# Advanced Engineering Mathematics
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## Lecture - 52
## Joint Probability Distribution - I

Hello friends. Welcome to my lecture on joint probability distributions. So far we considered the probability distribution of a single random variable. Now we see that in problems arising in economics, biology or social sciences, we study the statistical methods analyzing two or more variables.

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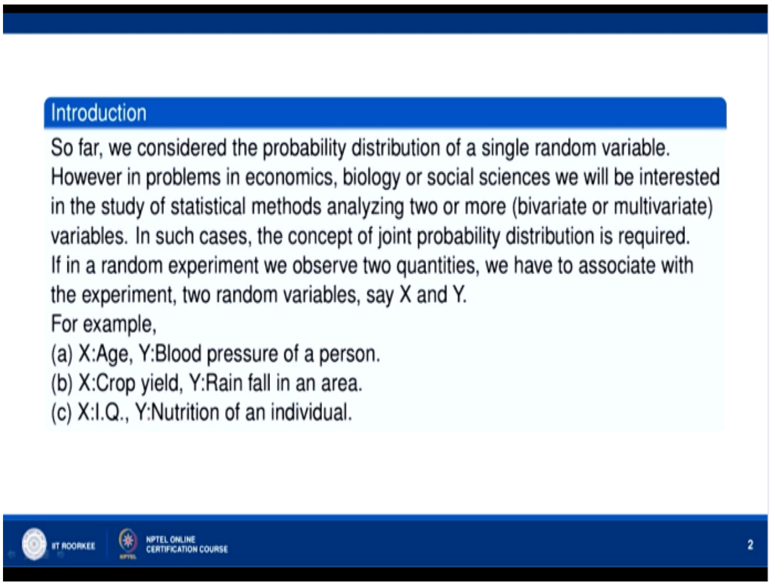


In such cases, the concept of joint probability distribution is required. If in a random experiment, we observe two quantities. We have to associate with the experiment two random variables. For example, let us consider a person we know his age and we measure the age and the blood pressure of the person. So X will denote the age of the person, Y will denote his blood pressure. In an area, let X denote the crop yield and Y denote the rainfall.

Then, again it is a case of two random variables. Now in an individual, one can have the value of his IQ and then the value of his nutrition. So X and Y again it is a case of two random variables.

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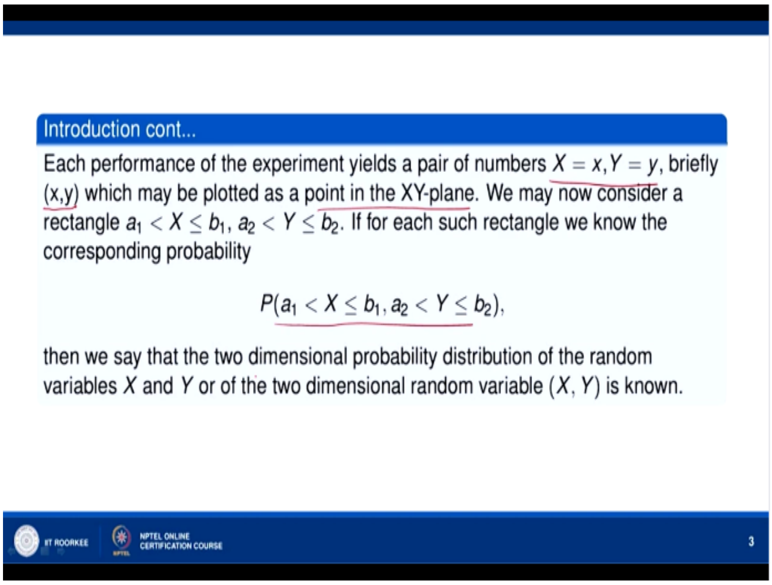


Now each performance of the experiment okay yields a pair of numbers X=x, Y=y or briefly we can say ordered pair x, y which may be plotted as a point in the XY-plane. We may now consider a rectangle a1<X<=b1, a2<Y<=b2. Then, for each such rectangle, we know the corresponding probability okay. P a1<X<=b1, a2<Y<=b2. Then, we say that if we know the probability then we say that the two dimensional probability distribution of the random variables X and Y or of the two dimensional random variable X, Y is known.

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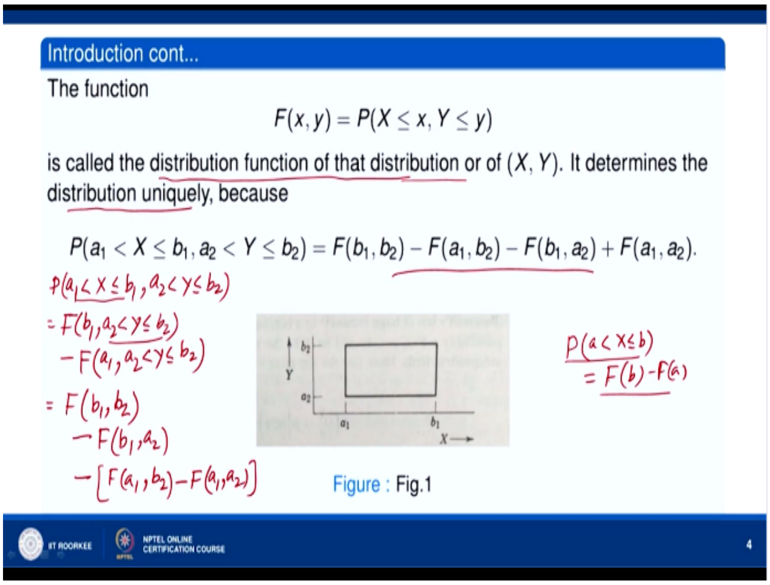


The function F x, y where probability X<=x, Y<=y. The function F x, y is given by the probability that X<=x, Y<=y is called the distribution function of the joint distribution X, Y or cumulative distribution function of the joint distribution of X and Y. It determines the distribution uniquely because probability that a1<X<=b1, a2<Y<=b2 can be obtained from the cumulative distribution function.

It is given by F b1, b2-F a1, b2-F b1, a2+F a1, a2. We can easily derive this formula. P a1<X<=b1, a2<Y<=b2 okay. Let us first consider the case where a1<X<=b1. Then, this is F b1 okay F b1, a2<=b2 let us first find this okay -F a1, a2<Y<=b2 we can write like this okay. Then, further okay so what we are doing is that we first consider the case when Y is not varied, Y it remains between a2 and b2.

X varies from a1 to b1, so it is F this can be written in two parts, probability that X<=x, Y<=y can be written as F b1 okay, so F b1 then we take up this one okay. So F b1, b2-F b1, a2 okay because F of b1, a2<Y<=b2 is F b1, b2-F b1, a2 and then –F a1, b2-F a1, a2 okay because we know that probability that X for a single variable, for a single variable we know that probability a<X<=b okay in the case of a single variable it is Fb-Fa okay.

So we keep this constant for the time and just consider the one random variable. It is the case of one random variable we can consider. So probability that a1<X<=b1 is F b1-F a1 and then we take up the second variable. So F b1, a2<Y<=b2 we can use this and then we have F b1, b2-F b1, a2. Similarly, F a1, a2<Y<=b2 is F a1, b2-F a1, a2 and this gives you -F a1, b2-F b1, a2+F a1, a2.

So this formula we get okay using the case of the cumulative distribution function for a single variable okay.

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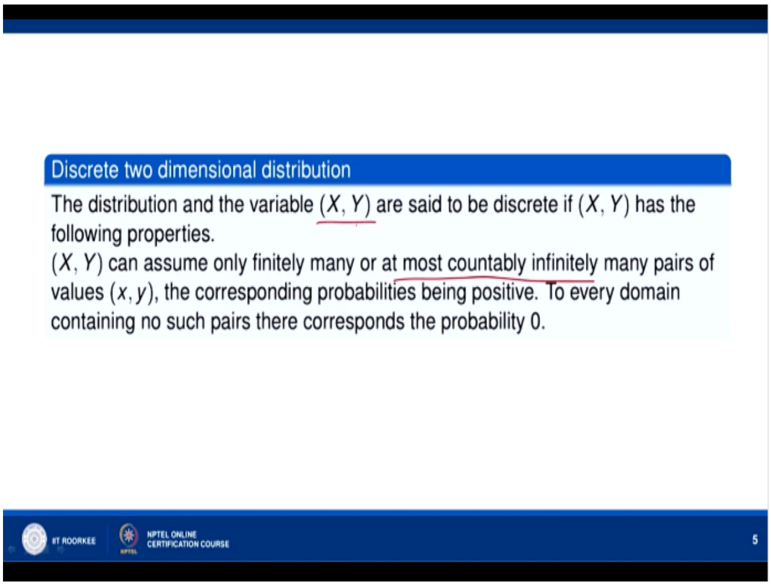

Now the distribution and the variable X, Y are said to be discrete if X, Y has the following properties. X, Y can assume only finitely many or at most countably infinitely many pairs of values x, y, the corresponding probabilities being positive. To every domain which does not contain any such pairs there corresponds the probability 0.

So if you say xi, yj is any point belonging to the joint distribution and xi, yj does not lie in that range okay of values of X and Y then we can say that the probability of X=xi, Y=yj will be=0.

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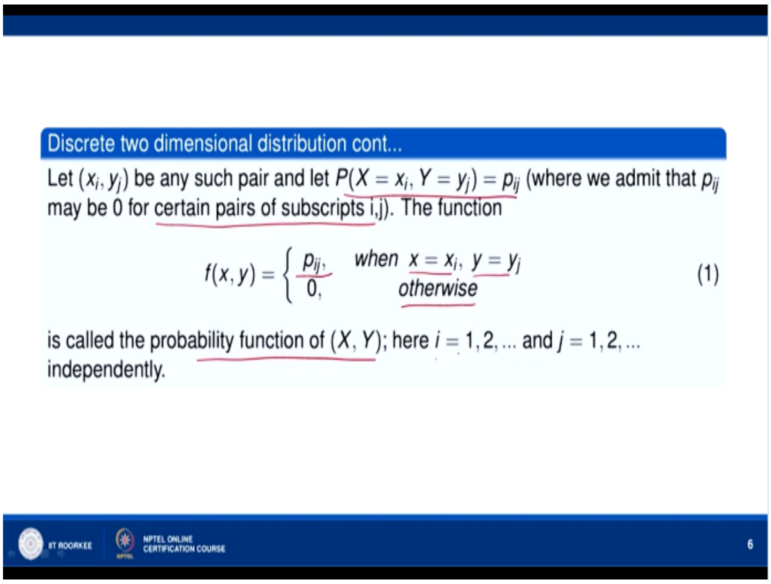


Now let xi, yj be any such pair and let P X=xi, Y=yj be pij okay where we admit that pij may be 0 for certain pairs of subscripts i, j okay. Now the function f x, y=pij when x=xi, y=yj otherwise it is 0. If xi, yj does not belong to the set of values taken by X and Y okay then the probability of f x, y will be 0 for such pair. So this is called the probability function of X, Y, here i takes values 1, 2, 3 and so on and j takes values 1, 2, 3 and so on.

In case of the finite when X and Y takes finite number of values say i runs from 1, 2, 3 and so on up to n and j runs from 1, 2, 3 and so on up to m in that case and if X, Y takes the values which are countably infinitely many, then we may have to take xi tending i taking 1, 2, 3 and so on up to infinity, j taking values 1, 2, 3 and so on up to infinity.

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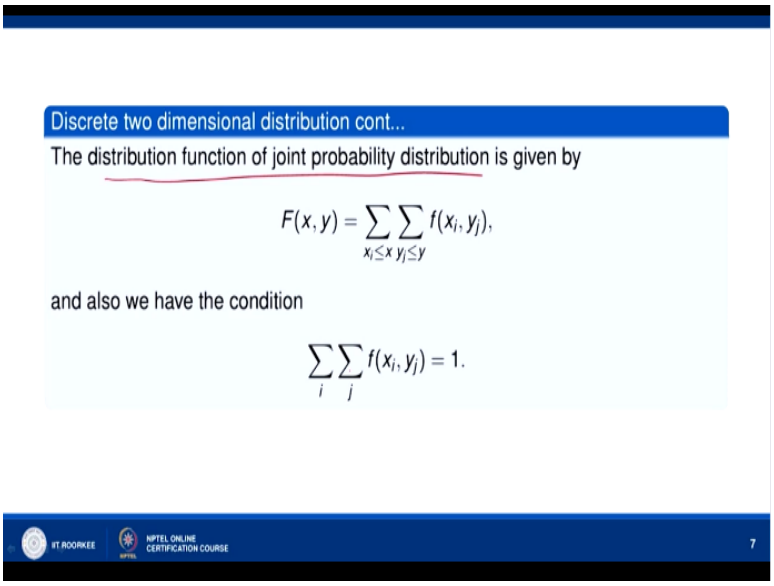


So let us then consider the distribution function of the joint probability distribution. Distribution function F x, y will be given by double sigma f xi, yj where xi is<=x, yj<=y okay where x takes the value of xi, y takes the value yj and moreover the sigma f xi, yj where i runs over 1, 2, 3 and so on up to infinity, j runs over values 1, 2, 3 and so on up to infinity that sum is=1 in case of countably infinite X, Y.

In case of finitely many, i may run from 1 to n, j may run from 1 to m. So what do we mean to say is that the total probability is=1.

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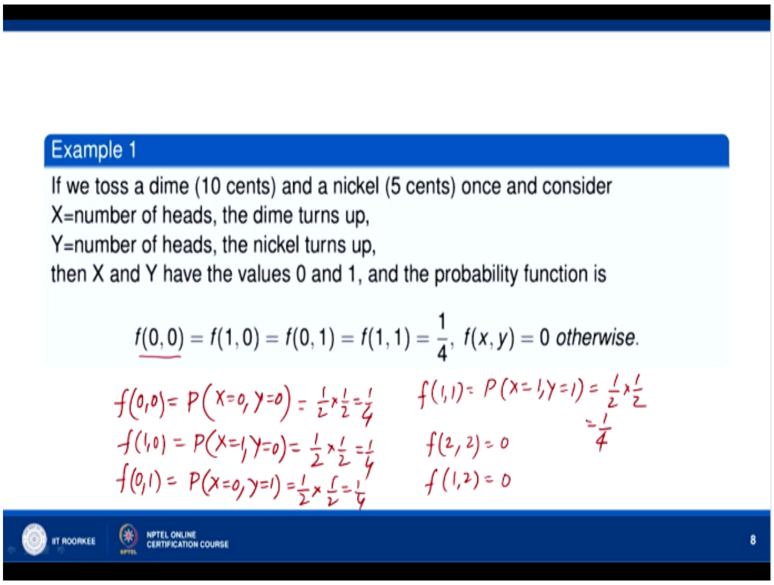


Now suppose we have this question. If we toss a dime, dime means American 10 cents coin and nickel means American 5 cents coin. If we toss a dime and a nickel once and consider X as number of heads, the dime turns up by the number of heads the nickel turns up. Then, X and Y can have the values 0 and 1 okay. So X takes values 0, 1 and Y can also take value 0, 1 and probability of having a head or not having a head is equal okay.

So that is half each, so f 0, 0 okay, f 0, 0 means that we want the probability that X=0, Y=0, f 0, 0 means probability that X takes the value 0, Y takes the value 0 means when we toss the dime okay and the nickel we do not get head okay on the 10 cents coin as well as on the 5 cents coin. So in both the cases, the probability is 1/2, 1/2 so we get 1/2*1/2 means it is 1/4 okay. F 1, 0 is the probability that X takes the value 1, Y takes the value 0.

That means on the dime, we get the head while on the nickel we do not get the head. Again, we have 1/2*1/2, so 1/4. Likewise, f 0, 1 which is probability that X=0, Y=1 is also 1/4 okay. F 1, 1 means we get heads on both the coins okay, so we get probability that X=1, Y=1 means we get 1/2*1/2, so it is 1/4 okay. We do not have any other values of X, Y. If you say that I want the probability that f of 1, 2 okay then f of 1, 2 is=0 because f of 2, 2 for example okay f of 2, 2 is=0, even f of 1, 2 is=0.

Because probability that X takes the value 1, Y takes the value 2 is not possible, so these values are all 0s okay.

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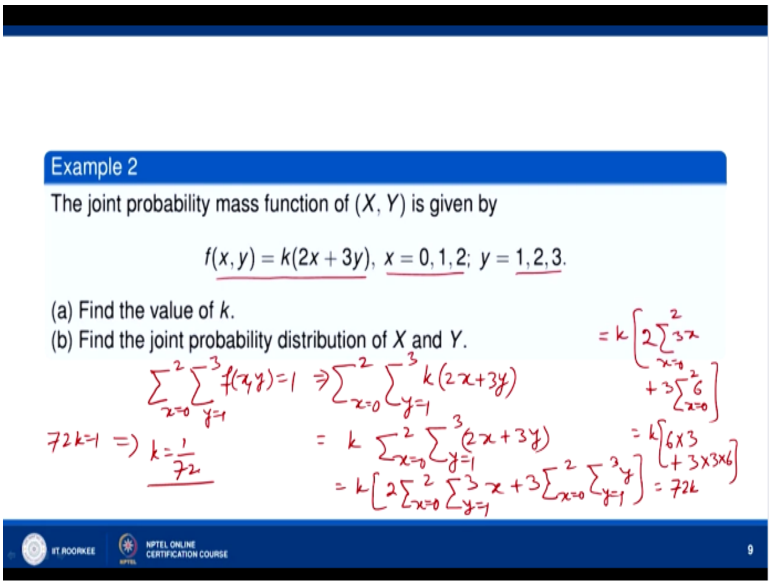


The joint probability mass function of X, Y is given by f x, y=k times 2x+3y where x takes values from 0, 1 and up to 2 and y takes values 1, 2, 3. Find the value of k. We know that the total probability is=1 that means sigma x=0 up to 2 and sigma y=1 to 3 okay f x, y is=1, so this will give us the value of k okay. So this is sigma x=0 to 2 y=1 to 3 k times 2x+3y okay. This is=k is a constant, we can write it outside.

Then, sigma x=0 to 2, sigma y=1 to 3, 2x+3y, so k times. So we can write it as 2 times sigma x=0 to 2, y=1 to 3 okay, x+3 times sigma x=0 to 2, y=1 to 3 y okay. Now y runs from 1 to 3, there is no y here, so this will be 3 times x okay, x will be added 3 times, so this is 3 times x and then x varies from 0 to 2. So this is=k times okay 2 sigma x=0 to 2 okay, x will be added 3 times, so 3x+3 times now y=1 to 3 y, so 1+2+3, so 3+2+1 that means 6 okay.

So sigma x=0 to 2 6 okay. So here what we will get 3*2 that is 6, k times 6. Now sigma x=0 to 2 x that means x is 0, 1 and 2. So 1+2 that is 3, so 6*3 and here what we will get 6 will be added 3 times for x=0, then for x=1, then for x=2. So 3 times 3*6 okay, so this will be 3*3 that is 9, 9*6=54, 54+18, so 72 k okay, so 72 k=1, this gives you k=1/72. So that is the answer to the first part. Now let us find the joint probability distribution of X and Y okay.

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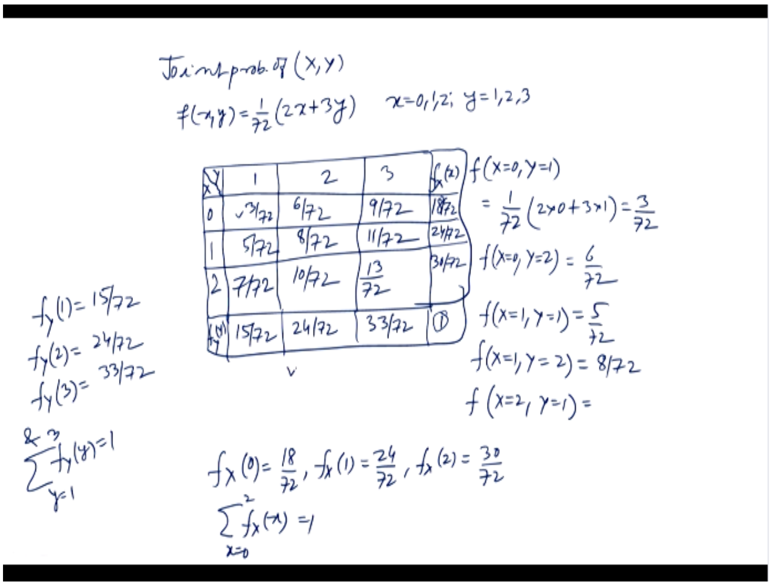


So let us find joint probability distribution of X, Y okay. So we have f x, y=now k times 2x+3y, k is 1/72 okay. So 1/72 2x+3y okay and x takes value 0, 1, 2, y takes values 1, 2, 3 okay, so let us make the table, we can write it in the form of a table okay. So this is x, let us say this is y okay, x takes values 0, 1 and 2 okay, y takes values 1, 2 and 3 okay. Now here we need the probability f x=0, y=1 okay in this box okay.

So x=0, y=1 we get 1/72 x is 0, so 2*0+3*1. So we get 3/72. So this is 3/72 okay. Then, x=0, y=2 okay. So f x=0, y=2, so this will be x is 0, y is 2 okay so that means 6/72 okay. Then, x is

0, y is 3 so 3*3=9, 9/72 okay this was 6/72. Then, in this box x is 1 okay, y is 1 okay so f x=1, y=1 that means x is 1, y=1 so 2+3=5/72. So we get 5/72 here. Now f x=1, y=2 okay, so x is 1 means 2x=2 and y is 2 so 2*3=6, 6+2 is 8/72 okay.

Then, x is 1, y is 3, so 3*3=9, 9+2=11/72 okay and then x is 2, y is=1. So this was how much x=1, y=2? This was 8/72. Now we have x=2, y=1 so we get x=2 means 2*2=4, 3*1 is 3, so 3+4 that is 7/72 and x is 2 that is here 4, 3*2 that is 6, so 10/72 okay and here we get x=2, y=3, so 2*2=4, 3*3=9, 9+4 that is 13/72 okay. So here x is=0 this gives you the probability distribution of x, so fx, x okay. So fx, x you can add these values, 6+3=9, 9+9=18/72 okay.

When x is 0, this gives us marginal distribution which we shall talk about in the next lecture fx, x now here we have 5+8=13, 13+11=24/72 okay and we get here 7+10=17, 17+13=30, 30/72. So this means that fx, x okay; fx, 0 is=18/72, fx; 1 is=24/72; fx, 2=30/72 and sum of these probabilities should be=1. So fx, 0+fx, 1+fx, 2, 18+24 that is 42+30 that is 72/72=1, so sigma fx, x where x varies from 0 to 2 you can see is=1.

This is marginal distribution of X. Now similarly we can find marginal distribution of Y okay. So fy, y, so we can add the column now column entries, 3+5=8+7=15/72 okay, here 6+8=14+10=24/72, here we get 9+11=20, 20+13 that is 33/72 and you can see total of these is also=1. So 13+24 that is 39+33 okay that is 72/72, so this value is 1 okay. So fy, y, y=1 fy, 1=15/72, marginal distribution of Y, fy, 2=24/72 and fy, 3=33/72.

And we can say that sigma fy, y where y runs from 1 to 3 is=1 okay. So this is the joint probability distribution of X and Y.

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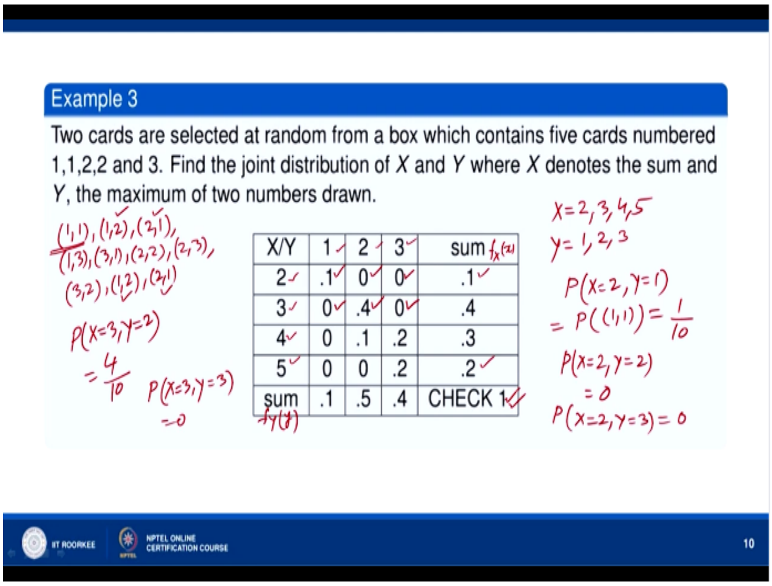


Now two cards are selected at random from a box which contains five cards numbered 1, 1, 2, 2, and 3. Find the joint distribution of X and Y where X denotes the sum and Y denotes the maximum of two numbers drawn. So when we draw two cards okay, the ordered pairs will be 1, 1; 1, 2; 2, 1; 1, 3; 3, 1; 2, 2; 2, 3; 3, 2 okay, so 1, 2, 3, 4, 5, 6, 7, 8 and we shall also have 1, 2; 2, 1. So this we shall have total 10 pairs okay.

So we shall now find the joint distribution of X and Y. So joint distribution of X and Y means now X denote the sum of the two numbers okay. So X can take the values starting with 2, 3, 4 and 5, X can take these values and Y denote the maximum of the two numbers, so Y will take value 1, 2, and 3 okay maximum of the two numbers okay. Then, this is the table actually. Let us see how we get this. So X takes values 2, 3, 4, 5; Y takes values 1, 2, 3 okay.

Now this entry, this entry gives the probability that X takes the value 2, Y takes the value 1 okay. X takes the value 2, Y takes the value 1 let us see how it comes. So we want that ordered pair where sum of the two values is 2 and the maximum is 1 that is only one case okay 1, 1 where the sum of the two values 1, 1 is 2 and their maximum is 1. So probability we have 1, 1 okay.

Now we have 10 ordered pairs and so each one equally likely to come so we have the probability 1/10; 1, 1 occurs only once okay. Then, this is probability that X=2, Y=no, X=2, yeah Y=2, probability that X=2, Y=2 okay. We want sum to be 2 and maximum value to be 2 okay which is not possible okay, so the probability is 0. Then, similarly probability that X=2, Y=3 okay.

So we got this, we got this. X=2, Y=3 is also not possible from here okay. So we get probability 0. So we get this and then similarly probability that X=3 okay Y=1 maximum is 1 but sum is 3 that is not possible. So the value is 0 okay. Here it is not possible. Then, probability that X=3, Y=2 okay, so X is 3 means sum must be 3 and maximum must be 2. Now there are how many cases, 1, 2; 2, 1; 1, 2; 2, 1 four cases, so probability is 4/10.

So we get this okay. Then, probability that X=3 and Y=3 okay. So sum must be 3 and maximum value must be 3, it is not possible here, so probability 0, so we get this. Similarly, you can see the probabilities of the cases when X is 4, Y takes values 1, 2, 3; when X is 5 Y takes values 1, 2, 3. They are given like this. This is the marginal distribution of X where fx, x okay this is fx, x.

So fx, x when X=2 takes the value 0.1 sum of these probabilities and when X=3 fx, x takes value 0.4 which is the sum of 0.4 and 0; fx, x when X takes value 4 is 0.3 sum of these 3 and fx, x is=0.2 when X is=5. So this is what we get and here we have fy, y okay. So that is the marginal probability distribution of Y. So this is sum of column entries, 0.1+0.4=0.5 0.2+0.2 is 0.4 and you can see sum of column entries, sum of row entries, 0.1+0.4+0.3+0.2 is 1; 0.1+0.5+0.4 is also 1 okay.

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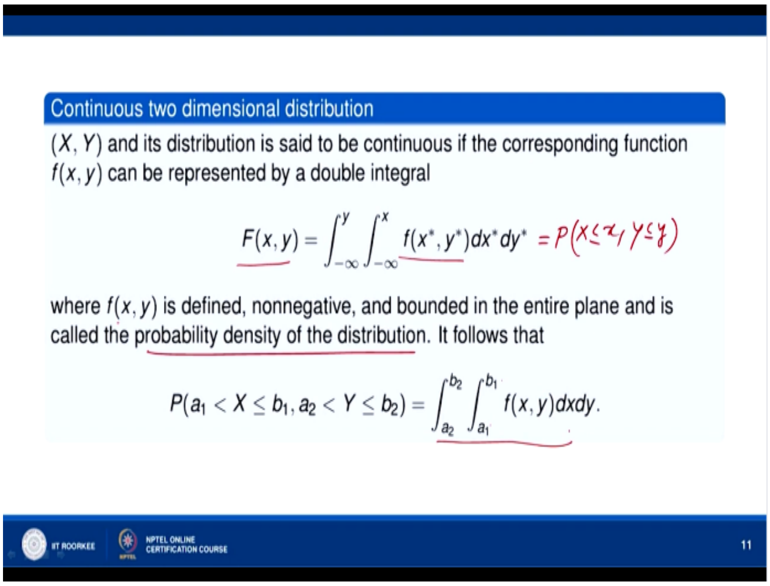


Now we go to continuous two-dimensional distribution. The random variable X, Y and joint distribution of X, Y and its cumulative distribution okay is said to be continuous if the corresponding function f x, y can be represented by a double integral. This is cumulative distribution okay. It is given by –infinity to y integral over –infinity to x f x star, y star dx star dy star. It is nothing but probability that X takes the value<=x, Y takes the value<=y.

So it is the cumulative distribution function in the case of a continuous variables X and Y where f x, y is defined nonnegative. So this f x, y is defined nonnegative and should be bounded in the entire plane and it called the probability density function of the distribution. In the continuous case, when X and Y are continuous random variables f x, y is called the probability density function of the distribution.

It follows that probability that a1<X<=b1, a2<Y<=b2 is a2 x goes from a1 to b1 okay, so this is the values of integral over a1 to b1. Here integral over a2 to b2 f x, y dxdy, elsewhere the value of f x, y, so probability that when X lie between a1 to b1 and Y lies between a2 to b2 okay we get the probability from this expression.

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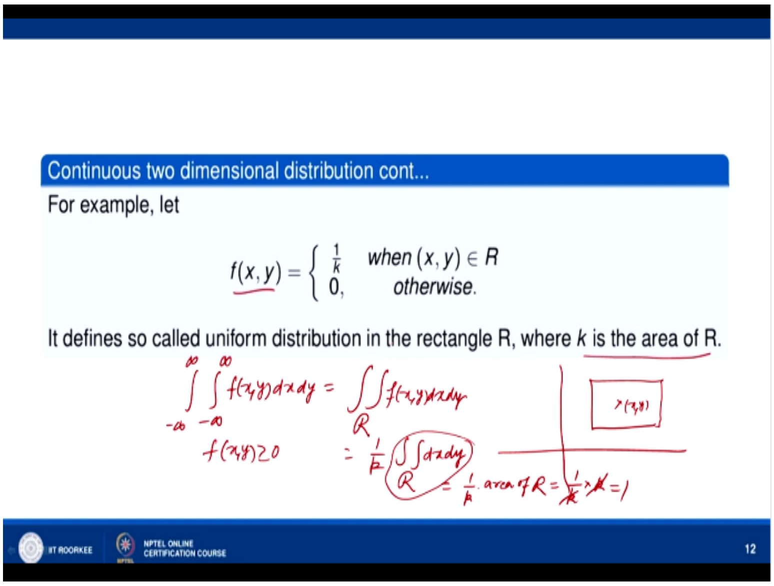


Now for example let us consider f x, y=1/k when x, y belongs to R and 0 otherwise. It defines the so called uniform distribution in the rectangle R where k is the area of R. So you can take a rectangle. Suppose this is our rectangle okay, f x, y is 1/k at every point x, y belonging to R okay. Take any point x, y belonging to R, f x, y takes the value 1/k and k is the area of R okay.

So total probability you can see integral over –infinity to infinity integral over –infinity to infinity f x, y dxdy is=double integral over R because elsewhere it is 0 leaving R it is everywhere 0, so f x, y dxdy and this is nothing but f x, y=1/k, so 1/k constant it will come out double integral over R dxdy okay. This gives you area of R okay, so 1/k*area of rectangle. So 1/k*k and we get 1 okay.

So total probability is 1 and you can see because k denotes the area of R so f x, y is>R=0, so it is the probability mass function for the distribution. Such a distribution is called uniform distribution.

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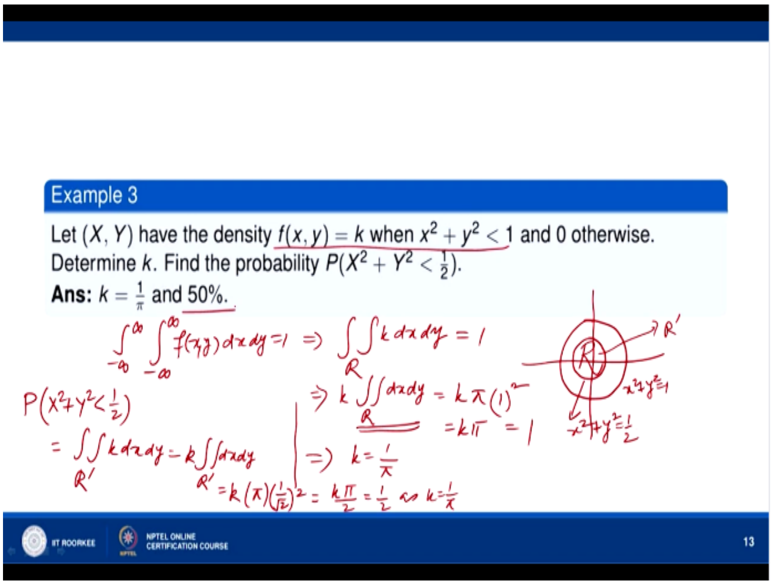


Now let X, Y have the density f x, y=k when x square+y square is<1. So we have this disk okay, x square+y square=1, let us draw this circle, interior of the circle okay is the disk. So f x, y=1, f x, y=k when x square+y square is<1 and 0 otherwise. So on the circle and outside it okay f x, y is 0 everywhere. We have to determine k okay. So integral over –infinity to infinity integral over –infinity to infinity f x, y dxdy is=1, this we know okay.

Because f x, y is density function, so this will reduce to integral over the region bounded by the circle okay R, double integral over R okay f x, y is k*dxdy where R is the region bounded by x square+y square=1, k is constant, so I can write it outside, double integral over R dxdy. Now this is area of the region bounded by the circle. We know the area of the region bounded by the circle of say it is of radius R, it is pi R square okay.

So k times pi*radius of the circle is 1, so 1 square and we get k pi okay. So k pi=1, so this implies k=1/pi okay. Now k we have found. Find the probability of X square+Y square<1/2. Now this is of radius 1, so we have another region okay. The region bounded by the circle X square+Y square=1/2 okay. We want the probability over the region bounded by this circle, so I can call that as R dash okay.

So probability that X square+Y square<1/2 is the double integral over R dash okay and since R dash lies inside R okay R dash lies inside R f x, y is=k. So k dxdy, so k times double integral over R dash dxdy. So this is k times area of region R dash. So area of the region bounded by the circle X square+Y square=1/2. So this radius of the circle is 1/root 2 okay. So pi times 1/root 2 whole square. So k*pi/2, now k is=1/pi, so this is 1/2 okay. So the probability is 50% okay.

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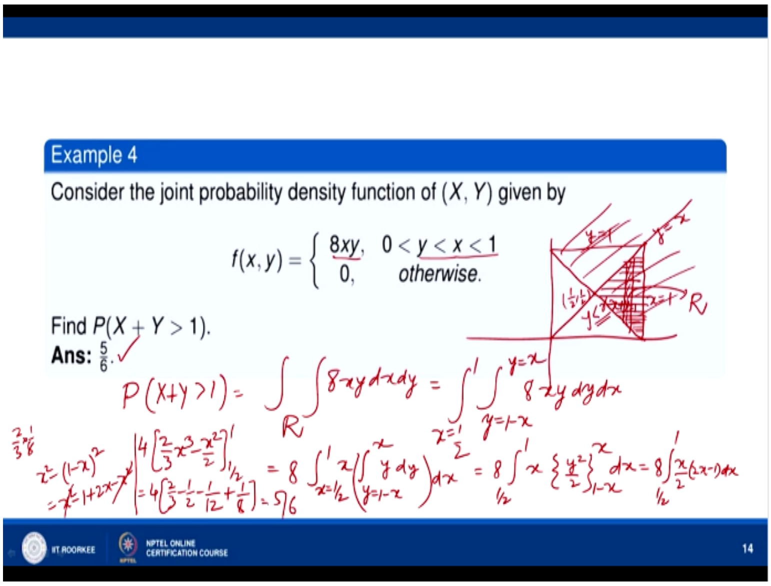


Consider the joint probability density function of X, Y given by f x, y=8xy, 0<y<x<1 and 0 otherwise. We have this region. See this is y=x okay, x is>0, y is also>0 but y is<x and x is<1 okay. So this is say x=1 okay. Now we have P X<and this is say y=1 okay. We want the probability that X+Y should be>1 so let us join the line. If this is our line x+y=1 okay. Now probability that X+Y is>1, X is>0, Y is>0 otherwise it is 0.

We are dealing in the first quadrant. Now which portion satisfies P X+Y>1, x+y is>1 here okay in this region, x+y is>1 okay right. So the required probability then now in this part okay we see that f x, y is 8xy when y is <x<1, so which part is y<x, y is <x in this part okay. So y<x and X+Y>1 means this portion okay. We need to consider this portion. So this means that let me denote it by R okay.

So we need the probability over R okay or in this region f x, y is 8xy okay. So let us cover this region. We can take vertical strip here okay. For the vertical strip, y varies from 1-x to y=x. Now at this point okay at this point Y+X=1 and Y=X, so this point is 1/2, 1/2 okay, so x varies from 1/2 to 1 okay 8xy dy dx we write now. So this is 8 times double integral, so x varies from 1/2 to 1 and y varies from 1-x to 1 okay, x I can write here y dy dx we can write.

So this is 8 times 1/2 to 1 okay x and then we have y square/2 okay, y varies from 1-x2 y=x not 1 okay, y varies from 1-x to y=x. So here also okay, so 1-x to x dx okay and we can get this as 8 times 1/2 to 1 x into now this is by 2 we can write here by 2, y square is x square-1-x whole square okay, x square-1-x whole square is how much? So this is x square-1+2x-x square, so this cancels and we get 2x-1, so let me write there 2x-1 and then dx okay.

Now it is not difficult to calculate. So this is 4 times what we get is this is 4, 2 times 8/2 is 4 and then we can get 2x square, 2x square means 2/3 x cube okay, when we integrate we get 2/3 x cube and then –x square/2 and we can put the limits 1/2 and 1 okay. So how much is that, when you put 1, you get 2/3-1/2, when we put 1/2 we get 1/8 here, 2/3*1/8 so this is 1/12 okay and then +1/2 whole square/2, so 1/8 okay. So we get this.

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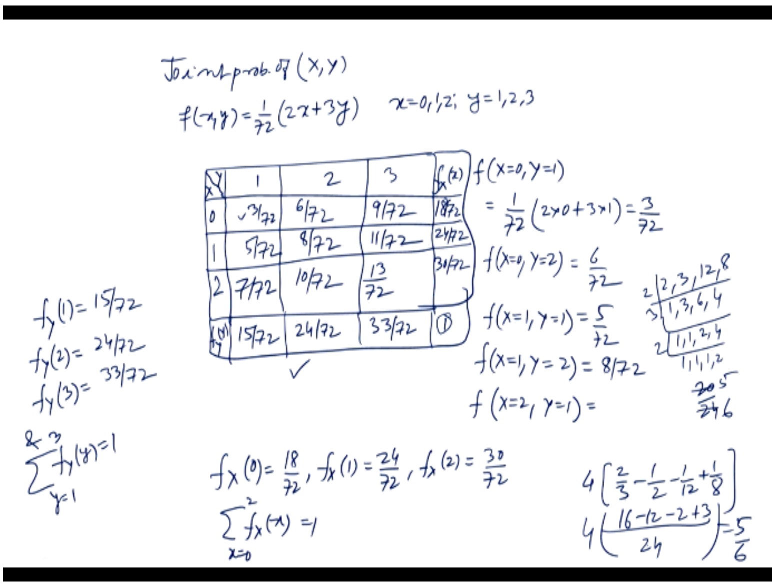


So if we want to calculate, this is 4 times 2/3-1/2-1/12+1/8 okay. So how much is the LCM? 2, 3, 12, and 8, so we get here okay, so 2*2=4*3=12*2=24, so this is 24 here and we get here 3*8=24, 8*2=16-12 and then we get -2, we get 3 okay. So 16+3=19, 19-14 is 5, 5*4=20/24 okay, so 5/6 okay, so this is 5/6 okay. So we get this okay. So this comes out to be 5/6.

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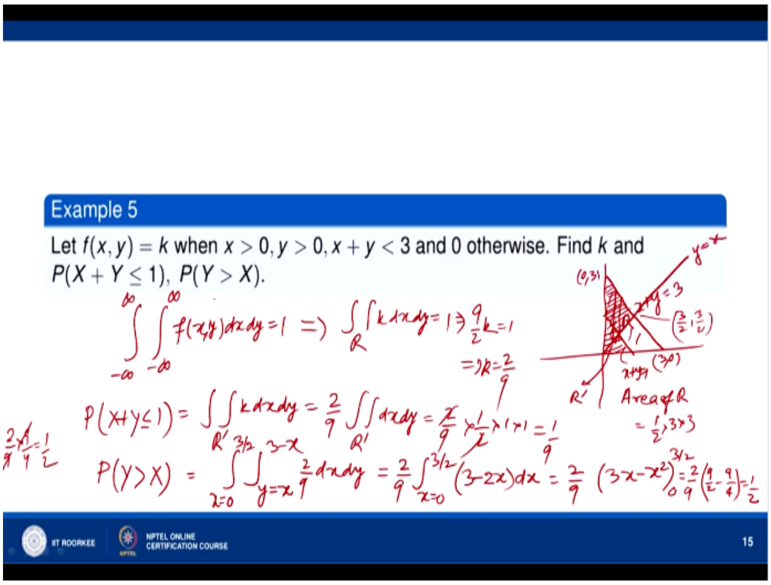


Now we have another one. Let f x, y be k when x is>0, y>0, x+y<3. So this is the region. This is your line x+y=3. So we have this region okay. In this region, f x, y is k everywhere else it is 0. Find k okay. So total probability=1. So we get here double integral –infinity to infinity f x, y dxdy=k=1 okay which implies let us say this region is R. So double integral over R f x, y is k, k dxdy=1. Now this is 3, 0, this is 0, 3 okay.

So R the area of R is area bounded by the triangle, so 1/2*3*3=9/2. So 9/2 k=1 implies k=2/9 okay. P X+Y<=1 okay. So X+Y=1 is a line parallel to x+y=3. This is the line X+Y=1 it is parallel to that okay. So we want the probability of this part okay because elsewhere it is 0. So this is let me call this as R dash okay. So double integral over R dash okay k dxdy, k is=2/9 and so double integral over R dash dxdy that is 2/9 area of R dash, area of R dash is 1/2*1*1, so we get 1/9 okay.

And then P Y>X okay, this is the line y=x and Y>X above the line okay. So we have to consider this part okay. So here what we will do? Let us take again a vertical strip okay. Let us take a vertical strip. So y will vary from x to 3-x okay. So y will vary from x to 3-x and x will vary from 0 to this point okay. So x+y=3 and x=y, so 2x=3 so this is 3/2, 3/2 point okay and we get x varies from 0 to 3/2 and k, k is=2/9, so 2/9 dxdy okay.

So this is=2/9 okay x varies from 0 to 3/2 okay integral of dy is y, so 3-x-x so 3-2x so 3-2x dx okay. So 2/9 3x-x square and we put the limits 0, 3/2 okay. So this is 2/9, 3*3/2, 3*3/2 means 9/2, 9/2-3/2 whole square that is 9/4 okay. So this is 9/2-9/4 is 9/4, so 2/9*9/4, so this cancels you get 1/2. So this is 1/2 that is it is 50% okay P Y>X. So with that I would like to end my lecture. Thank you very much for your attention.