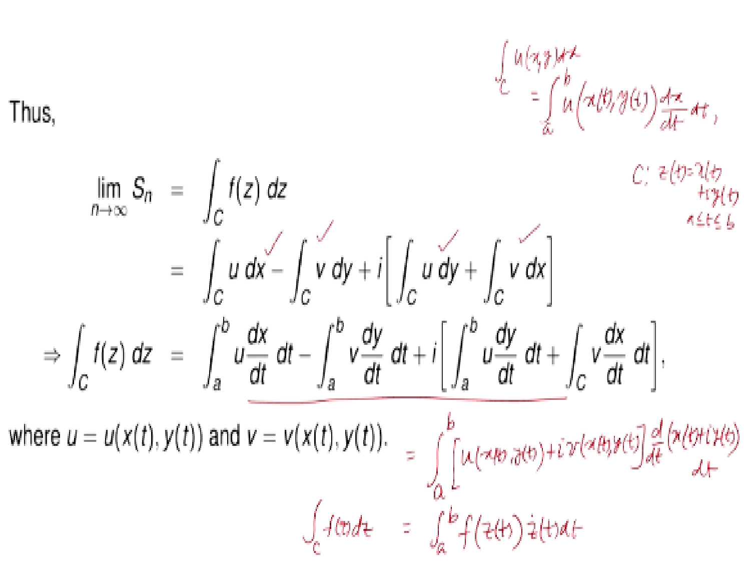

Now, and evaluation of these integrals, these real integrals; these 4 integrals; this one, this one, this one and this one can be done easily by using the parametric form of the curve C, okay so we replace integral over C u dx y integral over C u dx over dt, dt, integral over C uxy dx will be written as integral over a to b, u xt yt dx over dt, dt okay and we know that the parametric form of a curve C is zt = xt + iy t where a is < or = t < = b.

So, this is how we can evaluate the integral of u with respect to x, integral of v we can find, with respect to y and so on, okay so these 4 real integrals can be determined and we get the integral over C fz dz. Now, briefly this can be expressed further in a simpler form.

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Briefly, the above representation can also be expressed as

$$\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$$

where $z(t) = x(t) + iy(t)$.

$\dot{z}(t) = \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$

$f(z) = u(x,y) + iv(x,y)$

$f(z(t)) = u(x(t),y(t)) + iv(x(t),y(t))$

$i^2 = -1$

Example 1

Consider

$$\int_C \frac{1}{z - z_0}\,dz$$

$|z - z_0| = a$

$z - z_0 = ae^{it}, \; 0 \le t \le 2\pi$

where C is the circle of radius 'a' and center z_0 in the counterclockwise sense.
we may write $c := z - z_0 = ae^{it}, 0 \le t \le 2\pi$.

You can write, actually this is nothing but integral over a to b fz t dot z, this is z dot t dt, z dot t is dz/dt, okay and dz/dt is dx/dt + i dy/dt, okay, you can see fz t, fz = u xy + iv xy, so f of zt will be = u xt yt + iota v xt yt, okay, so when you multiply fz t, okay by z dot t okay, and u is iota square = - 1, okay and integrate over a to b, what you get is the expressions this one, okay, this is nothing but this quantity, this is nothing but integral over a to b u xt yt + iv xt yt * d over dt of xt + iy t dt, okay.

If you multiply these 2, you get this right hand side, so this is and this is nothing but integral over a to b f zt * z dot t dt, so integral over C, fz dz by using the parametric form of the complex of the curve C can be; the complex integration can be replaced by v definite integration, so we get this. Now, let us first, for example, consider integral over C 1 over z – z0 dz, where C is the circle of radius a and centre z0 in the counter, so we are moving along the curve C in the counter clock wise direction, okay.

What we do is; let us write the parametric form of the curve C, we can write the; the equation of the curve C is what; mod of z – z0 = a, okay because if a is the radius and z0 is the centre we know that the curve C can be written as mod of z – z0 = a, so we write in the parametric form, we write it as z – z0 = a times e to the power it, okay and we are moving in the counter clockwise direction, so we have 0 <or = t < or = 2 pi, okay.

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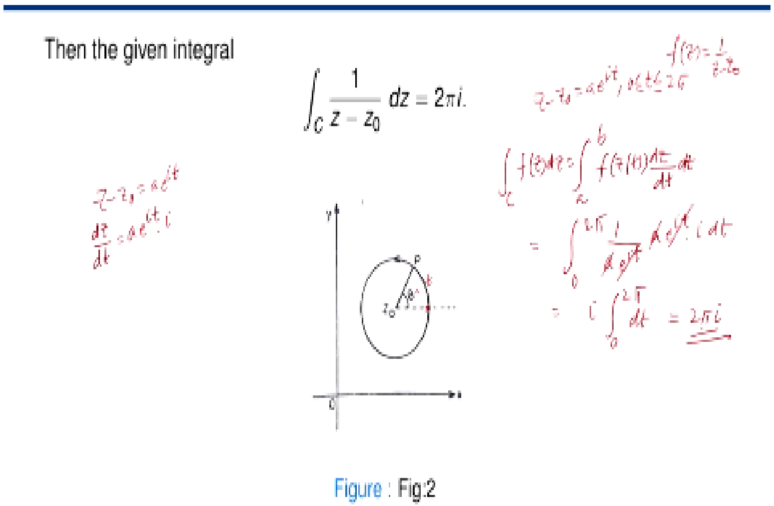

So, you can see here, the figure here you can see, this is your figure, okay we are moving the centre is z0, radius is a, if you take the point P here, theta, I have written z – z0 = a e raised to the power it, this theta = t for us, okay. So, you can see t is 0 here, okay, here t is 0 and it moves from 0 to 2 pi, when we go in the counter clockwise direction, okay. So, integral over C fz dz we have seen, this = integral over a to b f zt dz/dt * dt, okay.

So, replacing z by zt in the equation of the function fz, we will get fzt, fz is given as 1 over z – z0 here, okay, the complex form of the function fz is 1 over z – z0, so let us use the parametric form, so when we use the parametric form, fz becomes fzt and fzt will be = integral limits of 0 to 2pi, okay, f zt will be 1 over a e raised to the power iota t, okay, z – z0 is a e raised to the power iota t and dz/dt, let us find z – z0 = a e raised to the power it.

When we differentiate with respect to t, dz/dt, z0 is a constant, so if derivative with respect to t is 0, so we get a e to the power it * I, okay, so the value here a e to the power it * I dt, now e to the power it is never 0, so we can cancel out and what we get is; i times integral 0 to 2pi dt, so we get 2pi * iota, this is how we get the value of the given integral using the parametric form of the complex; parametric form of the curve C.

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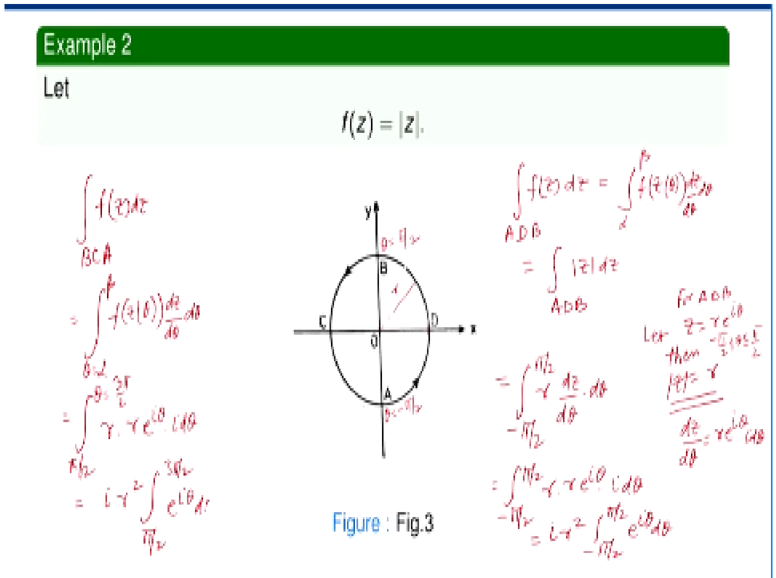

Let us take another problem, fz = mod of z, okay.

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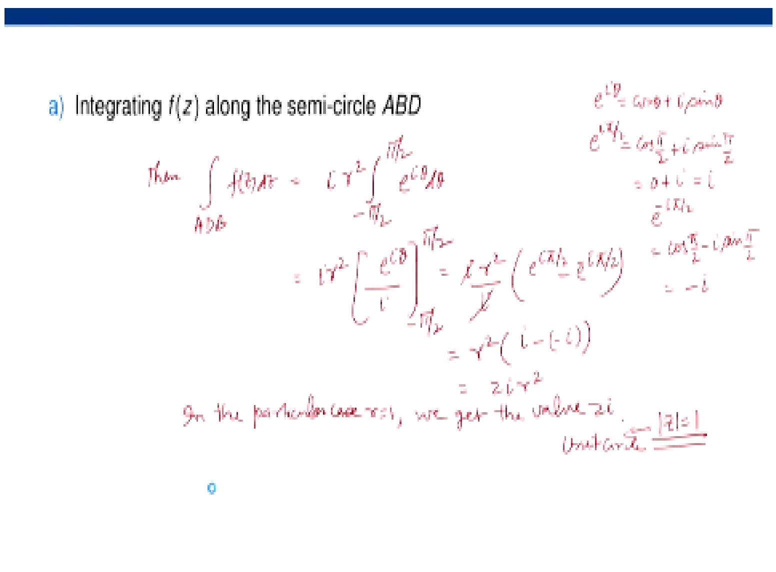

So, we have to integrate fz along the semi-circle ABD here, okay this semi-circle ABD, okay ABD, okay this semicircle ABD, so we have to integrate along the semi-circle ABD and it is centre is that origin, okay, radius we can take to be r, okay, so, we have to integrate fz along the semi-circle ABD

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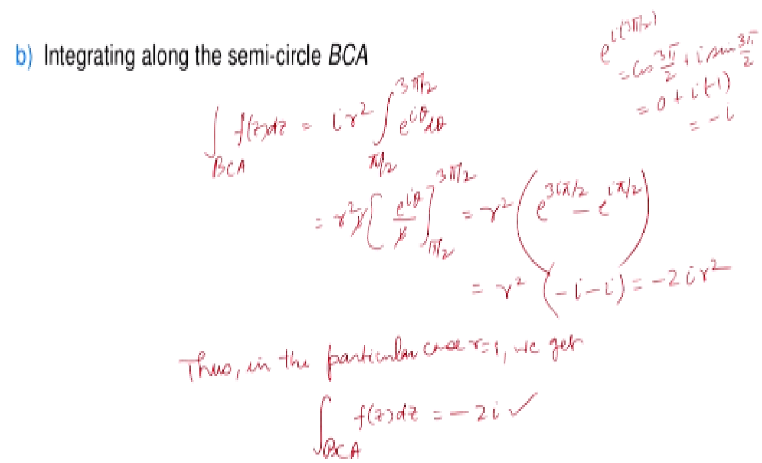

And then in the second case, we have to integrate along the semi-circle BCA, so semi-circle BCA is this one, okay, this is semicircle BCA, so first we integrate along ABD, okay, so let us see we have z = mod of z, so we have to integrate fz dz ADB, okay so what we will do is; fz = mod of z, so integral ADB mod of z dz, we have to find okay, so let us write dz = let z be = r e to the power I theta.

So, then mod of z = r, okay, now you can see here, theta varies from theta = -pi/2 here and theta = pi/2 here, so when you integrate mod of z along ADB, what you get; integral over; we can write z = r ei theta and theta varies from –pi/2 to pi/2, in case of ADB, for ADB z = r ei theta where theta varies from –pi/2 to pi/2, so –pi/2 to pi/2 and mod of z = r, so we get r times and then dz/ d theta * d theta.

Because fz is integral over limits of theta, say alpha to beta fz theta dz/d theta * d theta okay by over article on complex line integral, so this will be how much; this is dz/ d theta will be = r e to the power I theta * I d theta, okay so this is –pi/2 to pi/2 r times r e to the power i theta * id theta, r sqaure we can write it outside, I also we can write outside, so –pi/2 to pi/2 e to the power I theta d theta, okay.

Now, so then integral over ADB, fz dz = I r square, this is what we get, okay, so now, e to the power I theta we know it is cos theta + I sin theta, okay, so e to the power I pi/2 is cos pi/2 + I sin pi/2 and this =; cos pi/2 is 0, sin pi/2 is 1, so we get iota here and similarly, e to the power –I pi/2 = cos pi/2 – I sin pi/2, so this is –I, so what do we get here; r square times i - - i and we get 2i r square.

So, in the particular case, r =1, okay, we get the value 2i, okay, mod of z = r we have taken, so mod of z = 1 here, okay, mod of z = 1 is called the unit circle, so if you consider the unit circle, I mean to say that integral of mod of z along ADB = 2i. Now, let us consider the case of BCA, okay BCA is this, okay, so here, when you move in this direction, theta increases from 0 to pi/2, then 2pi and then 3pi/2, okay.

So, now we shall integrate along; when you move along BCA, okay then what you get is this, we start with theta = pi/2 and go up to theta = 3pi/2, okay, fz theta will be = mod of z, mod of z = r so r and dz/ theta; d theta we have already have found, r e to the power I theta * id theta, okay, so what do we get is; I times r square pi/2 to 3/2 e to the power I theta d theta, okay. So, this will give you this one, okay integral over BCA fz dz = ir square and this is = r square times.

So, this I cancels with this I and what we get; r square times e to the power 3i pi/2 – e to the power I pi/2, e to the power 3i pi/2 i= cos 3pi/2 + I sin 3 pi/2, we know that cos 3 pi/2 is 0 and sin 3 pi/2 is sin pi + pi/2, so – sin pi/2, so we get I times -1, so this is –I, so this is = r square times –I and e to the power I pi/2 is cos pi/2 + I sin pi/2, so then I, so we get –I, so we get -2i r square, okay.

And thus in the particular case, r = 1, okay we get, -2i, okay, so when you integrate along the semicircle BCA where the circle has radius 1 is that the value -2i, okay.

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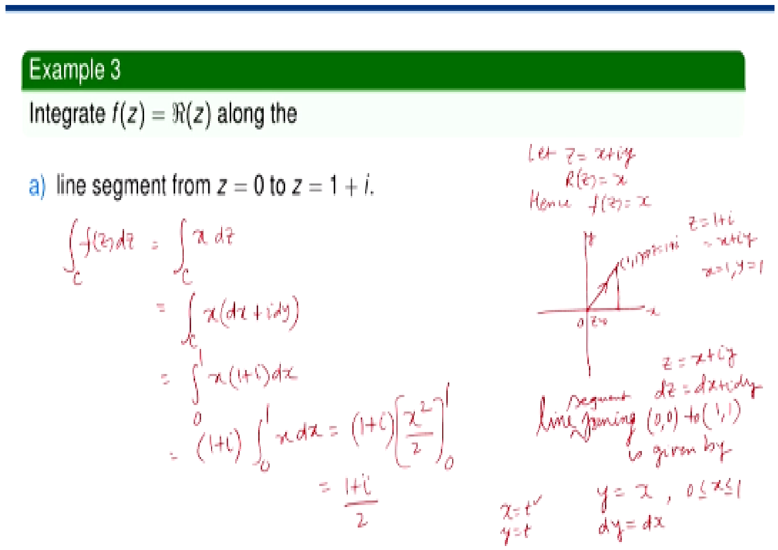

Let us consider one more example, fz = real part of z, okay, so let z be = x + iy, so then real part of z = x and hence fz = real part of z means x, okay, we are considering the line segment from z = 0 to z = 1 + I, so this is z =0, origin here z = 0 and z = 1 + I means, 1 1 point, okay, so 1 1 point, this is x = 1, this is y = 1, okay, so 1, 1 point, we are going from 0 to 1 + I direct, okay, so this is the line segment where we are integrating.

And we are integrating from z = 0 to z = 1 + I, this is nothing but equivalent to z = 1 +I, okay, so integral over fz dz here will be =; okay, fz is x dz, okay, now what do we do; z = x + iy, okay, so we can write here dz =; z = x + iy, so dz = dx + I dy, so I can write it as integral over C x dx + I dy, okay, the equation of the line, okay the line joining 00 to 11 is given by y = x, okay, so dy = dx, okay.

So, I can write integral over x and then dx + I dx that is 1 + I dx, okay and x varies from 0 to 1, okay, so 0 < or = x < or = 1, okay line segment joining 00 to 11 is given by y =x, where 0< = x < =1 from 0 to 1, so we get 1 + I is a constant, we can write it outside and 0 to 1 x dx we get, so this 1 + I times x square/ 2, 01, okay, so 1 + I times 1 over 2, so I can write 1 +; here what I have done is parametric form of the curve C I have taken, where in place of t I have written x, okay.

So, actually I am considering parameter t to be = x here, so x here we have taken as t or you can; and y is also = t, okay, so this is same as the complex integration, we have taken instead of the parameter t, we have taken x and y = x and y is also = t, so this is what we get, when we integrate from z = 0 to z = 1 + I along the line segment.

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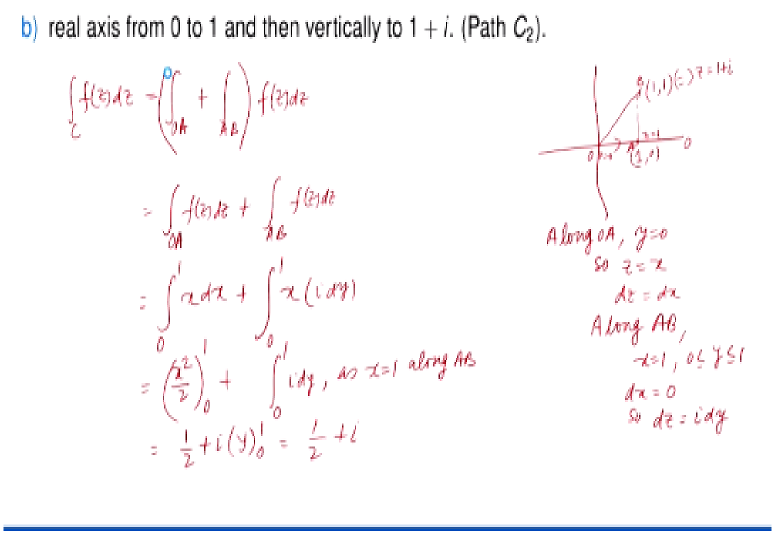

Now, in the second case, we integrate along this one, this is your z =0, this z = 1, 1 0 point and this is your 1 1 point, okay which is same as z = 1 + I, okay, so we have to move along the real axis, this is real axis, okay we have to move along the real axis from 0 to z = 0 to z = 1, this is z = 0, this is z = 1 and then vertically 2 1 + I, from z = 1 to z = 1 + I, we are moving vertically, okay. So, integral over C, fz dz = integral over; let us say this is I can write it as some let us say A point and this is B point.

So, integral over OA + integral over OB fz dz, okay, so integral over OA fz dz, but when we go from O to A, okay along the real axis, along OA, y =0, okay, so z = x, z =x means fz is real part of z, real part of z is x, so x times, dz = dx and x varies from 0 to 1, okay, so 0 to 1 x dx we have. Now, along; this is AB, we are going from O to A, then A to B, okay, so AB means now, along AB what is happening is; x = constant, x = 1, okay.

And x = 1 and y varies from 0 to 1, okay, so when x = 1, dx = 0, so dz = I dy, okay, so we will have here x times I dy and y varies from 0 to 1, okay, so we get here x = 1, we will put there, okay so we will get here x square/ 2, 0 to 1 and then iota times x = 1 along AB, so we have here iota dy, x = 1 along AB, okay, so what you get here; so we get here, 1/2 and then we had I times, we get 1 + 2/i.

Now, you can see here, the value that we obtained by integrating along the line segment from z =0 to z = 1 + I that was 1 + i/2, while the value that we have got from y integrating from 0 to A and then A to B is 1/2 + I, so both the values are not same, okay and therefore, the value of the integral is not does not always depend on the end points, it also depends on the geometric shape of the path.

So, we shall see later on because this function fz = real z, it is not analytic, okay, so that is why such a thing has happened, whenever a function fz is analytic, okay along whatever in the region of the analyticity of fz along whatever path you integrate from say z1 to z2, the value of the integral does not depend on the geometric shape of the path, it remains the constant, it only depends on the points z1 and z2 not on the geometric shape of the path.

But the function fz = real z here actually, is not analytic, so such a thing has happened.

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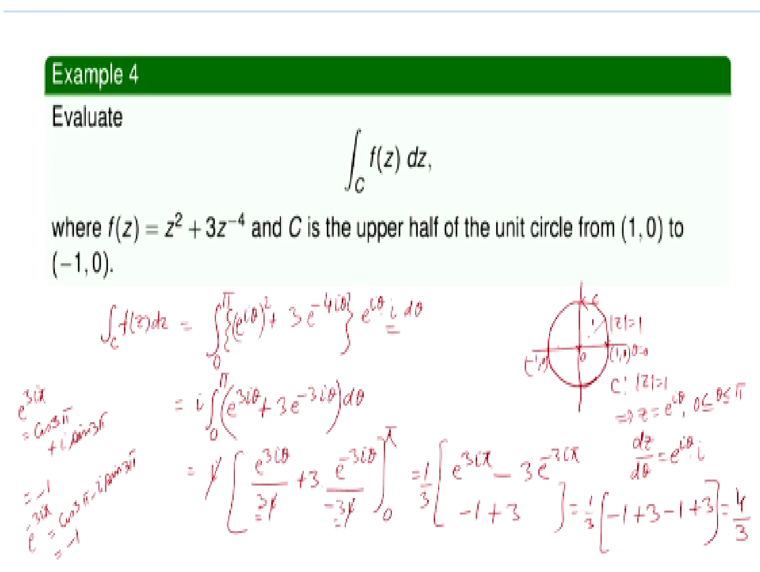

Now, let us consider another example, integral over C fz dz, where fz = z square + 3z to the power – 4, where C is upper half of the unit circle from z = ; from 1 0 to -1 0, let us draw the curve first, so this is your unit circle, mod of z = 1, this centre is 0 and radius is 1, okay, we are considering C is the upper half of the unit circle from 1 0, this is 1 0 point, so 1 0 to -1 0, so we have to integrate fz along the upper half.

So, this is parameter, so C is mod z = 1, the parametric form is z = e to the power I theta and theta varies here from, theta is 0 here at 1 0 and at -1 0 theta is pi, okay, so theta varies from 0 to pi, this is our curve C, okay. So, integral over C fz dz = integral over 0 to pi fz, when z is replaced by z theta is e to the power I theta, so e to the power I theta whole square + 3 times e to the power -4i theta and dz replace by dz/d theta; dz/d theta is e to the power I theta * i.

So, this is e to the power I theta * I d theta, so what we get here then; 0 to pi e raised to the power I theta whole square * e to the power 2i theta * e to the power I theta, this i; i can write it outside, so e to the power I theta * e to the power 2i theta * e to the power 3i theta, okay + 3 times e to the power -3i theta d theta. Now, this is = I times e to the power 3i theta over 3i 3 times e to the power -3i theta divided by -3i, 0 to pi.

So, this I can cancel with this I and I get e to the power 3i; okay, so what I get; 1 over 3, I can write outside, okay this 3, and this 3, so e to the power 3i pi + 3 times, so – okay, - 3 e to the power -3i pi, okay and then we put the lower limit, so – e to the power 0 that is 1 and then we get +3 e to the power 0 that is 1, okay. So, e to the power 3i pi; e to the power 3i pi = cos 3 pi + I sin 3 pi, cos 3 pi is cos 2 pi + pi which is cos pi, so cos pi -1 and sin 3 pi 0.

So, I get -1 here, e to the power -3i pi will be = cos 3pi – I sin 3pi and this is also = -1, so what I get; 1/3, so I get here -1, then here I get 3 here, then -1 + 3, so this is 4/3, so this is the value of the line integral of z along fz = z square + 3 times z to the power -4 along the unit circle from the point1 0 to the point -1 0 that is all in this lecture, thank you very much for your attention.