Advanced Engineering Mathematics
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Lecture - 49
Poisson Process

Hello friends. Welcome to my lecture on Poisson Process. The Poisson Process is one of the most widely used counting process. It is used when we want to count the occurrences of certain events that appear to happen at a certain rate but completely at random.

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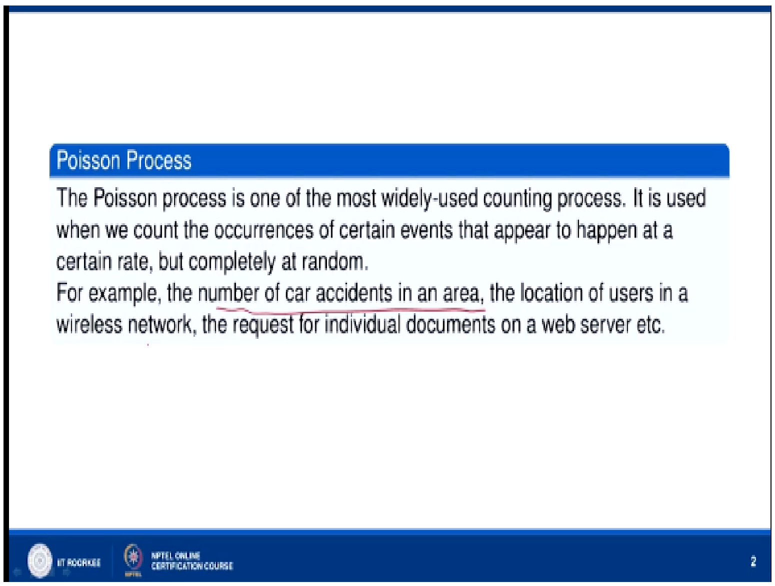

For example, the number of car accidents in area the location of users in a wireless network, the request for individual documents on a web server etc.

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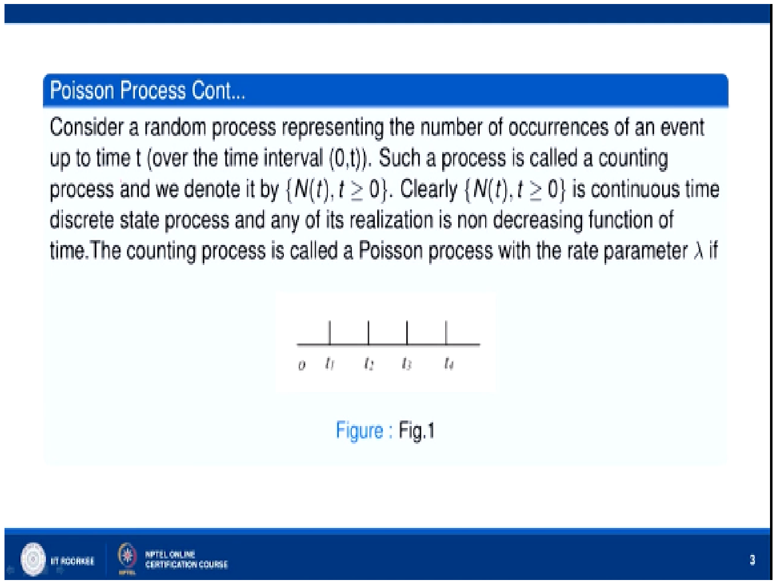

Consider a random process representing the number of occurrences of an event up to time t that is over the time interval 0, t. Such a process is called a counting process and we denote it by N(t) t > = 0. Clearly N(t) t > or = 0 is continuous time discrete state process and any of its realization is non decreasing function of time as time increases the value of N(t) increases, so it is non-decreasing function of time. The counting process is called a Poisson process with the rate parameter lambda if it satisfies the following conditions.

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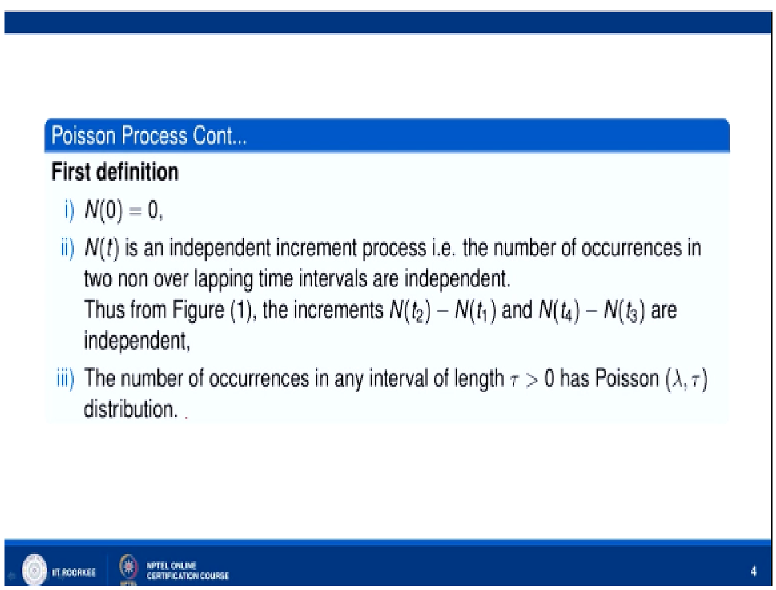

N(0)=0 that is at time = 0, okay. N(t)=0, N(t) is an independent increment process that is the number of credentials in two non over lapping time intervals are independent. Thus from figure 1 we can see that, in the figure 1 we are; the times are indicated, time t=0 then t=1, t=2, t=3 and t=t4. So in the time interval t1 to t2 the increments N(t2) – N(t1) and in the time interval t3 t4 the increment N(t4) – N (t3) are independent, okay.

So in two non over lapping intervals the number occurrences are independent. The number of occurrences in any intervals of length Tau > 0 follows a lambda Tau distribution.

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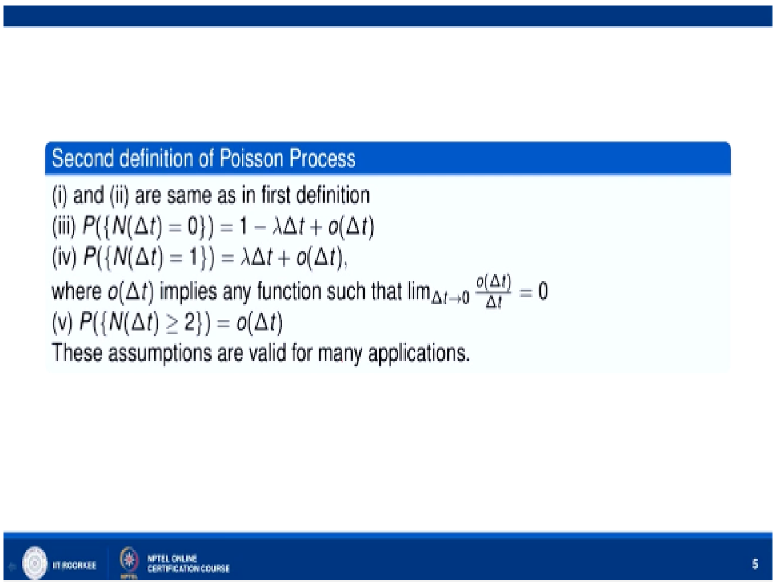

Now there is second definition of this Poisson process. In the second definition 1 and 2 are same thus, N (0)=0 and N(t) is an independent increment process that also occurs in the second definition. But the third condition, okay is replaced by these one, P N delta t=0 = 1-lambda delta t+ small order delta t. P N delta t = 1 is lambda delta t + small order delta t where small order delta t implies any function of delta t such that limit delta t goes to 0 smaller delta t over delta t=0.

Then, P N delta t > or = small order delta t, okay. So these assumptions are valid for many applications. So there are two definitions and this is definition 1, okay this is definition 2.

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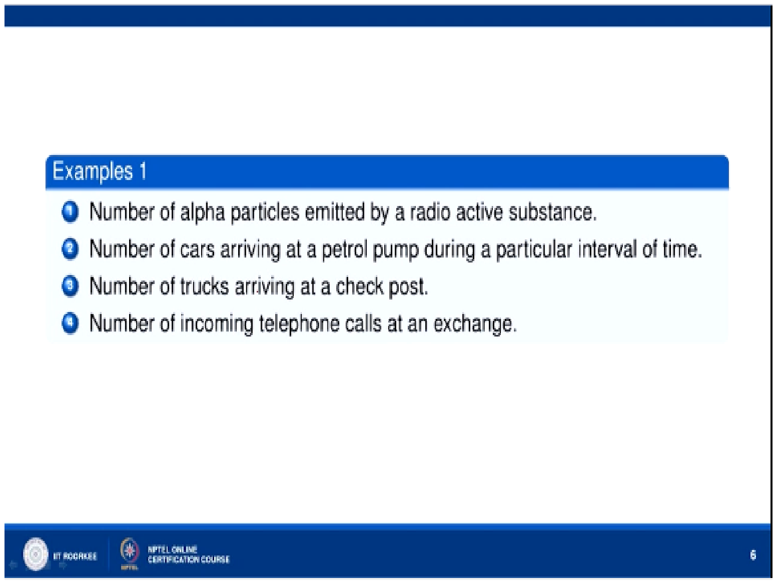

Now let us see some examples on Poisson process, number of alpha particles emitted by a radioactive substance; number of cars arriving at a petrol pump during a particular interval of time; number of trucks arriving at a check post; number of incoming telephone calls at an exchange.

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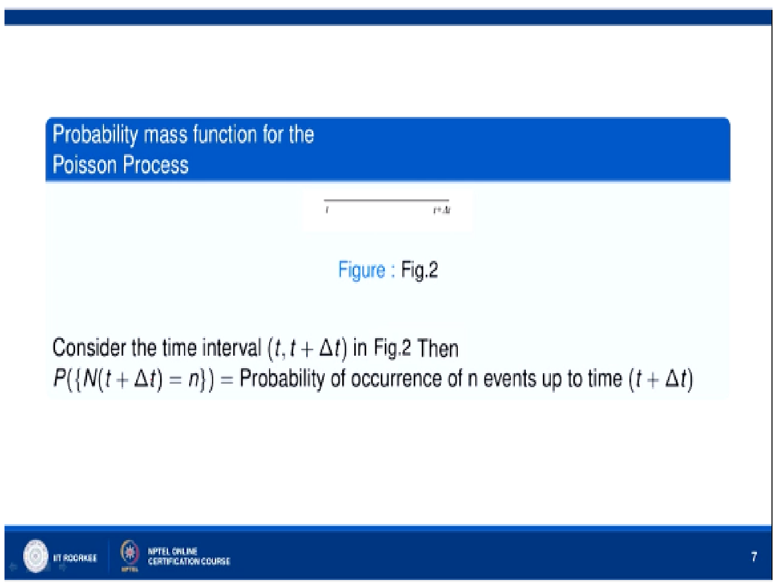

Probability mass function for the Poisson Process, so let us find the probability mass function for the Poisson process. This figure shows the time t and time t + delta, so consider the time interval t to t + delta t in this figure, okay. Then P N t + delta t = n, P N t + delta t = n means over the time period time interval 0 to t + delta t number of occurrences = n. The probability of N t + delta = n is probability of a occurrences of n events up to time t + delta t.

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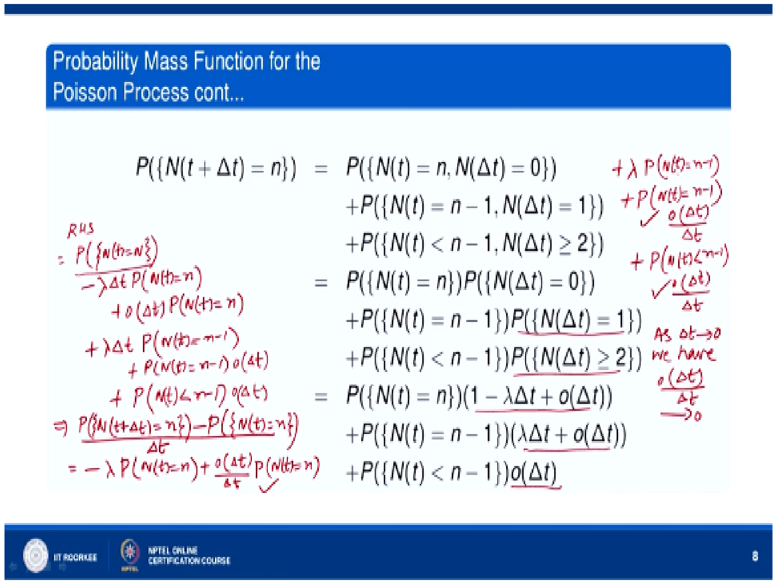

So P N t + delta t = n can be written as P N t = n, N delta t=0. This n occurrences in the time interval 0 to t + delta t may occur during the time 0 to t and then there is no occurrence between t and t + delta t, okay. The other situation can be that during the time period 0 to t number of occurrences are n-1 and from t to t + delta t there is one occurrence. Then, third situation is number of occurrences in the time interval 0 to t < n-1. And between t and t + delta t number of occurrences is more than or equal to 2. So there can be three such possibilities, okay.

Now, probability N t = n, and delta t=0 since we have assumed that the time intervals the; we have assumed that N t is an independent increment process that is number of occurrences in two non over lapping intervals are independent, okay. So probability of since N t = n and delta t=0 are independent events so probability of N t = n delta t=0 will be = product of their probabilities so probability of N t = n probability and delta t=0.

Similarly, for the second possibility case, probability N t = n-1 probability N t = 1 and delta t = 1. Then third possibility is probability that N t < n-1 probability that N delta t > or = 2, okay. So this is because as I said this is because the N t = n and delta t=0 these events are independent. Now probability N t = n * probability n delta t=0; n delta t=0 for probability that n delta t=0 it is we have 1-lambda delta t + small order delta t. So let us put that here.

So 1-lambda delta t + small order delta t then probability that n delta t = 1 can be replaced by lambda delta t + small order delta t. So let us put it here. Lambda delta t + small order delta t then probability that n delta is > or = 2 this one, okay. That we can replace by small order delta t, okay. So we put it here, okay. So we have P and t + delta t = n=P N t=n * 1-lambda delta t + small order lambda delta t + P N t=n-1 lambda delta t + small order delta t + P N t < N-1 * small order delta t.

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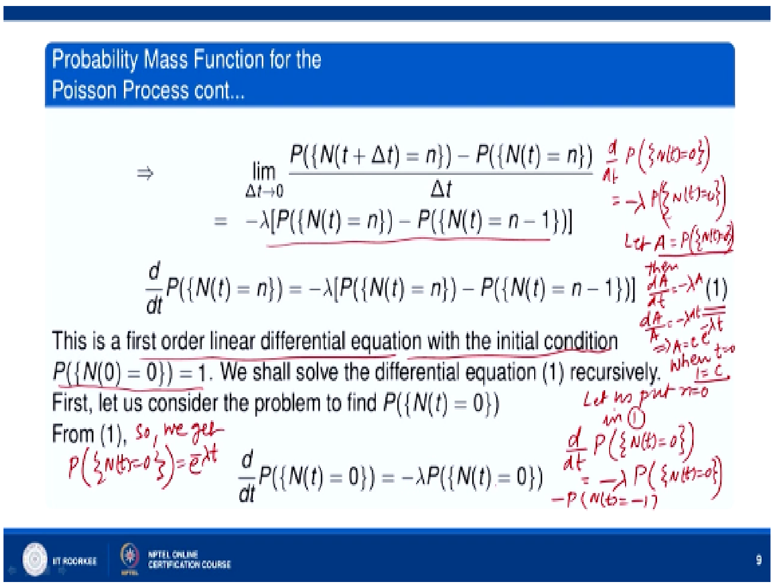

Now let us find the limit, okay. Limit of P N t+delta t = n – P N t=n / delta t. Let us see how we get this. So if you multiply on the right side what you get? Right side is equal P N t=n N t=n * 1 so P N t=n – lambda delta t P N t=n, okay + small order delta t * P N t=n, okay. Then here what we have, P N t=n-1 * lambda delta t and then P N t=n-1* delta 0; small order delta t, okay. And then P N t= N t < n-1 * small order delta t, okay.

Now this term, okay this term we can subtract on the left side and that divide by delta t. So what we will get? P N t+delta t = n – we subtract this value P N t=n, okay. And then divide by delta t. So what do we get then? –lambda P N t=n + small order delta t/delta t * P N t=n, okay. Plus this value lambda delta t P N t=n-1 is divided by delta t and we have + lambda times, we are dividing by delta t so P N t=n-1 and then we have P N t=n-1 * small order delta t/delta t and then we have P N t < n-1 * small order delta/delta t. Okay.

Now as delta t goes to 0, okay. As delta t goes to 0 small order delta t over delta t goes to 0. So let use this. So as delta t goes to 0, okay we have small order delta / delta t goes to 0, okay. So what will happen, this term okay this term will go to 0; this term will go to 0 and this term will go to 0, okay because P N t=n P N t=n-1 P N t<n-1 are finite quantities they are probabilities so they lie between 0 and 1, okay.

So they will go to 0 and therefore, as delta t goes to 0 P N t+delta t=n – P N t=n/delta t will go to –lambda P N t=n+lambda P N t=n-1, so we get; lambda we can take common, -lambda we can take common then we will have P N t=n – P N t=n-1. So as delta t goes to 0 we get this, okay. Now by definition of derivative the left hand side denotes d/dt of P N t=n. So d/dt of P N t=n-lambda P N t=n-P N t=n-1.

You can see this is the first order linear differential equation, okay with the initial condition P N 0=1, because we have this condition with us N0=0, okay. So probability of 0=0, okay. So probability of N0=0 is 0 okay. So let us solve this differential equation. We can put N=1, okay. Let us put N=; let us put N=0 first, okay. So let us put N=0 in equation 1, okay in this equation. Then d/dt of P N0=0. N t=0 will be equal to lambda – lambda P N t=0. –probability that N t < -: is equal to -1, okay.

Now probability that N t=-1 is 0 because N t is always > or = 0. So this value becomes 0. So thus, what we have d/dt of P N t=0 = -lambda P N t=0. Okay. So let us say, P N t=0 be A. Let A be equal to P N t=0. Let us say this value is A, okay. Then dA/dt = -lambda A. So we can solve it easily. So dA/A=-lambda dt. This gives you A=e to the power –lambda t * some constant say C, okay. A= some constant t times e to the power –lambda t. When t=0 we put we have to find the value of A at t=0, okay.

So at t=0 probability that N0=0 is 0, okay. Probability that N0=0 is 1, okay. Probability that N0=0; when t=0, okay yeah and P t=0 probability that N0=0 is 1, okay so what we will get, 1=C, okay we will get 1=C when you put t=0, we get right hand side as C and A, A is this okay at t=0. Since we have assumed that N0=0, so N0=0 is certain and therefore it is probability is 1, so C=1, okay. So we get A=e to the power –lambda t, okay. So we get probability N t=0 = e to the power –lambda t, okay. So this is how we solve this and this is what we get here, okay.

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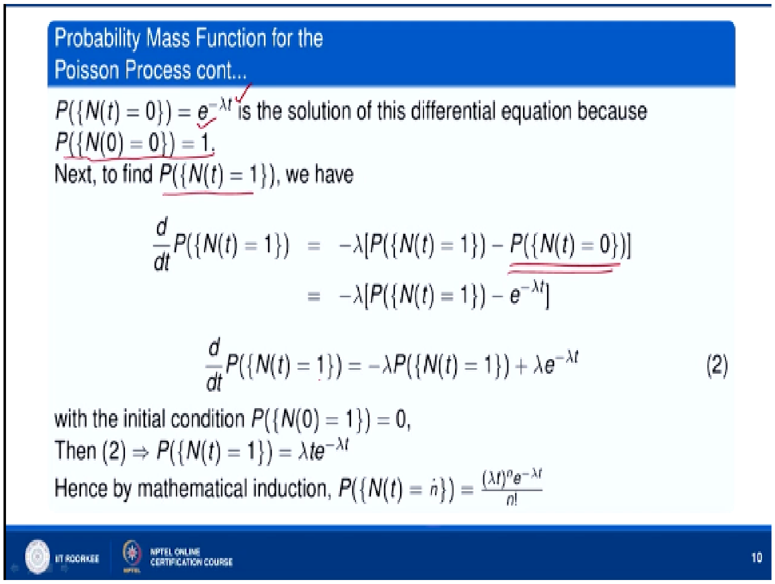

Probability that N0=0 is 1, okay so probability that N t= 0 comes out to be e to the power –lambda t, okay. Now let us find probability of N t=1. So let us take t=; n=1 here. So let us take n=1 here then we will get d/dt P N t=- lambda times P N t=1-P N t=0. And P N t=0 we have already found e to the power –lambda t so let us put its value here, so –lambda P N t=1 – lambda – e to the power –lambda t, okay.

So, then what do we get d/dt of P N t=1 = -lambda P N t=1+lambda e to the power –lambda t, okay. Now let us find the; let us solve this differential equation, okay. So this is of the type; okay this is of the type let us if you take P N t=1 as some say B then this is dB/dt = -lambda B + lambda e to the power –lambda t. So let us solve this.

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Let $P(\{N(t)=1\}) = B$

then $\frac{dB}{dt} = -\lambda B + \lambda e^{-\lambda t}$

$$\frac{dB}{dt} + \lambda B = \lambda e^{-\lambda t} \quad \text{--- (1)}$$

$$I.F. = e^{\lambda \int dt} = e^{\lambda t}$$

Multiplying (1) by $e^{\lambda t}$ and then integrating with respect to t we have

$$e^{\lambda t}.B = \int \lambda e^{-\lambda t} e^{\lambda t} dt = \lambda t + D$$

when $t=0$ then $D = B = 0$

thus

$$e^{\lambda t} B = \lambda t$$

$$B = \lambda t e^{-\lambda t}$$

$$P(\{N(t)=1\}) = \lambda t e^{-\lambda t}$$

So let P N t=1 I call as B okay. So then, dB/dt = -lambda B + lambda e to the power –lambda t, okay. So I can write it as dB/dt + lambda B = lambda e to the power –lambda t, okay. Now, this is a linear differentials equation, okay. So integrating factor is e to the power –lambda dt, okay. So this is e to the power lambda t. So we multiply by this integrating factor in this equation and then integrate with respect to t, okay.

So multiplying, let me call it as equation number say 1, okay multiplying 1 by e to the power lambda t and then integrating with respect to t. We have e to the power lambda t * B= lambda e to the power –lambda t * e to the power lambda t dt, okay. So this is equal to lambda t + some constant, let us say d, okay. Now when t=0, what we get? We get this as B and here this is 0, this is d so then D=B, okay. And when t=0 okay probability that N t=1, okay.

So we get n0=1, probability that n0=1 is equal to 0; n0=1 means, because N t is the probability that number of; N t denote the number of occurrences in the time interval 0 to t, okay. So when t=0 okay probability that n0=1 is 0 so B=0, okay. So we have B=0, okay. So thus, e to the power lambda t e * B = lambda t so B=lambda t e to the power –lambda t, okay. This means probability that, N t=1 = lambda t e to the power –lambda t. Okay.

So we get P N t=1=lambda e to the power –lambda B. Now we can carry on this process, okay we can take now n=2 and so that when P N t=2 is lambda t whole square * e to the power –

lambda t/2 factorial, okay. So by mathematical indexation we can, we then have probability that N t=n okay, N t=n is lambda t e raise to the power to the power n * e to the power –lambda t over n factorial, okay. So we prove this result, probability that N t=n is lambda t e raise to the power n* e to the power –lambda t/n factorial.

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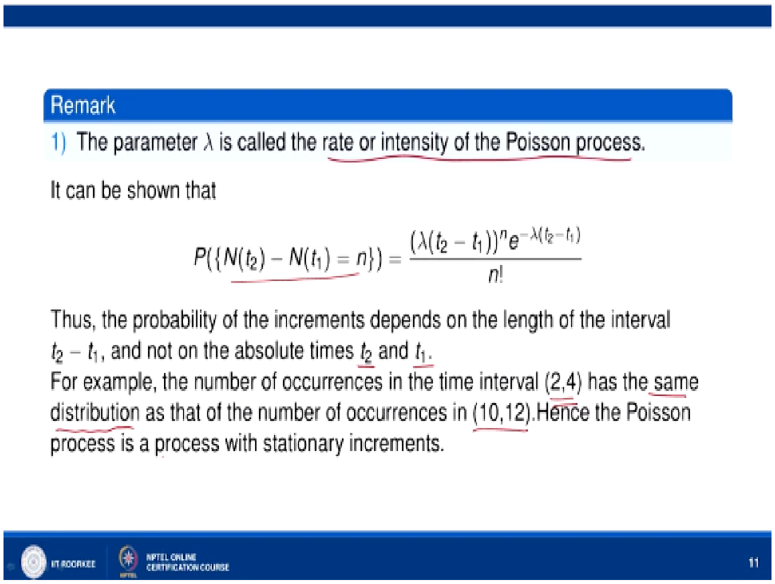

Now the parameter lambda here is called the rate or intensity of the Poisson process, okay. Now it can be shown that probability that N t2 – N t1 = n, okay. During the time interval t1 to t2 number of occurrences is n probability of that is equal to lambda t2-t1 to the power n* e to the power –lambda t2-t1/n factorial. Thus, the probability of the increments depends on the length of the interval t2-t1 and not on the absolute times t2 and t1.

For example, the number of occurrences in the time interval 2, 4 as the same distribution as that of the number of occurrences in the time interval the 10 to 12, the Poisson process is the process with stationary increments, so we can say that.

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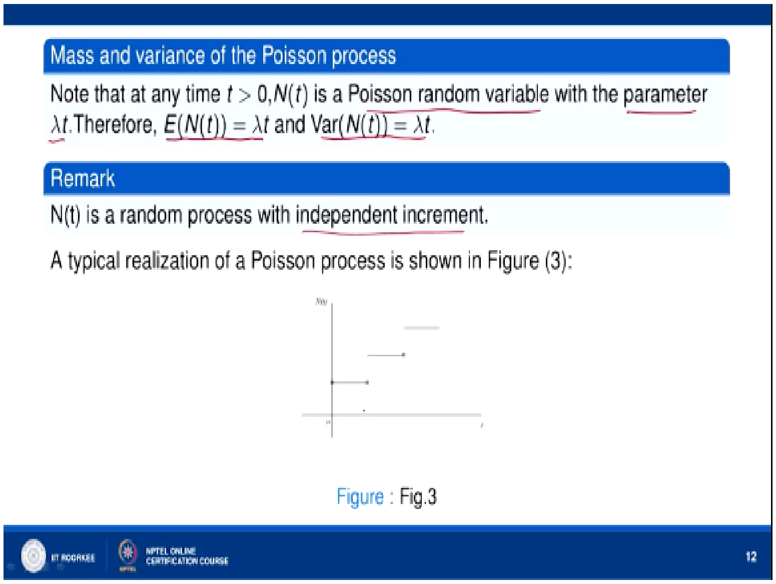

Now, let us find the Mass and variances of the Poisson process. Note that, at any time t=t > 0 N t is Poisson random variable with the parameter lambda t. We can see here. If you see this probability for N t=n it is lambda t raise to the power n * e to the power –lambda t/n factorial. Okay. In the case of Poisson random variable, the probability is that probability of n successes is lambda to the power n * e to the power –lambda t/n factorial because lambda is the mean there okay.

So here we can say that it is the Poisson random variable with the parameter lambda t. There the parameter lambda here the parameter is lambda t. Now there lambda is the mean here, lambda t is the mean and their variance is lambda, here variance is lambda t. So we can say that, the mass and variance okay, the mean and variance of the Poisson process is given by lambda t and lambda t again, okay.

So we can say that N t is a random process with independent increments. A typical realization of Poisson process is shown in this figure okay. Say this is 0, 0 to t1 okay then t1 to t2 then t2 to t3 so this is during the time period 0 to t1 it shows the number of occurrences and then t1 to t2 then t2 to t3 number of occurrences in the times of; during this period 0 to t1 number of occurrences here then between t1 to t2 number of occurrences here and then, so N t denote the number of occurrences here.

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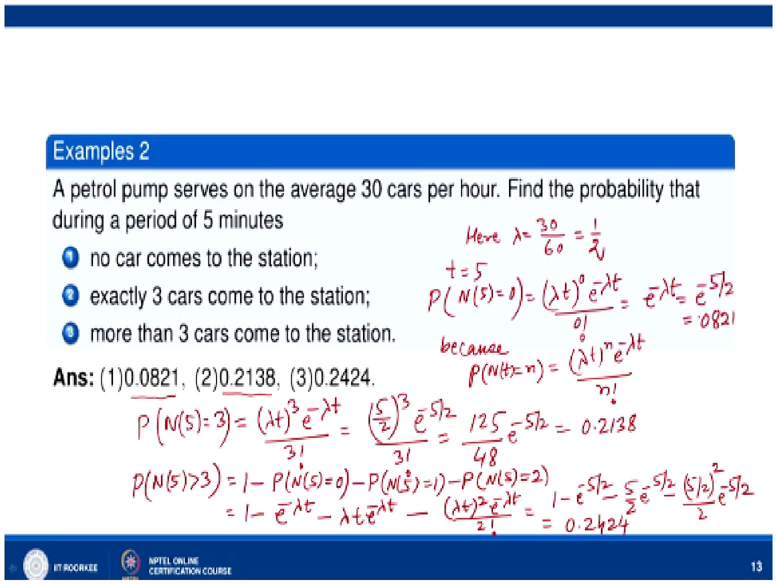

So let us consider this problem. A petrol pump serves on the average 30 cars per hour, okay. So here lambda = 30/60 the intensity, okay. Lambda, if you see here in the Poisson process lambda is what we denote lambda by a; it is a rate parameter, okay. So rate here, in this problem rate is 30/60 so this is lambda=1/2, okay. Now, so a petrol pump serves on the average 30 cars per hour, find the probability that during a period of 5 minutes, t=5. During a period of 5 minutes no cars comes to the station that means probability that n 5=0.

We have to find the probability that, during the time interval 5 minutes no car comes to the station so number of occurrences is 0. This is lambda t raise to the power 0 * e to the power –lambda t/0 factorial. Because, P N t=n we have this formula. P N t=n=lambda t raise to the power n * e to the power –lambda t/n factorial, okay. So here number of occurrences is 0. No car comes to the station. So this is equal to e to the power –lambda t. Lambda = 1/2, t=5 so e to the power -5/2 which means that the answer is 0.0821. Okay.

Then exactly three cars comes to the station, so probability that n5=3, okay. Probability that n5=3 means lambda t raise to the power 3 * e to the power –lambda t/3 factorial. So this is lambda is 1/2, t is 5. So 5/2 raise to the power 3, e to the power -5/2/3 factorial. Okay. And this value is then 5 cube means 125, 125/2 cube is 8, 8*6 is 48 * e to the power -5/2. So this value comes out to be 0.2138, okay.

Then probability that, more than 3 cars comes to the station yeah. So n5 > 3. Now this is equal to 1-probability that n5=0, no cars comes to the station – probability that n5=1-probability that n5=2, okay. So this is equal to 1-; n5=0 means e to the power –lambda t. And n5=1 means lambda t to the power 1, e to the power –lambda t/1 factorial. So lambda t e to the power – lambda t, and here n5=2 is lambda t whole square * e to the power –lambda t/2 factorial. Okay.

So this is equal to 1-e to the power -5/2 – 5/2 e to the power -5/2 – 5/2 whole square/2*e to the power -5/2 which comes out to be; if you calculate it, it is 0.2424. Okay. So this is a problem on Poisson process, okay which deals with the probability of the number occurrences during a time interval 0 to t. In our next lecture, we shall discuss exponential distribution which is based on the Poisson process. So with that I would like to end my lecture. Thank you very much for your attention.