**Advanced Engineering Mathematics**
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**Lecture – 05**
**Applications to the Problems of Potential Flow - II**

Hello friends, welcome to my second lecture on applications to the problems of potential flow, as we have seen in our last lecture, harmonic functions play an an important role in hydrodynamics.

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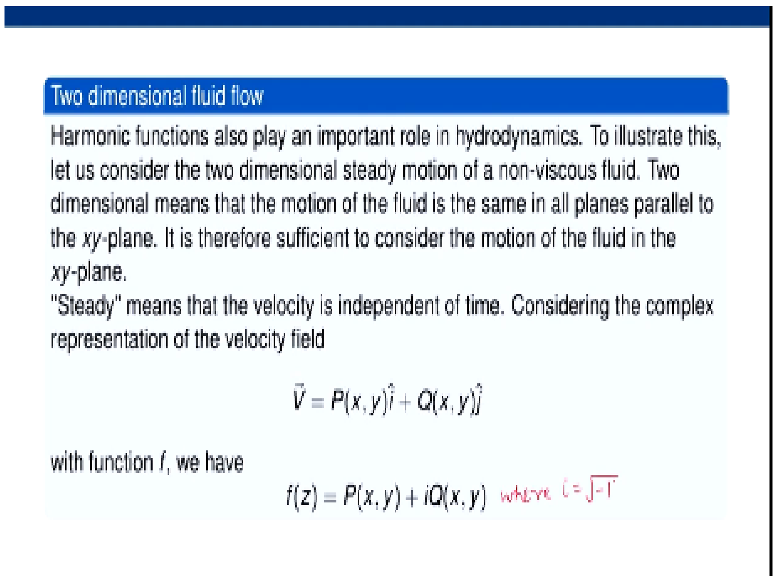


To illustrate this point, let us consider the 2 dimensional a steady motion of a non-viscous fluid, 2 dimensional means, that the motion of the fluid is same in all planes parallel to the XY plane and therefore, it is sufficient to consider the motion of the fluid in the XY plane, by steady we mean that the velocity is independent of time now, considering the complex representation of the velocity field; velocity field is given by v = P xy * unit vector I + Q xy * j.

With the function, we have fz = P xy + iQ xy, I is iota here, iota = this I is = iota root – 1, so when you consider the complex representation of the velocity field, you get the complex function, fz = P xy + iQ xy.

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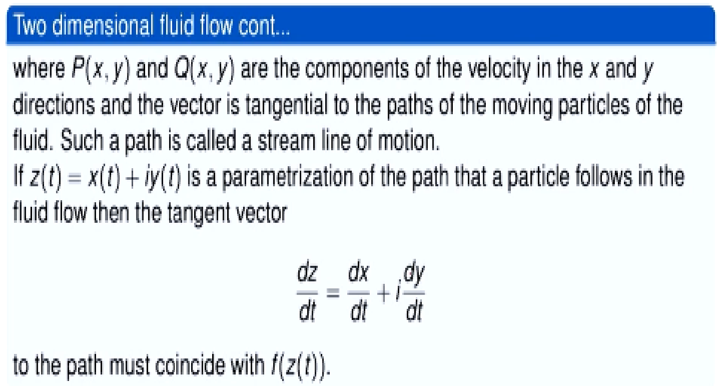


Where P xy and Q xy are the components of the velocity, you can see here, P xy and Q xy are the components of the velocity vector v in the X and Y directions, so and the vector is tangential to the to paths of the moving particles of the fluid, such a path is called a streamline of motion. If z = xt + iyt is a parameterisation of the path that a particle follows in the fluid flow then the tangent vector to this path will be given by, dz/dt = dx/dt + I dy/dt must coincide with f zt.

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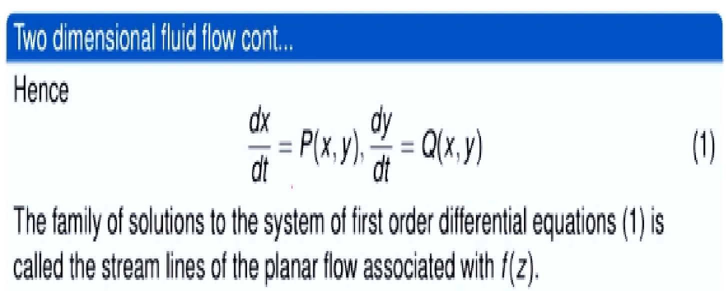


So, dx over dt = P xy, okay and dy/dt will be = Q xy, the family of solutions to the system of first order differential equations, this you can see, this is a system of first order differential equations, it is the family of the solutions of this system is called the streamline to the planar flow associated with the complex function fz.

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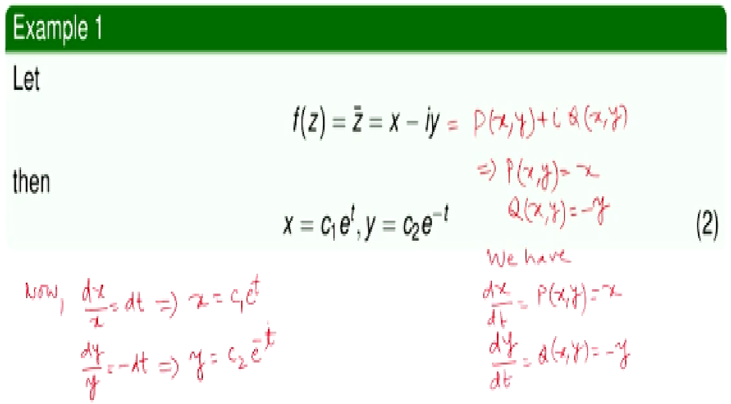


Now, let us consider fz = z conjugate = x – iy, so you can see here, x = z conjugate is; z = x + iy, so z conjugate is x – iy and now, we have written fz = P xy + iQ xy, where as you know P xy and Q xy are the components of the velocity in x and y directions, so comparing we get P xy = x Q xy = -y. Now, we know that we have dx/dt = P xy and dy/dt = Q xy, so Pxy z, so dx/dt = x and dy/dt = Qy which is –y.

And thus we have dx/dt, okay now, dx/dt = x; dx/dt = x gives dx/x = dt, implies that x = some constant times e to the power t, integrating both sides, we can see x = c1 e power t , similarly dy/dt = -y gives dy over y = -dt, so this implies y = c2 times e to the power –t, so we get when fz = x – iy, z conjugate, then x an dy are given by c1 e to the power t and y is given by c2 e2 power -t.

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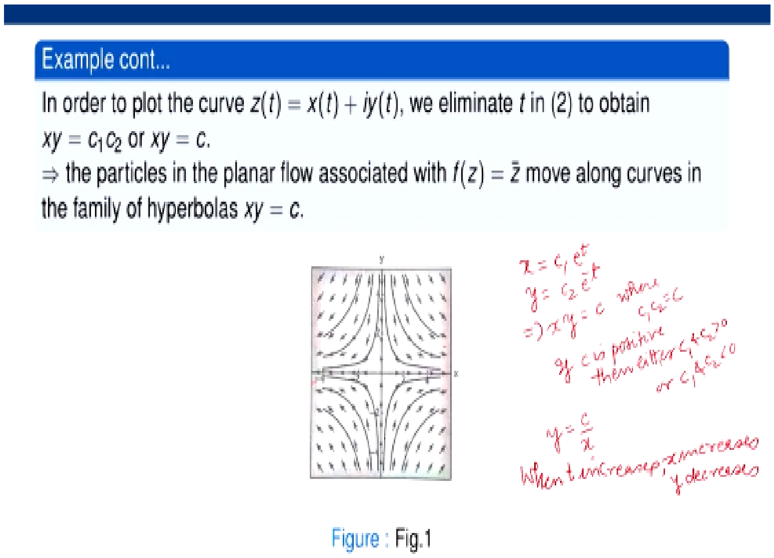


Now, in order to plot the curve, zt = xt + iy t, we eliminate t in the equation 2, okay, you can see here we have x = c1 e to the power t, y = c2 e to the power –t, so from here you can see when you multiply x and y, xy = c1 c2, c1 c2 we can write as a new constant c, so eliminating t, we get xy = c and therefore, xy =c for different values of c, give us rectangular hyperbolas, so the particles in the planar flow associated with fz = z conjugate move along curves in the family of hyperbolas xy = c, you can see here xy, these are xy =; if your c is constant, then you get the hyperbola in the first and third quadrant.

If your c is negative, you get the hyperbolas in the second and fourth quadrant okay, now you can also see how mention the directions of the flow here, you can see that from here, x = c1 e to the power t, y = c2 e to the power –t, okay so let us see here, x = c1 e to the power t, y = c2 e to the power –t, okay so that xy = c, okay where c is 1c2 = c, so e to the power t is always positive, we know, e to the power t is always positive, e to the power –t is always positive, okay.

So, if c1 and c2 >; if c is positive either c1 and c2 both are positive or c1 and c2 both are negative okay, so what will happen; if c1 is positive, c2 is positive, okay, you can see here, okay if xy = c, so when x is positive, it means c1 is positive, so y xy = c, so when your; the direction, yeah, when your x increases, you see y = some constant divided by x, so when x goes to; when x increases okay, x increases y decreases.

So, x = c1 e to the power t, y = c2 e to the power -t goes, t increases, x increases, okay and here what happens you see when x increases, when t increases, x increases because of c1 e to the power t, x increases and when t increases, y decreases, okay because y = c2 e to the power –t, so when t increases, e to the power –y decreases, so what happens when time increases okay, the fluid flow is in this direction like this, okay.

You see when t increases, y decreases, so the value of y decreases and the value of x increases also, the fluid flow is in this direction, okay similarly here in this quadrant, what happens when c1 and c2 both are negative, so here what will happen x will be a negative of e to the power t, so when t will increase, okay, x will be negative, here y; when c2 is negative, y will also be negative, x and y are negative.

So, when t increases, okay what will happen, x will go towards; x will go to – infinity, when t will go to 0, when t will go to; t will be increasing, x will go to –infinity and here when x; t will go to; t will go on increase and then this will go to 0, so y will go to 0 and therefore, the fluid flow is in this direction, so the fluid, directions of the fluid flow are determined by the set that as t increases, what happens to the values of x and y.

So, the values of x and y as t increases, I mean determine the direction of the fluid flow, so this is how we decide the directions of the fluid flow, the particles of the fluid flow move along the family of hyperbolas xy = c here.

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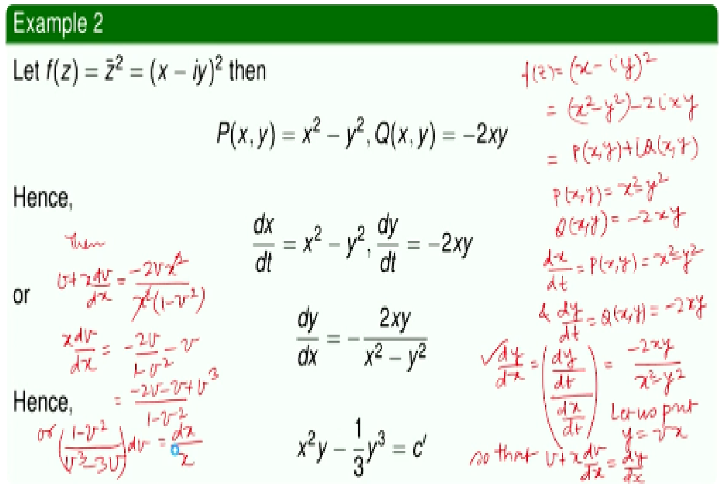


Now, let us take another function fz = z conjugate square which is x – iy, z conjugate is x – iy, so x – iy square and when we square this, you get x square – y square, x - y the whole square gives you x square + I square y square, I square is -1, so x square – y square – 2i xy. Now, fz = P xy + i Q xy, so P xy = x square – y square and Q xy = -2xy, so this is how we get the values of P xy and Q xy. Now, dx/dt = P xy, dx/dt = P xy, so we get dx/dt = x square – y square.

And dy/dt = Q xy, which gives us -2xy, okay, so what do we get; we get 2 equations; dx/dt = x square – y square, dy/dt = -2xy, so what we get; dy/dx = dy/dt divided by dx/dt, okay, so this = -2xy over x square – y square, okay, now we have to solve this equation, so we can solve this equation, this is a homogenous equation, you can see homogenous equation, so let us put y = v * x, so that v + x dv over dx = dy over dx, okay.

And then v + x dv over dx, this equation okay, this equation, dy/dx = - 2xy over x square – y square gives us v + x dv over dx = -2 vx square divided by x square times 1 – v square, okay. So, we can cancel out x square and then what we get; x dv/dx = - 2v upon 1 – v square – v, okay, so what do you get; when you take the LCM, 1 – v square we have – 2v and then we have –v, then we have + v cube, so v cube – 3v upon 1 – v square, now separating variables x and v, we have 1- v square divided by v cube – 3v dv = dx/x, okay.

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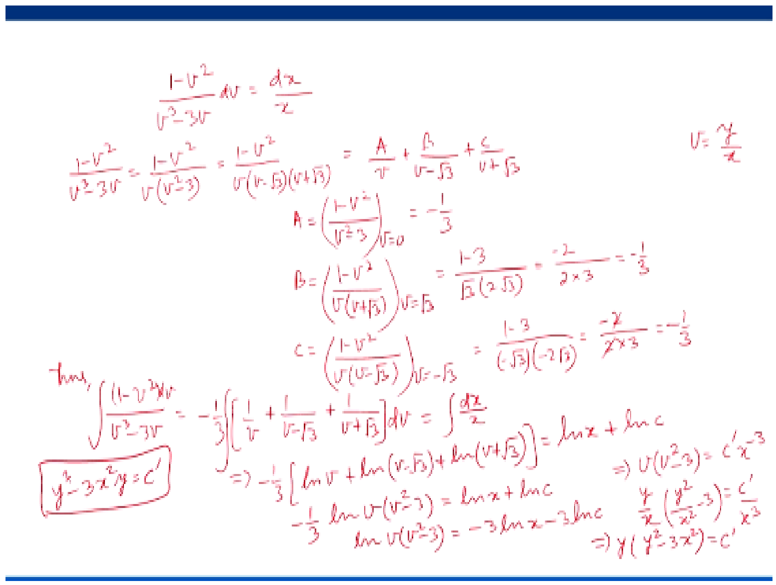

Now, you can see we have dx 1 – x v square up on v cube – 3v, dv = dx over x, okay, so we can bracket into partial fractions and then integrate, so let us take consider 1 – v square upon v cube – v 3v, I can write it as 1 – v square upon v times v square – 3 which is 1 – v square v times v – root 3 and v + root 3, then I can write it as A over v + B over v – root 3 + C over v + root 3, the values of A, B, C we can find.

A = 1 – v square divided by; you remove v from here, remaining is v square – 3 and put v = 0, so we get, this is 1 and this is -3, so – 1/ 3, B = 1 – v square divided by 1 – v - root 3 here, so v * v + root 3 and then put v = root 3, so we get 1- 3 divided by root 3 * 2 root 3 and we get here -2 divided by 2 * 3, so what do we get, -1/3 here also and then C similarly is 1 – v square over v * v – root 3 and we put v = - root 3.

So, what do we get, 1 – 3 divided by – root 3 * - 2 root 3 and what we get here, - 2 divided by; this is how much, 2 * 3, so you can see -1/3 here also, okay, so thus 1 – v square over v cube – 3v, this is = -1/3, A B C all are equal, so 1 upon v + 1 upon v – root 3 + 1 upon v + root 3, okay, so integral of this, okay, we integral here = integral of dx/x, so what we get; -1/3 and we get ln v + ln v – root 3 + ln v + root 3 = ln x + some constant let us write ln c, okay.

So, what we get here, 1/3; – 1/3 ln v + ln A ln B ln C is ln ABC, so we have ln v * v – root 3 * v + root 3, so v -; v square – 3, okay = ln x + ln c and you can multiply by -3 here and what do we get here, so ln v * v square – 3 =- - 3 ln x – 3 ln c, okay, so what we get here, this gives you, v * v square – 3 = x to the power – 3, some constant c dash * x to the power -3. Now, by v = y over x, so I can write here, y over x, y square over x square – 3 = c dash over x cube.

So, this gives you y * y square – 3x square = c dash, okay, so you can see here, x square/ -1/3 y cube = c dash, this same as that here you see y * y square or you can say y cube – 3x square y = c dash, okay, so this solution is same as this solution here we have, 3x square / - y cube = c dash.

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Taking $c' = \pm\frac{2}{3}, \pm\frac{16}{3}, \pm 18$, the stream lines are as shown in the figure:

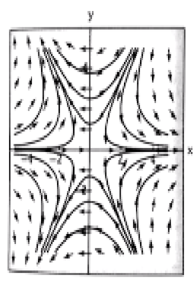


Figure : Fig.2

Now, taking c dash = +-2/3, in this equation here, let us take the particular values of c dash, okay and then we have plot it here, so c dash = +- 2/3, +-16/3, +-18, these streamlines are as shown in the figure. So, these are the streamlines for the values of c dash = + -2/2, +- 16/3, +-18/3, the direction as we have said earlier, the directions are decided accordingly by taking a parameter.

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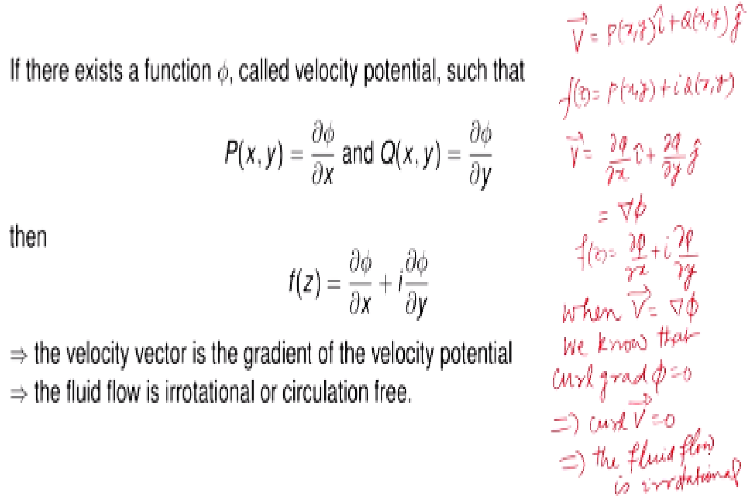


Now, if there exists a function phi called velocity potential such that all P xy = partial derivative of phi with respect to that is phi x, P xy = phi x,Q xy =phi y, then fz; fz was P xy + iQ xy, so fz will be = phi x + iphi y, the velocity vector is the gradient of the velocity vector; velocity potential because velocity vector was this, velocity vector was P xyi + Q xy * j which we have written in the complex form P xyi; P xy + I iota Q xy, okay.

So, v is actually P xy is phi x, Q xy phi y, so velocity vector = this, which is nothing but grad phi in 2 dimensions, okay and the equivalent form of this is fz = iota this okay, so this is the velocity vector is the gradient of the velocity potential, this means that the fluid flow is irrotational or circulation free because when the velocity vector is gradient when v = del phi, we know that curl of grad phi = 0, okay.

So, curl of v velocity vector will be 0 and this implies that the fluid flow is irrotational or we call it circulation free.

**(Refer Slide Time: 21:55)**

If the fluid is incompressible i.e. the density is constant then the divergence of the velocity vector is zero hence

$$\frac{\partial}{\partial x}P(x,y)+\frac{\partial}{\partial y}Q(x,y)=0$$

or

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$\Rightarrow \phi$ is a harmonic function.

$$\text{div}\,\vec{V}=\nabla\cdot\vec{v}=0$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}\right)\cdot\left(P(x,y)\hat{i}+Q(x,y)\hat{j}\right)$$

$$=\frac{\partial P(x,y)}{\partial x}+\frac{\partial Q(x,y)}{\partial y}$$

$$=\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_y}{\partial y}$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Now, if the fluid is incompressible that means the density does not change the time, okay, the density is constant then the divergence of the velocity vector is 0 and hence P xy del over del x P xy + del over del y Q xy =0 because when the divergence of the velocity vector is del dot v, okay, del dot v means, oh this we have written as this okay, so this is del P xy over del, del p xy over del y.

So, if the fluid is incompressible that is the density is constant, then the divergence of velocity vector is 0 and hence we have p xy is phi x, so and P xy is phi y, so we will get this, this is nothing but this, okay, which is = 0, divergence of v = 0, okay, so this is = 0 and this implies that phi is a harmonic function.

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Let $\psi$ denote the conjugate harmonic function of $\phi$ then $\Omega(z) = \phi(x, y) + i\psi(x, y)$ is analytic. The function $\Omega(z)$, of fundamental importance in characterizing a flow, is called the complex velocity potential.

The one parameter family of curves

$$\psi(x, y) = c \Rightarrow \frac{dy}{dx} = -\frac{\psi_x}{\psi_y} = \frac{\phi_y}{\phi_x}, \quad \text{(by C-R equations)}$$

$\phi_x = \psi_y$

$\phi_y = -\psi_x$

$\vec{V} = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

$\Rightarrow$ the resultant velocity of a particle is along the tangent to the curve $\psi(x, y) = c$. So the particle moves on the curve $\psi(x, y) = c$. These curves are called stream function.

Now, let us assume that psi denote the conjugate harmonic function of phi then we can write the associated analytic functions, omega z; omega z = phi xy + I psi xy, this function omega z is very important in characterising the flow, is called the complex velocity potential, the one parameter family of curves, psi xy = constant, consider the conjugate harmonic function and then the level; psi xy = constant, psi xy = constant gives you dy/dx = - psi x over psi y.

But since the function omega z is analytic, phi and psi that is phi Cauchy- Riemann equations, so using Cauchy- Riemann equations, phi x = psi y and phi y = - psi x, we can write psi x = -phi y divided by psi y = phi x, so this – psi x over phi y becomes phi y over phi x and therefore phi y is Q xy and phi x is P xy, so dy/dx = Q xy over P xy and therefore, the resultant velocity, okay the resultant velocity of a particle is along the tangent to the curve.

Because the velocity of the particle has the direction given by; see the velocity v is what; velocity vector is, so the direction of the velocity vector v is given by the slope, by the Q xy over P xy and here what do we notice; dy/dx = Q xy over P xy, so the resultant velocity of a particle is this is the slope of the tangent to the curve psi xy = c, so resultant velocity of a particle is along the tangent to the curve, psi xy = constant.

So, the particle moves on the curve psi xy = c, these curves are called the stream lines, okay and the function psi xy is called stream function.

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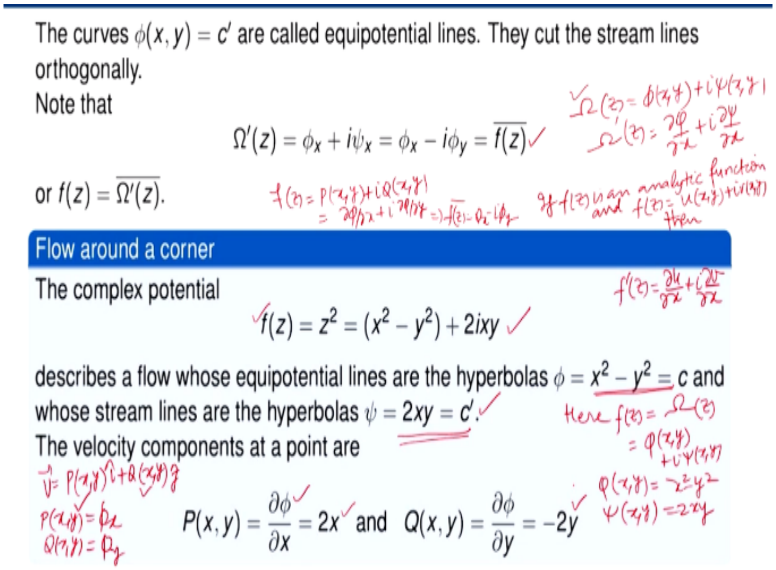


Now the curve phi xy = c dash, the level curves phi xy c dash are called equipotential lines and we know that phi xy and psi xy is double curves, phi xy= constant and psi xy = constant, they cut each other at right angles, so the stream lines cut the equipotential lines orthogonally, okay, so note that now let us consider omega dash z, omega dash z = phi xy + I psi xy, we know that if fz is an analytic function and u and v are its real and imaginary parts.

If fz is an analytic function, and is real and imaginary parts and fz = u xy + iv xy, then f dash z, f dash z = ux + I vx, this we already know okay, so here omega dash z will be phi x + i psi x okay but using again Cauchy- Riemann equations, psi x = - phi y, we get omega dash z = phi x – i phi y. Now, fz =; we have fz = P xy + iota Q xy; P xy is phi x iota Q xy is phi y, so fz conjugate will be phi x – i phi y, okay, this implies fz conjugate = phi x – i phi y okay.

So, this is a omega z dash = f dash conjugate or we can say fz = conjugate of omega dash z, now let us discuss the flow around a corner, the complex potential fz = let us consider the complex potential fz = z square, when you put z = x + iy, z square gives x square – y square + 2y xy, this describes a flow which equipotential lines are the; now equipotential lines are given by phi xy = constant, let us compare this with this function, okay.

This fz is the complex potential function, omega z and this fz is same as this omega z, so fz =; here fz = omega z, okay and omega z is phi xy + i psi xy and so phi xy = x square – y square and psi xy = 2xy, okay. Now, so equipotential lines are given by phi xy = constant, so x square – y square = constant which are the hyperbolas, they are given us the equipotential lines and the streamlines are given by psi xy = constant.

So that means, 2xy = c dash, so these give the stream lines and this give equipotential lines, the velocity components at a point are; now, velocity vector v okay, = P xy i + Q xy j, these are the velocity components, P xy and Q xy and we know that P xy = phi x, Q xy = phi y, so P xy = phi x and P xy =; phi x = 2x, okay, so Pxy = 2x and phi y = -2y, so Q xy = -2y, okay.

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So the magnitude of the resultant velocity $= 2\sqrt{x^2 + y^2}$.
The figure shows the flow in a channel bounded by the axes (stream lines $2xy = 0$) and a hyperbola $xy = a^2$. The continuous lines are the stream lines and the dotted ones, the equipotential lines.

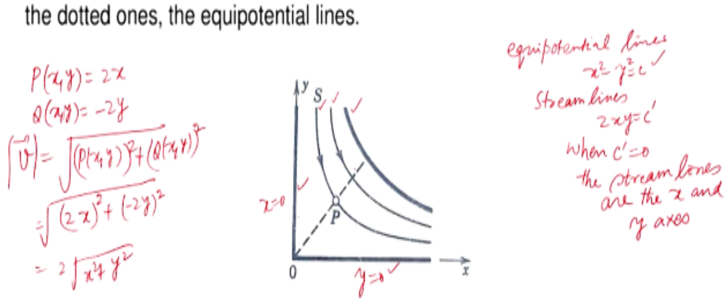


Figure : Fig.3

And so, magnitude of resultant velocity, now we know P xy = phi x which is 2x, Q xy = phi y which is -2y, so magnitude of resultant velocity will be velocity vector, magnitude of resultant velocity vector will be P xy square + Q xy square which is 2x whole square + - 2y whole square under root, so this is 2 times under root x square + y square, so the magnitude of the resultant velocity is 2 under root x square + y square.

Now, you can see in this figure, we have a flow in a channel, okay bounded by the axes, this is; these axis is y = 0, this axis is x = 0 and we have seen equipotential lines are x square – y square = constant and a streamlines are given by 2xy = constant, okay, so when you take the constant c dash = 0, you get this streamlines as x = 0 and y = 0, these 2 streamlines are correspond to the value of c dash = 0.

So, the figure shows the flow in a channel bounded by the axes and axes are the streamline, so when c dash = 0, the streamlines are the x and y axis, okay, the continuous lines are the streamlines, these are streamlines, you can see, these are streamlines okay, given by the hyperbola, you can see here c dash you can take as because we are taking the pi 2 quadrant, so you can take c dash to be positive, so xy = a square.

It will; when c dash is positive, I can write xy = a square, so these are xy = a square, these continuous lines and the continuous lines are therefore the streamlines and the dotted line, these dotted line okay, this is the your equipotential line, this one, okay, so and the dotted lines are the equipotential; equipotential lines are what; equipotential lines are given by x square – y square = 0, so this equipotential line is obtain when you take c = 0, okay.

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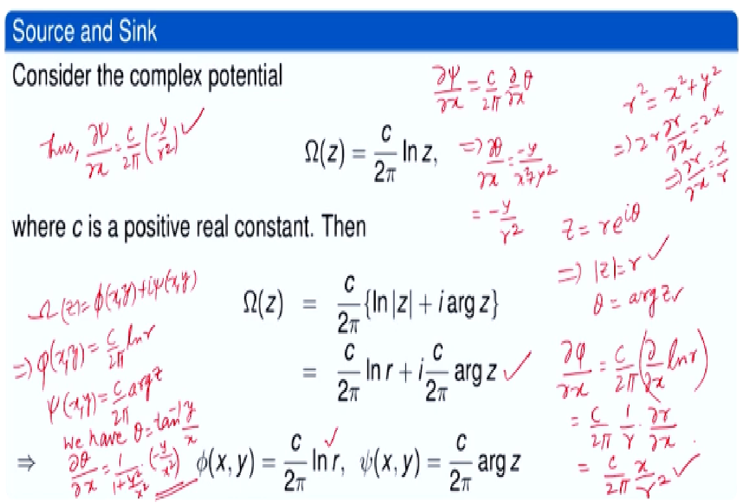


C = 0 means, y = + - x, so this is y = x plane, consider the complex potential omega z = c over 2 pi ln z, let us consider this complex potential, omega z = c over 2 pi ln z, where c is the positive real constant then omega z = c over 2, let us write the polar form of the; let me function z, we can write it as ln mod of z + i argument of z, so omega z = c over 2 pi ln mod of z + argument of z and I can write that as c over 2 pi ln r mod of z =; z we are taking as r e to the power i theta which implies that mod of z = r and theta =;

So, using polar form of the complex number z, w have c over 2 pi ln r + iota c over 2 pi argument of z. now, omega z = phi x y + i psi xy, okay, so omega z = phi xy + I psi xy, so this gives you phi xy = c over 2 pi ln r and psi xy = c over 2 pi rz, okay.

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then

$$\Omega'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = \frac{c}{2\pi r^2}(x - iy)$$

Hence complex representation of the velocity field is given by

$$f(z) = \frac{c}{2\pi r^2}(x + iy)$$

⇒ the flow is directed radially outward

Hence the given complex potential corresponds to a point source at $z = 0$ (i.e. a source line $x = 0, y = 0$ in space)

The constant $c$ is called the strength or discharge of the source.

If $c < 0$, the flow is said to have a sink at $z = 0$, it is directed radially inward, a fluid disappears at the singular point $z = 0$ of the complex potential.

we have $f(z) = \overline{\Omega'(z)}$

$= \frac{c}{2\pi r^2}(x+iy)$

$\vec{v} = \frac{c}{2\pi r^2}(x\hat{i}+y\hat{j})$

$= \frac{c}{2\pi r^2}\vec{r}$

And then what we get is omega dash z, now omega dash z we know is phi x – i phi y, okay and what is phi x; just find phi x here, so phi x let us find, this gives you phi x = c over 2 pi partial derivative of ln r, okay, so this is c over 2 pi, partial derivative of ln r with respect to x is 1 over r * partial derivative of r with respect to x. Now, this gives us r square = x square + y square, so this gives you 2r, rx = 2x, r , rx = x/r.

So, this is c over 2 pi x/ r square okay and similarly, we find the partial derivative of psi with respect to x, so partial derivative of psi with respect to x is c over 2 pi, let us represent this rz, rz is theta, okay, so delta over delta x theta okay, what do we know that we have theta = tan inverse y over x, so we differentiated partially with respect to x, so we get this one, okay and this gives you - y over x square + y square, okay, so this is –y over r square.

So, let us put the values then thus del psi over del x = c over 2 pi – y upon r square, so this is the value of psi x and here is the value of phi x, let us put these values in the expression here okay, so I have put here phi x – I psi x, so you get the same thing when you find phi x - I phi y, so we get c over 2 pi r square x – iy okay and we know that fz okay, let us recall that fz = omega dash z conjugate.

So, we have omega dash z conjugate, and therefore, the conjugate of this is = c over 2 pi r square x + iy, okay, so you can see fz = this complex form of the potential okay, so flow; the flow, you see if you write the v, the vector, you see v vector will be; v vector is will be c over 2 pi r square xi = yj, okay, because this is the complex form of the velocity potential, okay, v velocity vector. So, v is; so this means that the velocity vector is along the direction of xi + yj vector, okay that is the flow is directed readily also.

This r vector, this 2c over 2 pi r square * r vector in the direction of r vector, the flow is in the direction of r vector that means it is directed readily outward, hence the given complex potential corresponds to a point source at z = 0, that is a source line x = 0, y = 0 in a space, the constant c is called the strength this constant c is called the strength or the discharge of the source. If c is < 0, then the direction of the fluid flow will be towards the source, okay.

Then the flow will be said to have a sink at z = 0 because the flow is directed readily inward, a fluid disappears at the in this case a fluid disappears at the similar point, z = 0 of the complex potential.

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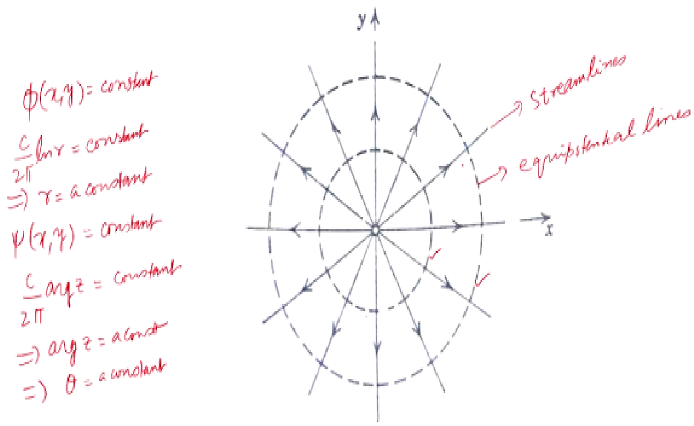


Figure : Fig.4

So, this is the figure, you can see the flow is; if there is a source here, the flows is directed readily outward, okay, so these are the; these are equipotential lines okay and these are streamlines, these are streamlines because flow of the fluid is along the streamlines and these are equipotential lines, this we can see here also, you can see we have phi xy = constant, when you take phi xy = constant, c over 2 pi is a constant, okay.

Ln, so ln r will be a constant and ln r is a constant when r is constant, okay, phi xy = constant implies that r is constant and psi xy = constant implies rz = constant that is theta = constant, okay, so phi xy = constant means, c over 2 pi ln r is a constant and which implies that r is a constant, okay r is constant means concentric circles, okay, so these are equipotential lines, okay and then similarly, psi xy = constant means, c over 2 pi argument of z is a constant which means that rz is a constant, okay, rz is theta, so theta is a constant, okay.

So, for the radial lines, theta is constant and therefore, the fluid flows along the radial lines, radially outward.

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**Gravitational Field**

Let a particle '$A$' of mass $M$ be fixed at a point $P_0$ and let a particle '$B$' of mass $m$ be free to take up various positions $P$ in space. Then '$A$' attract '$B$'. According to Newton's law of gravitational force $\vec{p}$ is directed from $P$ to $P_0$ and its magnitude is proportional to $\frac{1}{r^2}$, where $r$ is the distance between $P$ and $P_0$,say,

$$|\vec{p}| = \frac{c}{r^2} = \frac{GMm}{r^2},$$

where $G = 6.67 \times 10^{-8} cm^3/gm.sec^2$ is the gravitational constant. Hence, $\vec{p}$ defines a vector field in space. Let $P_0$ and $P$ be given by $(x_0, y_0, z_0)$ and $(x, y, z)$ respectively. Then $r = \sqrt{\sum(x - x_0)^2}$ and $\vec{r} = (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}$.

Now, let us consider another example that of a gravitational field, let a particle A of mass M be fixed at a point P0 and let a particle B of mass m be free to take up various positions P in space, then we know that A attracts B and according to Newton's law of gravitational force, the gravitational force P is directed from P to P0, the magnitude of this force is proportional to 1 over r square, where r is the distance between P and P0.

And its given by; magnitude of P is given by c over r square where c is a constant and the constant is G * M *; G * capital M * small m divided by r square, where G is gravitational constant, G is given by 6.67 * 10 to the power -8 centimetre cube divided by gram second square, this is the gravitational constant and the P vector defines a vector field in space. So, let P0 has coordinates x0, y0, z0.

And the point P has coordinates x, y, z then the vector r will be given by x – x0 * i + y – y0 * j + z – z0 * k and its magnitude r will be square root x – x0 whole square + y – y0 whole square + z – z0 whole square which we have written as sigma x – x0 whole square.

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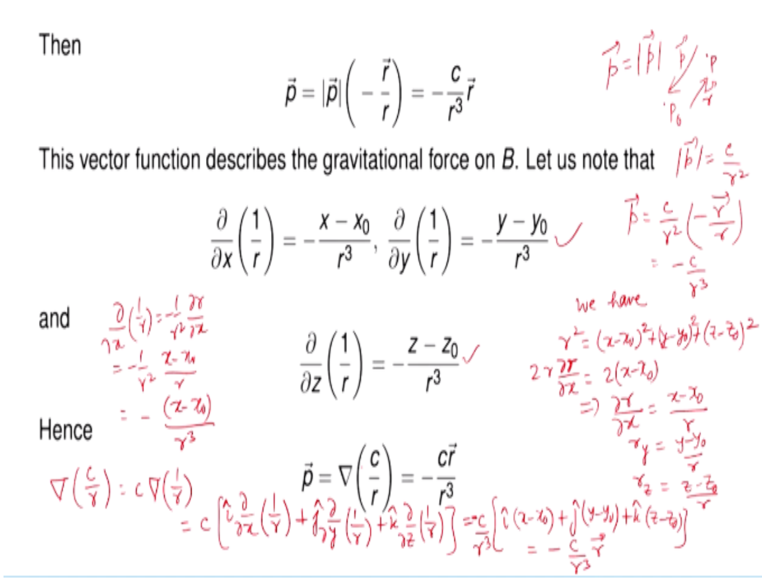


And the vector P then can be written as P = mod of P * a unit vector in the direction of vector P, okay, these P0 and here is P okay, P vector is in this direction that is opposite to the direction of r, okay, so we divide r by its magnitude to get a unit vector in the direction of r and then we put a negative sign, we get P = mod of P – r over r and we know that mod of P = c over r square, okay, c over r square, so what do we get?

P vector = c over r square * - r vector divided by r okay, - c over r cube, okay, now we know that we have r square = x – x0 whole square, y – y0 whole square, z – z0 whole square, the distance between P0 and P, okay, so when you differentiate this with respect to x, what you get; you get this which implies rx = x – x0 over r, similarly ry if you find yz, vz; y – y0 over r and rz = z – z0 over r we get, okay.

Now, partial derivative of 1 over r is -1 over r square * rx, so that is = -1/ r square x – x0 over r, so we get –x – x0 over r cube, similarly, the partial derivative of 1 over r with respect to y is y – y0, -y – y0 over r cube and partial derivative of 1 over r with respect to z is – z – z0 over r cube, okay, so what we get here, this vector P is nothing but del of c over r, why it is d over del of c over r, we will let us see that.

Del of c over r means c times del over del of 1 over r and del of 1 over r is what; c times i, okay this is what we have and this is how much; c times i del over del x 1 over r, del over del x 1 over r is – x – x0 over r cube, okay, so – 1 over r cube we can write here, okay – c over r cube we can write here then this is x – x0 + j, y – y0 + k z – z0 which is r vector, so this is – c over r cube r vector, okay.

So, del c over r okay, P; if you write P = del c over r, what you get is – cr over r, so this – c r over r cube can be written as del of c over r and this means that the gravitational force P can be written as the gradient of a scalar potential which is c over r, okay.

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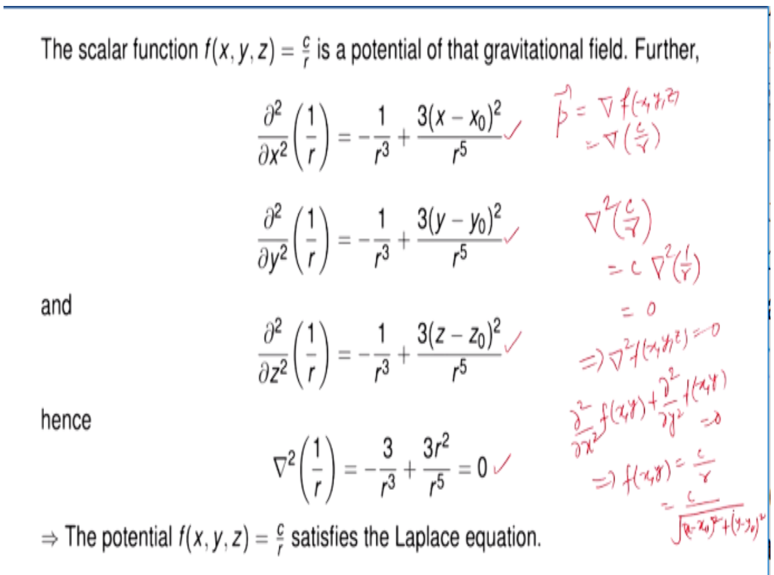


So, the scalar function f x y z, let us write P = del of f x y z, there f x y z is c over r, okay, so this fx, the scalar function f x y z = c over r is a potential of that gravitational field, further if you differentiate 1 over r, we have differentiated 1 over r with respect to x and got this okay, del over del x of 1 over r = this, if you differentiate it further, you get del square over del x square of 1/r, if you differentiated it further with respect to x, we get del square/ del x square of 1 over r.

And that comes out to be this, okay, similarly del square over del y square upon over r comes out to be this and del square over del z square upon r comes out to be this and when you adapt all of them, you get del square of 1 over r which is -3 over r cube + 3 r square over r to the power 5 which is = 0 and therefore, f x y z = c over r okay, del square of c over r = c times del square of 1/r, okay.

And this del square upon by r is 0, so this is 0, so del square of f x y z = 0, so this f x y z satisfies the Laplace equation and therefore, this gravitational; this function, scalar function is called the gravitational potential.

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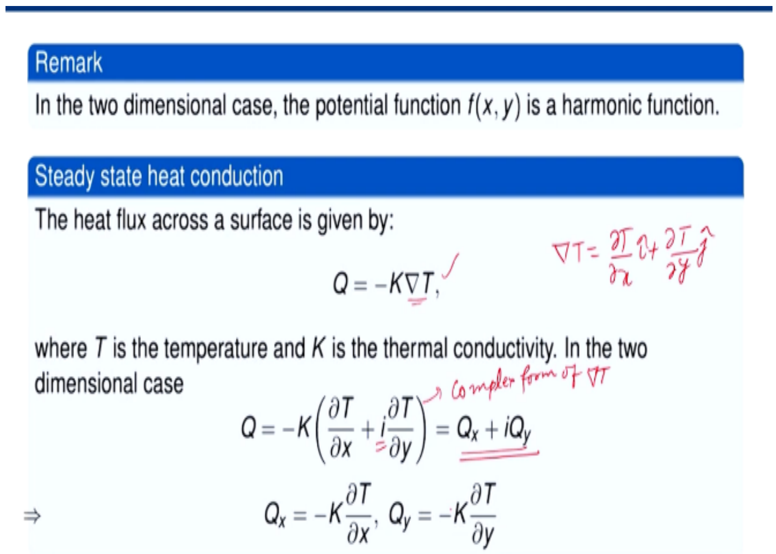


Now, in the 2 dimensional case, the potential function f xy okay is a harmonic function, here you can see we have taken in 3 dimensions, when you can see the 2 dimensional case, okay then you get f xy = c over r which satisfy the Laplace equation this one, in the 2 dimensional case, we have this and then we say that f xy = c over r okay that is c over under root r is what; x – x0 whole square + y – y0 whole square.

This is a harmonic function, okay, the heat flux across a surface, okay let us consider a steady state heat conduction, the heat flux across a surface is given by Q = - K grad t, okay, where T is the temperature and K is the thermal conductivity, in the 2 dimensional case, Q = - K, now in the 2 dimensional case, the del T; del T will be del T over del x * I + del T over del y * j, when you write the complex form of this, okay in the complex form del T can be written as del T over del x + iota, this is iota, okay * del T over del y.

This is the complex form of del T, okay, complex form of del T and so this is Q in the complex form is Qx + iQ y and so Qx = -K del T over del x, Qy is –K del T over del y.

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It follows that for a simple closed curve $C$ in the $z$ plane representing the cross section of a cylinder and for the steady state conditions (no net accumulation of heat inside $C$) we have

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$\Rightarrow T$ is a harmonic function. ✓

If $U$ is the conjugate harmonic function, the function

$$W(z) = T(x, y) + iU(x, y)$$ ✓

is an analytic function.

The families of curves $T(x, y) = \alpha$, $U(x, y) = \beta$ are called, respectively, isothermal lines and flux lines. $W(z)$ is called the complex temperature.

$Q_x = -k\frac{\partial T}{\partial x}$

$Q_y = -k\frac{\partial T}{\partial y}$

$\frac{\partial Q}{\partial x} = -k\frac{\partial^2 T}{\partial x^2}$

$\frac{\partial Q}{\partial y} = -k\frac{\partial^2 T}{\partial y^2}$

Hence $\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} = -k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] = 0$

Now, it follow that for a simple closed curve C in the z plane, okay representing the cross section of a cylinder and for the steady state conditions, okay that means no net accumulation of heat inside C we have, Qx; derivative of Qx with respect to x + derivative of Qy with respect to y = 0. Now, Qx = -K del T over del x, Qy is – del T over del y, so that gives you, okay, so del Q/ del y, some del y gives you this, okay.

Hence, this is = 0, which mean that del square T over del x square + del square T over del y square = 0 and this mean that T is the harmonic function. Now, if you use the conjugate harmonic function of T, then the function Wz = T xy + iU xy is an analytic function and the families of curves T xy = alpha, U xy = beta are then respectively called isothermal lines, okay because along those lines, so T xy = alpha, temperature remains constant.

So, they are called isothermal lines and U xy = beta are called the flux lines, Wx; Wz is called the; this Wz is called the complex temperature. With this, I would like to end my lecture, thank you very much for your attention.