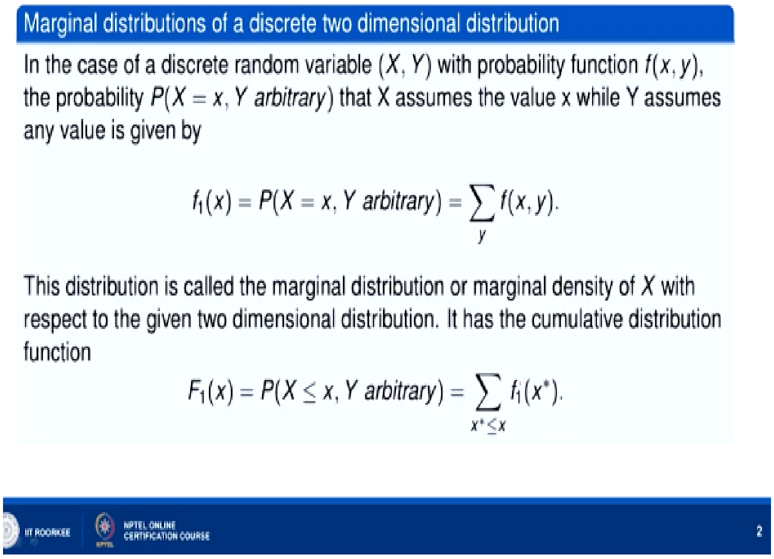
Advanced Engineering Mathematics
Prof. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 53
Joint Probability Distribution - II

Hello friends welcome to my second lecture on joint probability distribution. Let us first consider the discreet case of 2 random variables. So in the case of discrete random variable xy with probability function fxy, the probability that x takes the value x and y is arbitrary is given by f1x = probability that x takes the value x, y is arbitrary.

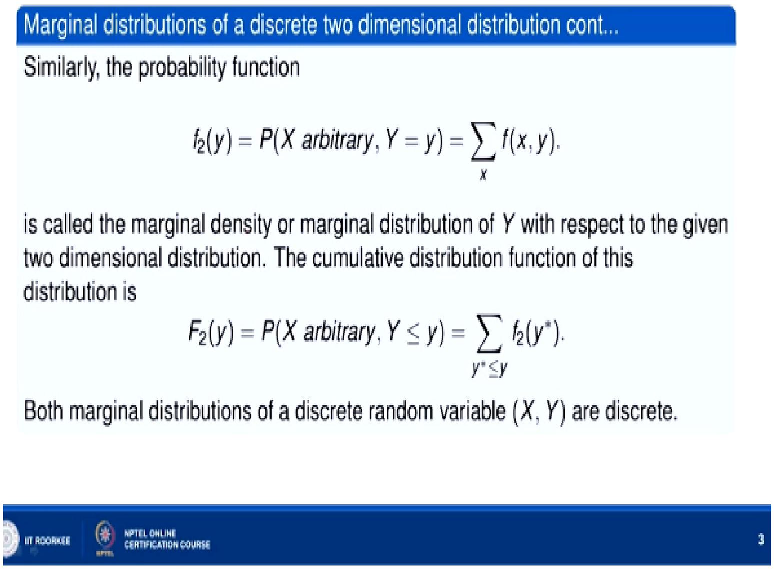
(Refer Slide Time: 00:54)



So we take the sum of the function fxy over all the values that the random variable y takes, this will give you the marginal distribution or marginal density of X with respect to the given 2 dimensional distribution. Now in the case of cumulative distribution function we have F1x = probability that X <= x and Y arbitrary which will be sigma f1 x star, where x star is <= x.

So we will take the sum of the marginal density function over all values of x star which are <= x to get the cumulative distribution function of x with respect to the given 2 dimensional distribution.

(Refer Slide Time: 01:45)

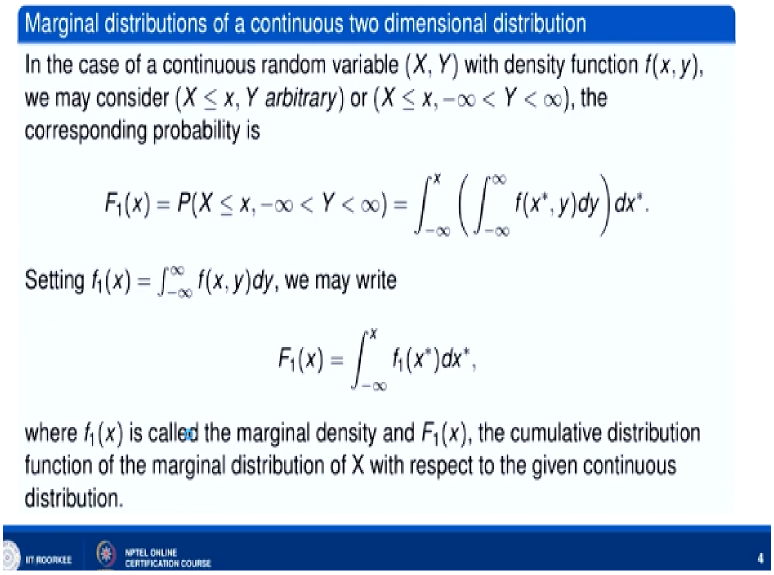


In the case of the marginal density function of Y with respect to the given 2 dimensional distribution we have F2y = probability that X is arbitrary, Y = y okay. This is defined in a similar manner as the case of marginal density function or marginal distribution of x with respect to the joint distribution xy. So probability that x is arbitrary, Y is = y is sigma, fxy where x sigma is taken over the values of x.

The value that x takes okay, so this is called the marginal density or marginal distribution of y with respect to the 2 dimensional distribution. The cumulative distribution function of this distribution that is cumulative distribution function of y with respect to the given 2 dimensional distribution is F2y = probability that x is arbitrary, Y is <= y, here we take the sum of the marginal density function over all values of y star that are <= y.

Both the marginal distributions you can see of the discrete random variable XY are discrete.

(Refer Slide Time: 02:58)

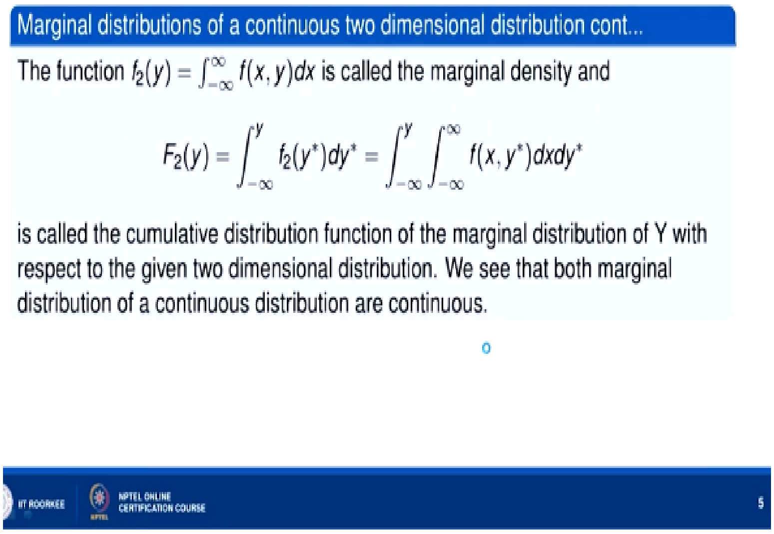


Now in the case of a continuous random variable XY, with density function Fxy, we may consider x <= x, Y arbitrary or X <= x – infinity < Y < infinity. The corresponding probability function okay is given by F1x = the corresponding probability will be F1x = PX <= x – infinity < Y < infinity this is integral over – infinity to x, integral over – infinity to infinity fx star y dy dx star.

This is the cumulative distribution function in the case of the cumulative distribution function of x with respect to the joint distribution xy. Here this integral over – infinity to infinity fxy dy is F1x which is the marginal density function of x with respect to the joint distribution okay. So using this marginal density function F1x we can write the cumulative distribution function F1x as integral over –infinity to x F1x star dx star.

Where f1x is called the marginal density and capital F1x is called the cumulative distribution function of the marginal distribution of x. So this f1x is called the marginal density or marginal distribution and F1x the cumulative distribution function of the marginal distribution of x with respect to the given continuous distribution.

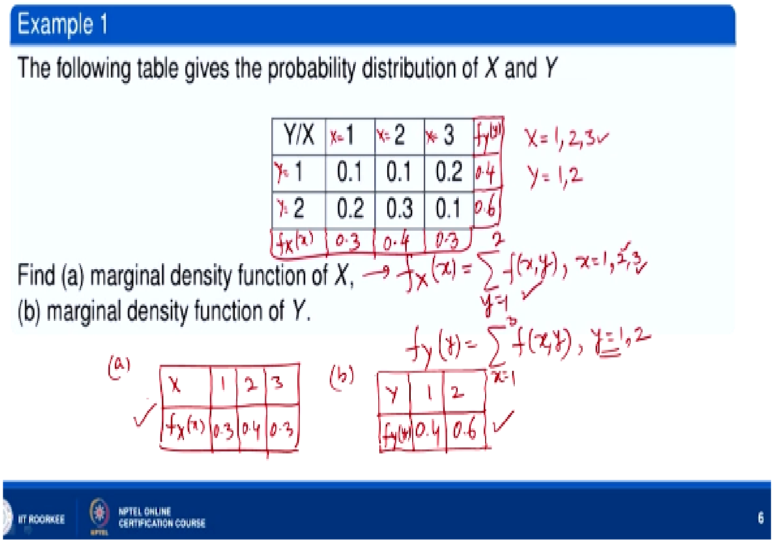
(Refer Slide Time: 04:34)



Now in a similar manner we can define the marginal density function or marginal distribution of y with respect to the joint continuous distribution xy and the corresponding cumulative distribution function. So the marginal density function of y with respect to the joint distribution will be given by f2y = integral over – infinity to infinity fxy dx it is called the marginal density okay.

And the corresponding cumulative distribution function is F2y = integral over – infinity to y f2y star dy star which is = integral over – infinity to y putting the value of f2 y star here, we get integral over – infinity to infinity fx y star dx, dy star okay. So this is called the cumulative distribution function of the marginal distribution of y with respect to the given 2 dimensional distribution and both you can see the marginal distributions, that is f1x, this f1x okay, this one and the next one F2y okay. They are continuous you can see.

(Refer Slide Time: 05:49)



Okay, let us consider the following example. The following table gives the probability distribution of X and Y okay. So this is Y, Y means these values are of Y, Y = 1, this is Y = 2 and here we have X = 1, X = 2, X = 3 okay, so we want to find the marginal density function of X first of all okay. Marginal density function of X means we want to determine fxx okay. Marginal density function or marginal distribution of x okay.

So here you can see X takes the values 1, 2, 3 okay, Y takes the value 1, 2 okay. So fxx we have to determine, fxx will be given by sigma fxy y = 1 to 2 okay, yeah, so this will be fx y and fyy, this is marginal density function of y. So fyy will be = sigma fxy, x varies from 1 to 3 okay. Because x takes the values 1, 2, 3. Now you can see here when, so this is fxx will be found for x = 1, 2, 3.

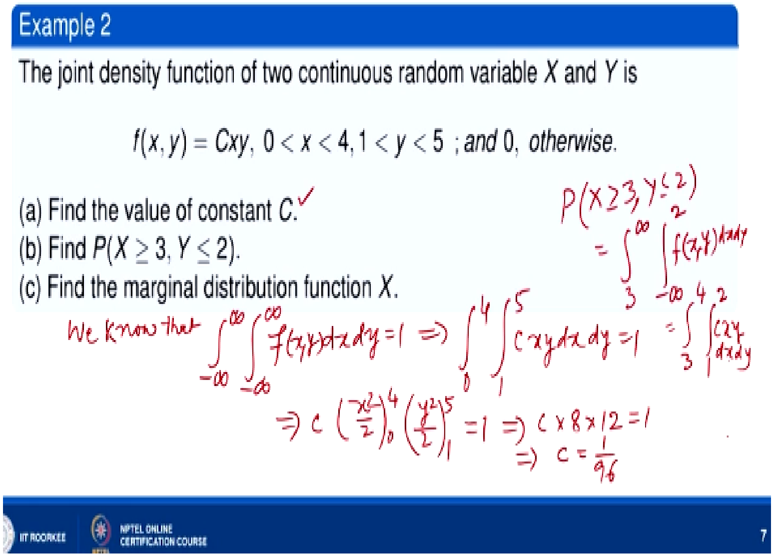
Okay and fyy will be found for y = 1, 2 okay now you can see here. If you sum the column entries, okay, if you sum the column entries then you get the value of fxx for x = 1 okay. So here you write this is fxx okay. You can see here in this table. So fxx, when x = 1, fxx will be summation over all the values of y. Y = 1 to 2, so this is 0.1 + 0.2 that is 0.3 okay. When x = 2, okay the values of y for 1 and 2 are 0.1 and 0.3.

So let us sum over the values of Y, so it is 0.4 okay and then when x = 3, the values of fx y are for y = 1 it is 0.2, for y = 2 it is 0.1, so we have 0.3 okay. So we can say the marginal density function of x. The part a okay, so we have values of x here and we have fxx here. Values of x are 1, 2, 3 and the values of fxx are 0.3, 0.4, 0.3 okay, so this is the answer for the part a. This is marginal density function of x.

Now part b, so now we want to find marginal density function of y with respect to the joint distribution, so we have this. So this is our y, as y is taking values 1 and 2 okay, we want fyy okay. So now what will happen fyy is summing of the values of fxy for x = 1 to 3, x goes from 1 to 3. So we now take the sum of entries in the rows okay. So we have this. So this is fyy. Okay when y = 1 okay, you can sum the entries 0.1, 0.1, 0.2.

So 0.1, 0.1, 0.2 is 0.4 okay, summing of the values of fxy okay when x runs from 1 to 3 means you take the sum of the entries in the row for y = 1 and then for y = 2 you take again the sum 0.2, 0.3, 0.1, so 0.6. So we have 0.4 and this is 0.6, so this is probability density function of, marginal density function of y and this is marginal density function of x.

(Refer Slide Time: 10:31)



Let us now consider we have considered just now a discrete case of 2 discrete random variables, let us consider now a continuous case, continuous joint distribution of 2 continuous random variables. So the joint density function of 2 continuous random variables X and Y is given by fxy = Cxy, x varies from 0 to 4 okay, y varies from 1 to 5, then fxy Cxy and 0 otherwise okay.

So we have to determine the value of the constant C first. Okay we know that fxy is the joint density function, so integral over – infinity to infinity, integral over – infinity to infinity fxy dx dy will be = 1 okay. This implies that by the definition of fxy integral 0 to 4, integral 1 to 5 or cxy dx dy = 1, because fx y is 0 elsewhere okay. Now c is a constant, we integrate with respect to x to get x square/2 and it is 0 to 4 and then we have integrated.

When we integrate y, we get y square/2 integral over 1 to 5 okay, this is = 1 okay. So this implies C times 16/2 means 8 okay and then we have 25/2 – 1/2, so 24/2 that means 12, so we have C = 1/96 okay, so this is the value of the constant C. Now let us find the probability that x is >= 3, y is <= 2 okay. So probability that x is >= 3 and y is <= 2, will be given by integral over 3 to infinity okay.

x varies from 3 to infinity, y varies from – infinity to 2, fxy dx dy okay. So by the definition of fxy, it is only cxy over the interval 0 to 4 for x and 1 to 5 for y elsewhere it is 0. So this integral will become the integral from 3 to 4 for x, because when x is > 4 it is 0, and for y it will be replaced by 1 to 2, because if it is x, y is < 1, it is 0 okay. So cxy dx dy okay. So let us now calculate it 3 to 4, 1 to 2.

(Refer Slide Time: 13:32)

We have

$$P(X \geq 3, Y \leq 2) = \int_3^4 \int_1^2 \frac{1}{96} xy \, dx \, dy$$

$$= \frac{1}{96} \left(\frac{x^2}{2}\right)_3^4 \left(\frac{y^2}{2}\right)_1^2 = \frac{1}{96}\left(8 - \frac{9}{2}\right)\left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{96} \times \frac{7}{2} \times \frac{3}{2} = \frac{7}{128}$$

Marginal density function of X

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_1^5 cxy\, dy = \frac{1}{96} x \left(\frac{y^2}{2}\right)_1^5 = \frac{x}{96} \times 12 = \frac{x}{8}, \quad 0 < x < 4$$

Thus $f_X(x) = \begin{cases} \frac{x}{8}, & 0 < x < 4 \\ 0, & \text{elsewhere} \end{cases}$

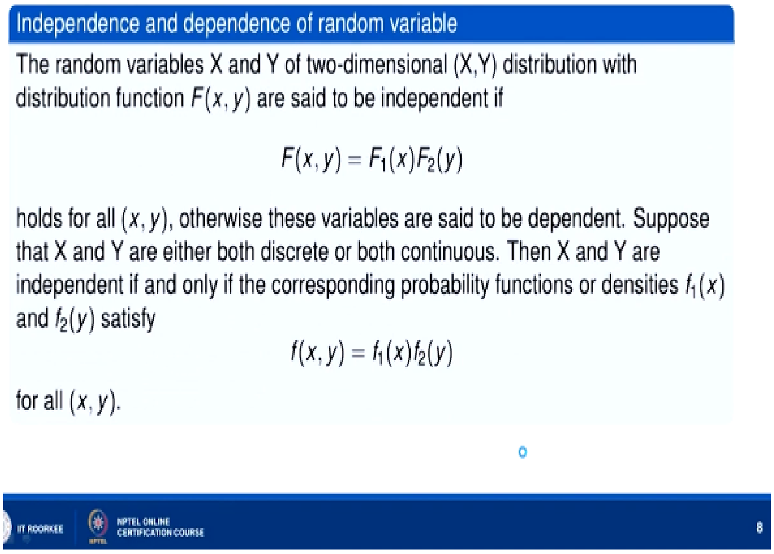
So we have P X >= 3 y <= 2 is given by this integral okay. So we integrate 1/96 * x square/2 integral over the limits are 3 to 4, then we have y2/2 limits are 1 to 2. So this gives you 1/96 and we have here 16/2 means 8-9/2 and we have here 4/2 means 2 – 1/2 okay. So this is = 1/96 and then we have 16-9 so 7/2 and we have here 3/2 okay. So 3 when cancels with this we get 3 times 3 are 9 and 3 2s are 6.

So this is 7/128 okay, so this is the probability when x is >= 3, y is <= 2. Now find the marginal distribution function of x. So we have to find the marginal density function of x okay. Now marginal density function of x is given by is = fx x and this is given by integral over – infinity to x okay, fx y, it is given by integral over – infinity to infinity fxx = integral over – infinity to infinity fxy dy okay.

Yeah we have to because fxx is probability that x takes the value x and y is arbitrary. So this is integral over y varies from 1 to 5 okay, elsewhere it is 0 for y, fxy 0, so 1 to 5 okay and we have c xy dy okay. So this c is 1/96 * x * y square/2 and we have limits 1 to 5 okay. So x/96 and we have 25/2 when we put y as 5, 25/2 – 1/2 so we have 12 okay. So 12 8s are 96, so we have x/8 okay.

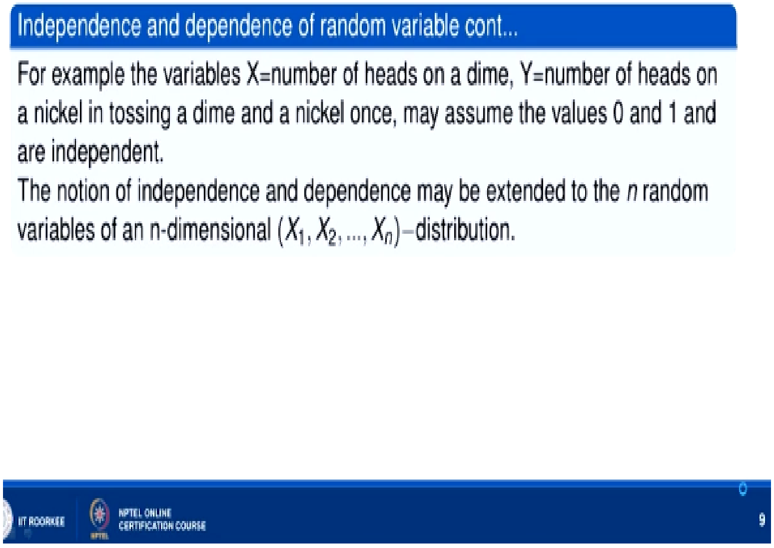
So if x lies in the interval 0 to 4 okay, if x lies in the interval 0 to 4 then fxx is x/8 okay. So thus fx x = x/8 when 0 is < x, < 4 and 0 elsewhere, because if your x does not lie in the interval 0 to 4 okay fxy will be 0 okay. So this is for 0 < x, < 4, so this is the marginal density function of x okay for example 2.

(Refer Slide Time: 17:06)



Now let us discuss independence, dependence of random variable. The random variables X and Y of 2-dimensional distribution X Y with distribution function Fxy, cumulative distribution function Fxy are called independent okay if Fxy = F1x * F2y, it holds for all xy, otherwise these variables are said to be dependent. Suppose that x and y are both discrete okay, are both continuous then x and y are independent if and only if the corresponding probability functions are densities f1x and f2y satisfy fxy = f1x * f2y for all xy.

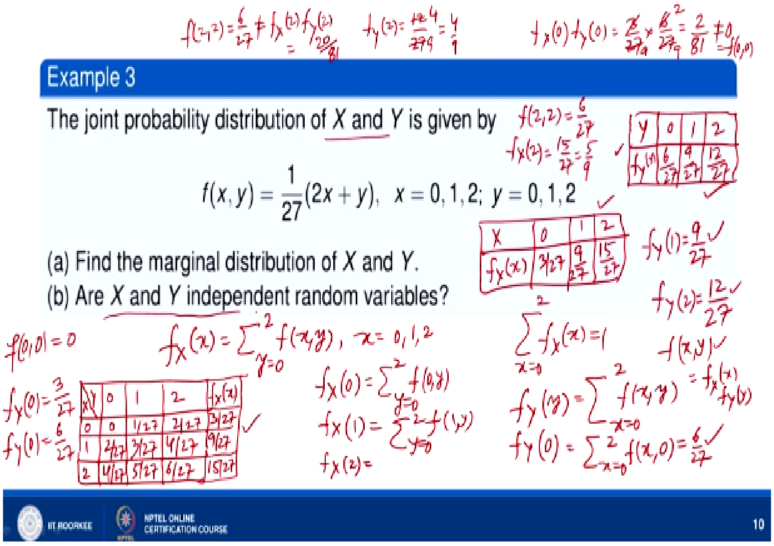
(Refer Slide Time: 17:49)



For example, you consider the case of random variables X and Y where X denotes the number of heads on a dime, okay and Y denotes the number of heads on a nickel okay then when you toss a dime and a nickel okay and their x denotes the number of heads on the dime and by the number of heads on the nickel, then they will assume values 0 and 1 okay and so they are both independent, the notion of independence and dependence may be extended to n random variables of an n-dimensional distribution.

In a similar manner we can extend it. So here the number of heads if you calculate the number of heads on a dime and denoted by X and number of heads on a nickel that denote by Y, then either head will come, then number will be 1, if X head does not come, then the value will be 0 okay. So number of heads will be either 0 or 1 okay and they are independent events. So this is the case where random variables X and Y are independent to each other.

(Refer Slide Time: 19:02)



Let us look at the joint probability distribution of X and Y, fxy = 1/27 times 2x + y, x = 0, 1, 2, y = 0, 1, 2, find the marginal distribution of x and y. So let us first find the marginal distribution of x. So fxx let us find, fxx is sigma y = 0 to 2 fxy okay and x takes the values 0, 1 and 2 okay. So let us form the table for fxx, you can see, we will have to first form the table for all x and y okay.

So let us say like this, this is suppose x and this is y okay. So y takes value 0, 1, 2 okay, x also takes value 0, 1 and 2 okay. So when x is 0, y is 0, x is 0, y 0 f00 = 0, so this is 0 okay. When x is 0 y = 1, x is 0, y = 1 so it is 1/27. When x is 0, y = 2 it is 2/27 okay. When x = 1 y is 0, 2/27. When x is 1, y = 1 it is 3/27. When x is 1, y is 2 it is 4/27 okay. When x is 2, y is 0 okay, so it is 4/27. When x is 2 and y is 1, so it is 5/27.

And when x is 2, y is 2, so it is 6/27 okay, let us make the total row wise okay. So when you row wise you consider the total okay, then what will happen, you are summing over fxy for y = 0, 1, 2 okay, x remains fixed okay, suppose x is 0 okay, so fx0 will be sigma y = 0 to 2 okay f0y okay that means f00 + f01 + f02. So this is 3/27 okay and when fx 1 you want to find, fx1 will be sigma y = 0 to 2 f1y okay.

So 2, 3, 5, 4, 9, 9/27 okay and then fx2 similarly fx2 will be 4, 5, 9, 9, 6, 15/27 okay, so we have 3 + 9 12, 12 +15, 27/27. So sum of fxx when x varies from 0 to 2 must be = 1 okay that is a check, so this is fxx okay. Similarly, we can find fyy, fyy = sigma x = 0 to 2 okay and we have fxy okay. So fy0 okay, fy0 is sigma x = 0 to 2, f okay this okay, so when you take x = 0 we have f 00, when you take x = 1 we have f10.

When you take x = 2 we have f20, that is column sum okay. So 0, 2/27, 4/27 so 6/27 okay and fy1 similarly okay column sum. So 1+3+ sum of entries in the column. So 1+3, 4, 4+5 9/27 and fy2 = you can see 2+4, 6; 6 +6, 12/27 and sum of the values of fyy for y = 0, 1, 2 must be = 1. So this is 6/27, this is 9/27, so 15/27 + 12/27, so 27/27 that is = 1. So marginal distribution of y. We have this table.

Y here, f/y here okay, y is taking value 0, 1, 2 okay, when y is 0 it is 6/27, when y is 1 it is 9/27, when y is 2 it is 12/27 okay, so this is marginal density function of y with respect to the joint distribution, this is marginal density function of x, this one fxx = you can write it in the form of the table. For 0 it is 3/27, for 1 it is 9/27, for 2 it is 15/27 like this okay, so this is marginal density function of x. This is marginal density function of y.

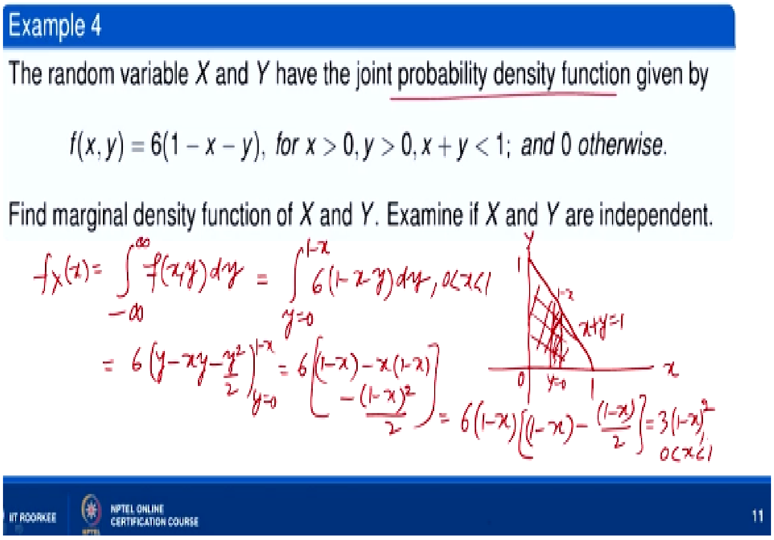
Okay are X and Y independent random variables? Now as we have seen in the definition of independence x and y will be independent random variables if fxy = fxx, the joint density function must be = the product of the marginal density functions, fxy must be = fxx * fyy for all xy belonging to 0, 1, 2 for all xy where x to x values from 0, 1, 2 and y to x value 0, 1, 2. Now let us see here.

We can notice that f00 you can see, f00 here is from the table you can see when x is 0, y is 0 it is 0 okay. So this f00 when x = 0, y = 0 is okay, what is the value of fx0? fx0 = 3/27 okay, what is the value of fy0 it is = 6/27 okay. So fx0 * fy0 = 18/27 *27 okay. So we get fx0 * fy0 okay 3/27 * 6/27 okay so this is 1/9 and this is 2/9, so this is 2/81 okay, which is not = 0 and 0 is the value of f00.

So f00 is not = fx0 * fy0 okay and therefore x and y are not independent okay, x and y will be independent provided fxy will be = fxx * fyy for all xy. Now we can see one more thing, f22 let us check for one more pair, when x is 2, y is 2, so f22 is how much here you can see f22 is 6/27 okay. So this is f22, the density function for the joint distribution, when x is 2, y is 2, 6/27, what is fx2? fx2 is = 15/27.

15/27 means 5/9 okay and fy2 will be = 12/27 fy2, so 4/9 okay. So 5/9 * 4/9 = 20/81 okay. So f22 is = 6/27 and this is not = fx2 * fy2 which is = 20/81 okay. So there is one more pair when x is 2, y is 2, fx y is not = fxx * fyy. So x and y are not independent random variables.

(Refer Slide Time: 29:06)



Let us take now the case of continuous random variable section Y. So the random variable section Y have the joint probability density function given by fxy = 6 times 1 - x - y when x is > 0, y is > 0, x+y <1 and 0 otherwise. So let us see in which part of the xy plane fxy is nonzero. So this is the part, this is 1, this is 1, this part okay. This is x + y = 1, this is x axis and this is y axis okay.

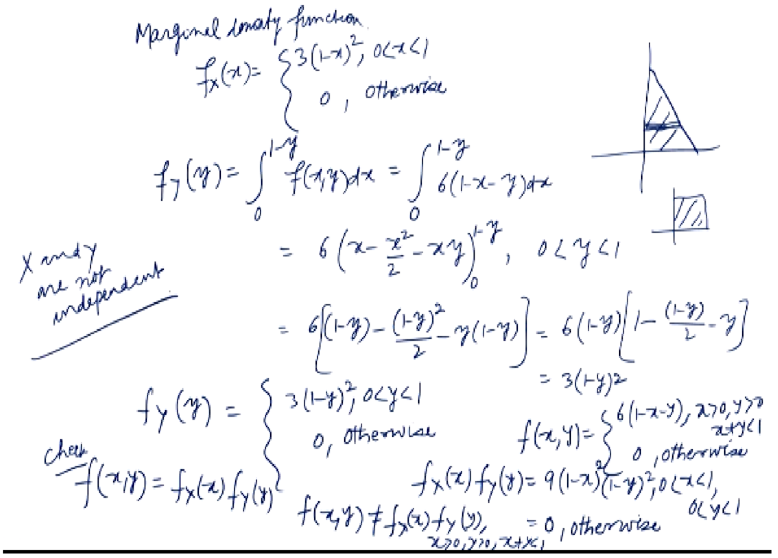
So x is > 0 here, y is > 0 and x + y is < 1, now we want to find the marginal density function of x and marginal density function of y. So fxx we have to find, fxx = when x will be > 0, y is > 0. So fxx is integral/- infinity to infinity okay, fxy dy okay. Now in this region fxy is given by 6 times 1-x –y. So this will be integral over y varies from, we have to integrate over y, x remains constant.

So we have to take a vertical step, along the vertical step x is constant, y varies and y varies from 0 to 1-x okay. So y varies from 0 to 1 –x and we have 6 times 1-x –y dy okay and this is the case when x lies between 0 and 1. If x lies elsewhere fxy will be 0. So this is = 6 times x – x square/2 – xy and when we put the limits y = 0 to y = 1-x what we get is 6 times x -, no we are integrating with respect to y.

So we should write here y – x * y – y square/2 okay. So y = 1-x, so we have 1-x here and then x times 1-x and we have -1-x whole square/2, okay, when y is 0 this is 0 okay. So this is how much, we can take 1-x common, so 6 times 1-x and what we get inside 1-x okay, 1-x – 1-x/2 okay. So 1-x/2 that is the value of the expression inside the bracket. So this is 3 times 1-x whole square.

So this is the fxx, the value of fxx when 0 is < x, <1. So we can write the marginal density function here.

(Refer Slide Time: 32:21)



Marginal density function as fxx = 3 times 1-x whole square when 0 is <x, <1 and 0 otherwise okay, now we, let us find the other marginal density function, that is marginal density function of y okay. When we want to find the marginal density function of y, now what will happen, we will integrate fxy with respect to x, keeping y fixed okay. So y is fixed along the horizontal strip okay.

So when y is fixed x varies from 0 to 1-x okay, x varies from 0 to 1-y, so we have this fyy = integral over 0 to 1-y, fxy dx okay. So this is 0 to 1-y and 6 times 1-x –y dx okay. So again what we will have, we will have the integral as 6 times x – x square/2 okay –x * y right and the limits are 0 1-y and the values of y vary from 0 to 1 okay, because we have this region and we are taking a horizontal strip in this region.

So y is fixed here, y varies from 0 to 1 okay. So this will be again 6 times 1-y -1-y whole square/2 – y * 1-y. So what we will get this is 6 times 1-y we can take common and then what we will get 1-y/2 – y okay, which is 3 times 1-y whole square, 0 < y <1. So fyy = 3 times 1-y whole square 0 <y <1 and 0 otherwise, okay. Now we have to answer whether the variables x and Y are independent or not.

So we have to see fx y even if they are independent, fxy must be = fxx * fyy okay. So let us check this. Let us check whether it is true or not okay. So fxy = we are given that fxy = 6 times 1-x –y when x is > 0, y is >0 and x + y is <1 and 0 otherwise. Here what is happening is fxy we have seen fxy = 6 times 1-x –y, x is > 0, y is >0 x+y < 1 okay and 0 otherwise and what we find is that fxx * fyy okay.

This is = 3 times 1-x whole square * 3 times 1-y whole square, so 9 times 1-x whole square * 1-y whole square when 0 is < x, <1, 0 is <y <1 okay, in this region and 0 otherwise. Okay so in the common portion, this portion, see when 0 is < x, <1 and 0, <y, <1 okay in this square okay and this region is a part of this okay. So over this region where x is > 0, y is >0 x+y is < 1 fxy 6 times 1-x –y while fxx * fyy is 9 times 1-x whole square * 1-y whole square.

So they are not equal okay, so it turns out that fxy is not = fxx * fyy when x is > 0, y is > 0 and x + y is < 1 okay. So they are not independent okay. So x and y are not independent okay. So we have found the marginal density function of x and y and also we have seen that x and y are not independent. So with this I would like to end my lecture. Thank you very much for your attention.