Advanced Engineering Mathematics
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Lecture – 48
Continuous Probability Distribution

Hello friends. Welcome to my lecture on Continuous Probability Distribution. Let s be a sample space which is an interval of the real line or have points defined in a continuum. For such sample spaces, we define the random variable in such a way that they take all values in an interval of values.

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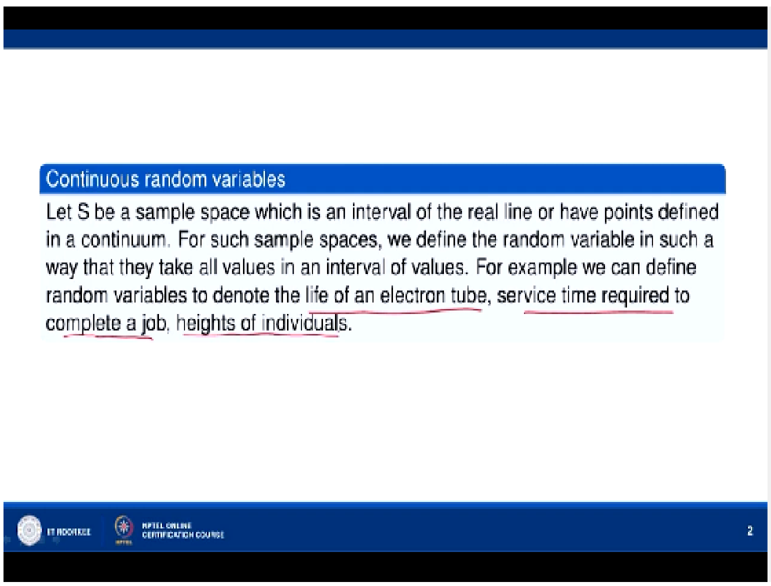

For example, we can define the random variable to denote the life of an electron tube, service time required to complete a job, heights of individuals, okay. They all come under continuous random variables. So such random variables are all continuous random variables.

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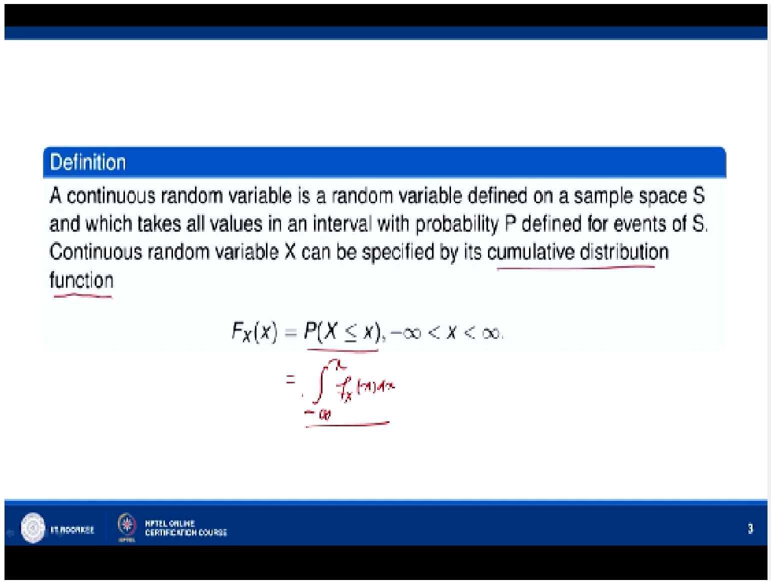

Now a continuous random variables is a random variables defined on a sample space S, okay and which takes all values in an interval with probability P defined for a events of S. Continuous random variables X can be specified by its cumulative distribution function. cumulative distribution function is the probability that, we denote it by FXX, X is the continuous random variable, okay. F denotes the cumulative distribution function.

So FXX is probability that X takes the value less than or equal to x where x belongs to the interval -infinity to infinity.

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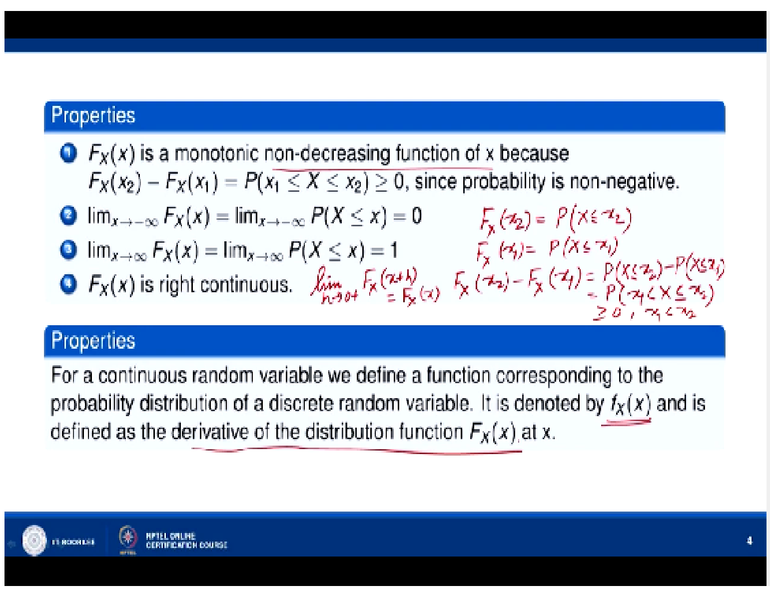

Now this FXX, is monotonically non-decreasing function of x, okay. Because you can find

FXx2-FXx1, this will come out to be probability that x1 is less than or equal to X less than or equal to x2. FXx2=Probability that X is less than or equal to x2 and FXx1=probability that x is less than or equal to x1, okay. So FXx2-FXx1, okay is the probability that X is less than or equal to x2-probability that X is less than or equal to x1, okay.

So this is probability that X is less than or equal to x2, okay. Since probability is a non-negative function, this is greater than or equal to 0, okay. So whenever x1 is less than, so here we are assuming that x1<x2. So when x1<x2, FXx2-FXx1 is greater than or equal to 0 that is FXx1 is less than or equal to FXx2. So FXx is a monotonically non-decreasing function of X. Now limit x tends to -infinity.

Limit x tends to -infinity FXx is limit x tends to -infinity probability that x is less than or equal to x, that is equal to 0, okay. Because probability that X is less than or equal to x when x tends to -infinity, okay, will be equal to 0 and when limit, because this is nothing but, this is probability X less than or equal to x. This is nothing but -integral over -infinity to x fxxdx, we shall see later on.

We shall see later on that this is, FXx is nothing but integral over -infinity to x fxxdx. So when x tends to -infinity, fxx tends to 0. So limit x tends to -infinity Px less than or equal to x is 0 and here limit x tends to fxx that is limit x tends to infinity, probability that x is less than or equal to x will be equal to 1, okay. Because it will be then the total probability. Total probability will be equal to 1.

Fxx is right continuous, okay. The right continuous means limit h tends to 0+, okay. Fxx+h=Fxx, okay. So it is right continuous. Now for a continuous random variable, we define a function corresponding to the probability distribution of a discrete random variables. For a continuous random variables, we define a function corresponding to the probability distribution of a discrete random variable.

It is denoted by fxx. So this is known as the probability density function for the random variables, continuous random variable x and it is defined as the derivative of the distribution function fxx. So this probability density function fxx is defined as the derivative of the distribution function fxx at x, okay.

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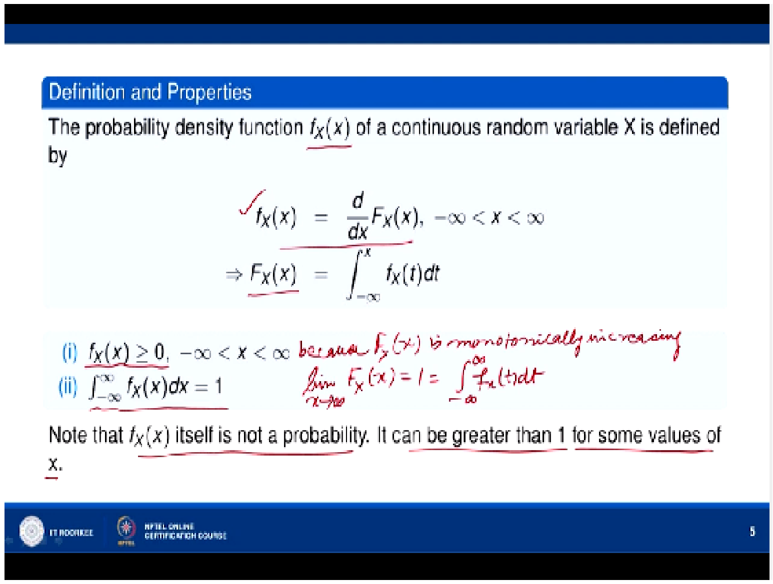

So the probability density function fxx of a continuous random variable is defined as d/dx of fxx, okay where x varies over -infinity to infinity. From this equation, one can see that fxx is integral over -infinity to xfxtdt, okay. So you can see since fxx is a monotonically increasing function, okay, its derivative must be non-negative, okay. This follows because fxx is monotonically increasing. So its derivative is non-negative.

So fxx is greater than or equal to 0 and further more limit x tends to infinity fxx=1, okay. We have assumed that limit x tends to infinity fxx=1, okay. So from there, okay, this will be equal to integral over -infinity to infinity fxtdt, okay. So when x tends to infinity, fxx tends to integral over -infinity to infinity of xtdt and limit x tends to infinity fxx=1, so integral over -infinity to infinity fxxdx=1. Now we can note that fxx itself is not a probability. It can be greater than 1 for some values of x, okay. The integral of fxx over -infinity tends to infinity is 1, okay.

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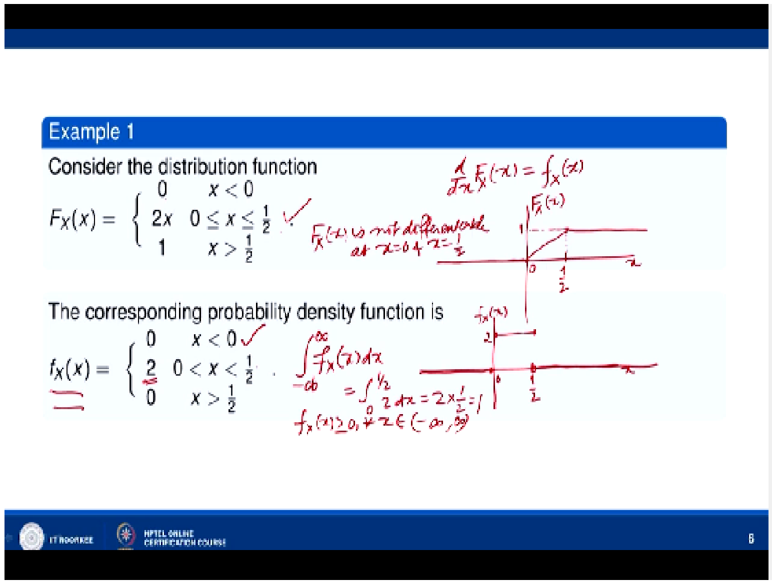

So consider the distribution function. So this is our distribution function, cumulative distribution function fxx=0 when x<0, 2x for 0 less than or equal to x less than or equal to 1/2 and 1 when x>1/2, okay. So let us find its derivative d/dx of fxx. By our definition, this is fxx, probability that the function. So fxx will be equal to 0 when x<0, okay, derivative of 0 is 0. And then 0 less than or equal to x less than or equal to 1/2, okay.

So here the derivative will be 2, derivative of 2x will be 2. And it will be over the interval 0<x<1/2. Why? Because if you draw the graph of this cumulative distribution function, so it is 0 for x<0, okay. And then 0 to 1/2, it is 2x. It is given by 2x. So that means it is a straight line. This is 1/2 here. So it is like this, okay. At x=2, it is 1. So this is 1, okay. So we have this, okay. And then x>1/2, it is 1, okay.

So like this, okay. Now you can see at x=0 and at x=1/2, it is not differentiable. Because the left hand derivative is 0, right hand derivative is 2, okay. So it is not differentiable at 0. It is not differentiable at 1/2, okay. So fxx is not differentiable, although it is continuous throughout over -infinity to infinity but is not differentiable at x=0 and x=1/2. Elsewhere it is differentiable. So in the interval, open interval 0<x<1/2, the derivative of fxx=2. And when x>1/2, its derivative is 0.

So you can see the probability density function is taking value 0 over the interval -infinity to 0. So that means here up to this, it is 0 everywhere and then it is 2/0 1/2 interval, okay. So it is over

0 1/2, it is 2, okay and this is fxx. This is fxx, cumulative distribution function, okay. And now here then it takes value 0. So from here onwards, it takes value 0, okay. So this fxx can take value other than 1.

It can take value greater than 1. Here it is taking value 2, okay. But the integral of fxx over -infinity to infinity must be equal to 1, that you can see. So fxxdx, this will be equal to integral, because over -infinity to 0, it is 0. Over 1/2 to infinity, it is 0. It is non-0 over the interval 0 1/2. So this is 0 to 1/2 and 2dx. This is equal to 2*1/2 which is equal to 1, okay. And moreover, it is non-negative, fxx is greater than or equal to 0 for all x. So it is probability density function.

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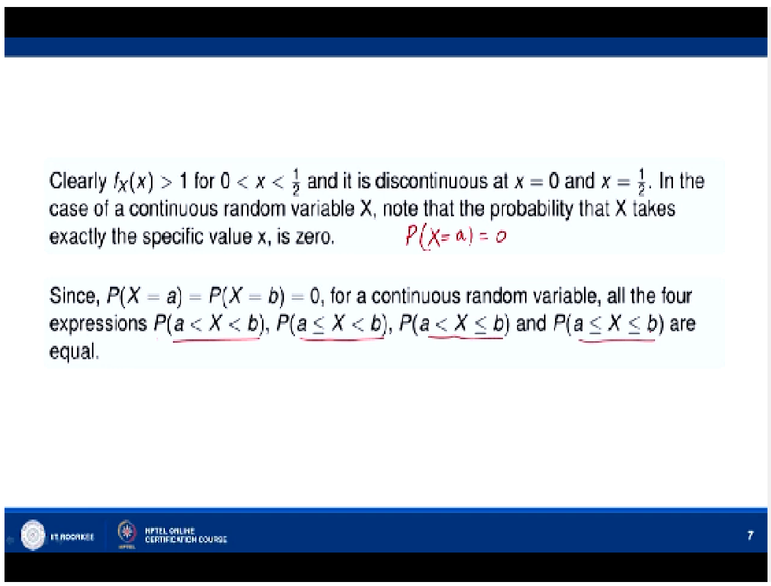

Now let us go to, so here I have explained clearly fxx>1 for 0<x<1/2. It is discontinuous at x=0. You can see, it is discontinuous at x=0 and also discontinuous at x=1/2. In the case of a continuous random variable x, note that the probability that X takes exactly the specific value x is equal to 0. That means probability that X takes the value a, okay, are x=0, okay.

So since PX=a=PX=b=0 for a continuous random variables, all the 4 expression probability of a<X>b, probability a less than or equal to x<b, z<X less than or equal to b, probability of a less than or equal to X less than or equal to b are all equal, okay.

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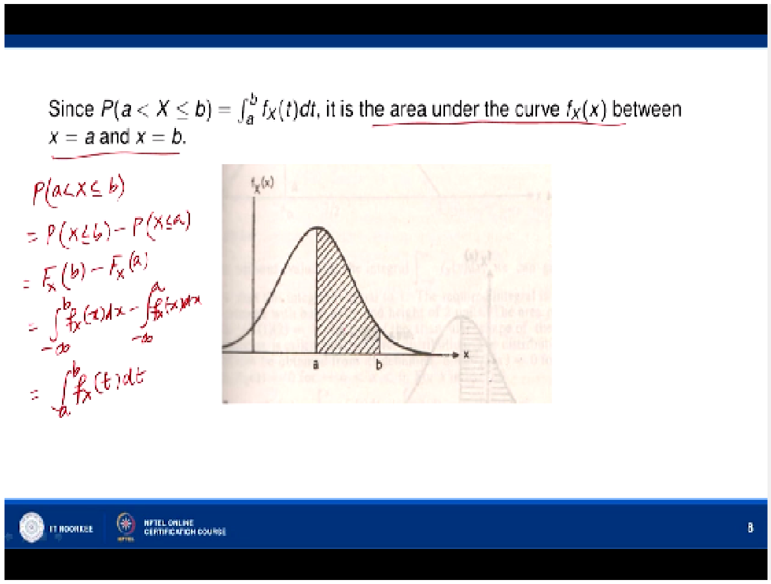

Now probability of a<X less than or equal to b is integral over a to b fxtdt because probability a<X less than or equal to b, this is equal to probability that X is less than or equal to b-probability that X less than or equal to a, okay. So this is fxxb-fxxa we can say, okay. And this is equal to integral over -infinity to bfxxdx-integral over -infinity to axxdx, okay. Since a<b, okay, is equal to integral over a to bfxxdx, okay.

So fxtdt we can write, okay. So it is the area under the graph of fxx function, okay from over the interval a to b. So it is the area under the graph of fxx between x=a and x=b, okay as shown in this figure.

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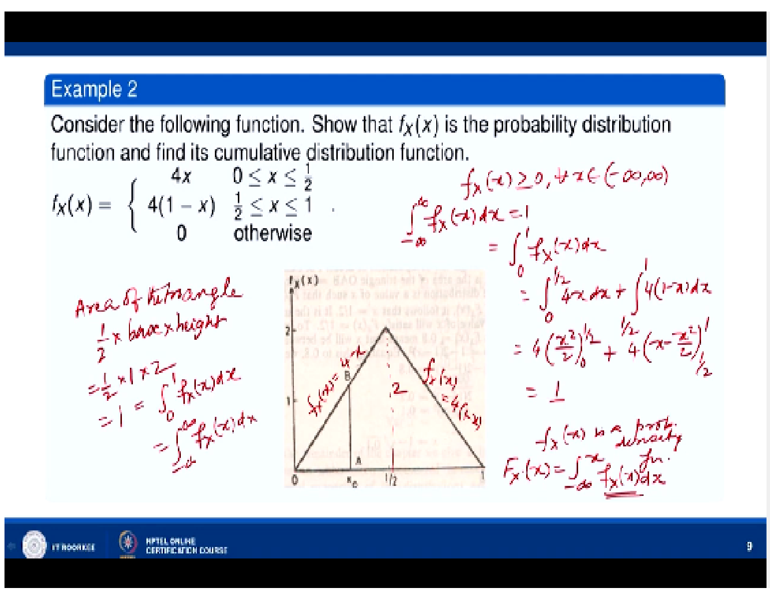

Now let us consider the following function. fxx=4x over the interval 0 less than or equal to x less than or equal to 1/2. 4*1-x 1/2 less than or equal to X less than or equal to 1 and 0 otherwise. So we have drawn the graph here over the interval 0, it is given by this. Over the interval 0 to 1/2, it is given by the straight line y=fxx=4x. Okay, over the interval 1/2 to 1, it is given by this straight line, 4*1-x.

So this is 4x, fxx=4x and this is fxx=4*1-x, okay. Now we have to show that it is a probability density function. First of all, let us note that fxx is greater than or equal to 0 for all x belonging to -infinity to infinity, okay. The graph of fxx lies above the x axis. On the negative side of x axis, it is 0 everywhere. On the positive side of x axis, after 1, okay, it is 0 and over 0 to 1, it takes non-negative values.

So it is greater than or equal to 0 for all x. Now it is a probability density function provided we prove that integral over -infinity to infinity fxxdx=1, okay. So over -infinity to 0, it is 0. Over 1 to infinity, it is 0. So we can write it as integral over 0 to 1 fxxdx which is equal to integral over 0 to 1/2. Over 0 to 1/2, it is 4x and then over 1/2 to 1, it is 4*1-x, okay. So we can integrate this 4*x square/2 0 1/2+4*x-x square/2 1/2 to 1.

We can substitute the limit and we shall see that it comes out to be 1. Otherwise also we can see, from the geometry also we can find the integral over -infinity to infinity fxdx. This is integral over 0 to 1 fxdx. That means the area bounded by this triangle, okay, integral over 0 to 1 fxxdx is the area bounded by the triangle.

And this area is how much? 1/2*base*height, area of the triangle. So this will be 1/2, base=1 and height is at x=1/2, okay. At x=1/2, we can see it is, fxx=4x. So at x=1/2, the height is 2, okay. So this is equal to 1, okay. So area of the triangle is integral over 0 to 1 fxxdx which is equal to integral over -infinity to infinity fxxdx. So fxx is a probability density function, okay. Now let us find its cumulative distribution function.

So cumulative distribution function is FXx and it is given by integral over -infinity to xfxxdx, okay. So first we consider the case when x<0. When x<0, fxx is 0. So this probability density function assumes value is 0. So cumulative distribution function is also 0, okay.

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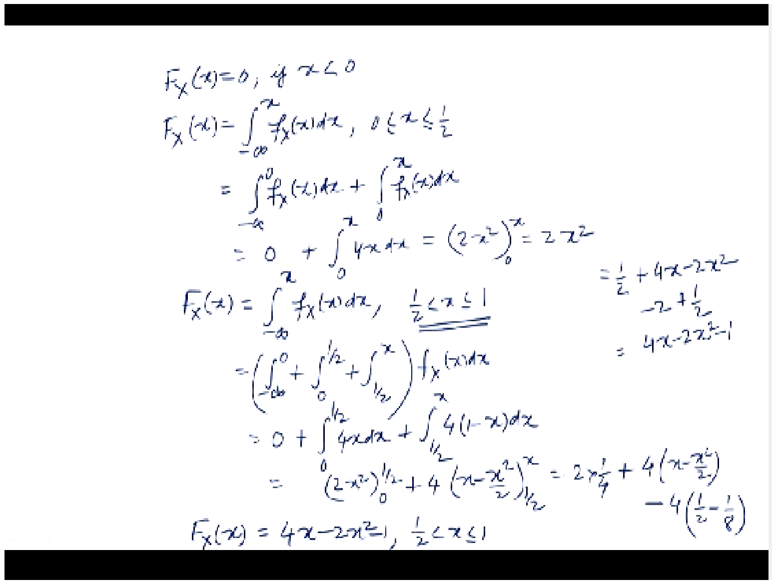

So FXx=0 if x<0, okay. Then let us consider the case when 0 is less than or equal to x<1/2, okay. So we consider the case 0, suppose x is greater than or equal to 0 but less than 1/2, okay. So we can assume, this is equal to, in fact all of x=1/2 also we can take. So we will have integral over -infinity to x, that means -infinity to 0 fxxdx+integral over 0 to 1/2 fxxdx, okay. So this is equal to 0 because over -infinity to 0 fxx is 0 and then 0 to 1/2, it is given by 4x, okay.

So this is 2x square. So 0 to, oh it is x, okay. Let me not 1/2. So we can write it as 0 to x, not 1/2, okay. So this will be 2x square 0 to x that is 2x square, okay. Then fxx=-infinity to xfxdx. Let us find fxx. When x>1/2 and less than or equal to 1. So 1/2<x less than or equal to 1, okay. Then we have -infinity to 0 0 to 1/2 and then 1/2 to x, okay, fxxdx. So we shall have here, integral over -infinity to 0 fxxdx will be 0 because the function fxx is 0.

Then 0 to 1/2, fxxdx. 0 to 1/2, fxx is 4x. So 4xdx. Then 1/2 to 1, 1/2 to x. fxx is 4*1-x, okay. So we will get this as, this is 2x square 0 to 1/2 and here we will get 4*x-x square/2 1/2 to x, okay. So this is 2*1/4+4*x-x square/2-4*1/2-1/2 square, that is 1/4/2, that is 1/8, okay. So how much it will be? This is equal to 1/2, okay, +4x-2x square and what we get here? 1/2-1/8 is or we can multiply by 4. So -2, okay, and +1/2. So what do we get?

This is 1/2, 1/2 is 1, 1-1, so 4x-2x square-1, okay. So when x lies in this interval, okay, fxx=4x-2x square-1, okay. So 4xfxx=4x-2x square-1, provided 1/2<x less than or equal to 1. And when x>1, okay.

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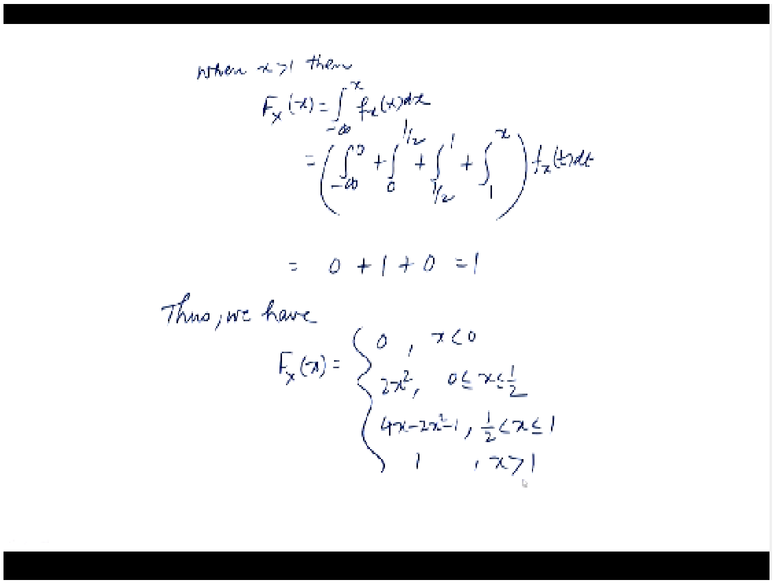

Now let us consider the case when x>1, okay. Then fxx=integral over -infinity to xfxxdx=integral over -infinity to 0, okay, +integral over 0 to 1/2+integral over 1/2 to 1, okay, +integral over 1/ to x fxtdt, okay. So this is equal to integral over -1/2 to 0, it is 0. Then integral over 0 to 1/2+integral over 1/2 to 1, fxt we have found is equal to 1. So we have 1+integral over 1 to x, okay.

Integral over 0 to 1/2+integral over 1/2 to 1, this is integral over 0 to 1/2+integral over 1/2 to 1 fxtdt=1. And integral over 1 to x fxtdt=0 because that is 0, okay. fxt is 0 for x>1. So fxx=1 when x>1. So thus we have fxx=0 when x<0. And we have fxx=2x square when x is lying between 0 and 1/2. And then we have 4x-2x square-1 when 1/2<x less than or equal to 1 and 1 when x>1. So this is how we find the cumulative distribution function, okay.

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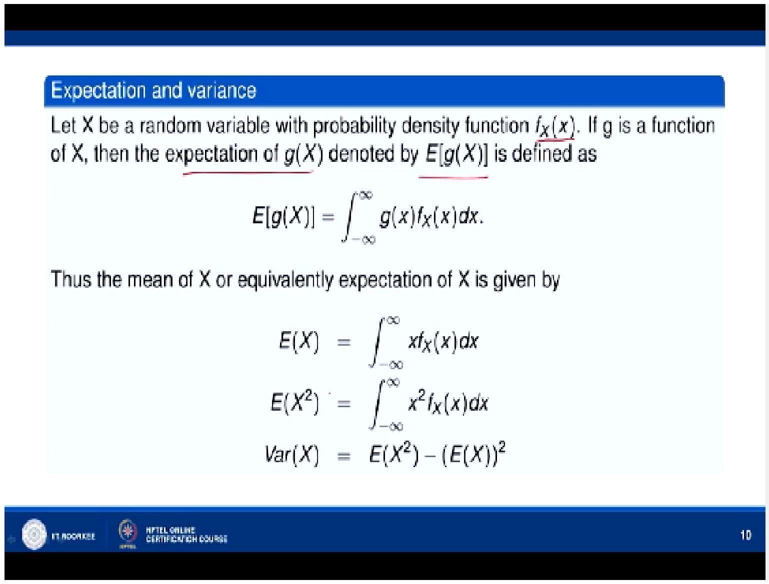

Now we go to the definition of expectation and variance. Let X be a random variable with probability density function fxx. If g is a function of X, then the expectation of gX denoted by EgX is defined as EgX=integral over -infinity to infinity gxfxxdx. The mean of fx or equivalently the expectation of X is given by integral over, EX=integral over -infinity to infinity Xfxxdx.

And expectation of X square is similarly defined integral over -infinity to infinity x square fxxdx. Now variance of X we know. Variance of X is expectation of x square-expectation of X whole square. So this is how we can determine variance of X once we have the values of EX and EX square.

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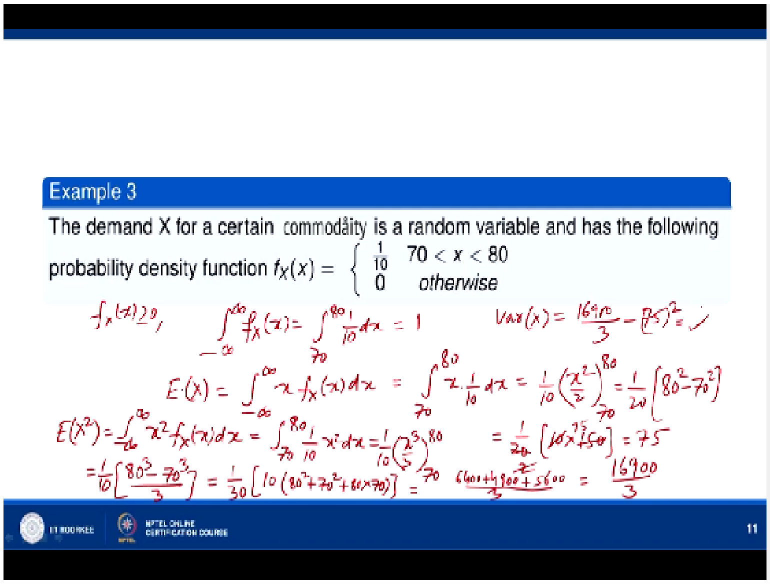

Now let us consider this question. The demand X for a certain commodity is a random variable and has the following probability density function fxx=1/10 70<x<80 and 0 otherwise. We can find here, we can see that integral over -infinity to infinity fxx=integral over 70 to 80 1/10dx, okay. This is equal to 1, okay. And also that fxx is greater than or equal to 0. So that we can see here, it takes non-negative values, okay.

So fxx is a probability density function. We can find mean here. So expectation of x=integral over -infinity to infinity x*fxxdx. So this will be integral over 70 to 80 x*1/10dx. And this will be equal to 1/10*x square/2 70 to 80. So we can find here, this is 1/20 and then we have 80 square-70 square, okay. So this is equal to 1/20*, this is t square, so 80 square-70 square is 80-70 that is 10*80+70, that is 150, okay.

So we have 75, okay. So expectation is 75. Expectation of x square if you find, this will be equal to integral over -infinity to infinity x square fxxdx and this will be equal to integral over 70 to 80 1/10 that is fxx, *x square dx. So this is 1/10x cube/3, okay, 70 to 80. So this is 1/10 80 cube-70 cube/3, okay. So this is 1/30, then we have, we can factorize this. So 80, a cube-b cube is a-b, that is 80-70 is 10, *a square, so 80 square, +b square means 70 square, +ab, means 80*70, okay.

So this will give you, equal to, this is 80 square is 6400+4900+, 80*7, 5600, /3, okay. So this will come out to be 00, 4 9, 13, 13 and 6 is 19 and then 7 4, 11, 11 5, 16, okay. So this is EX square.

Now once we have EX square, we can subtract from EX square EX whole square to get the variance of X. So variance of X will be equal to EX square that is 16900/3-75 whole square and we get the variance of X, okay. So we can calculate the variance of X here and from that variance of X, we can find the value of standard deviation also, okay.

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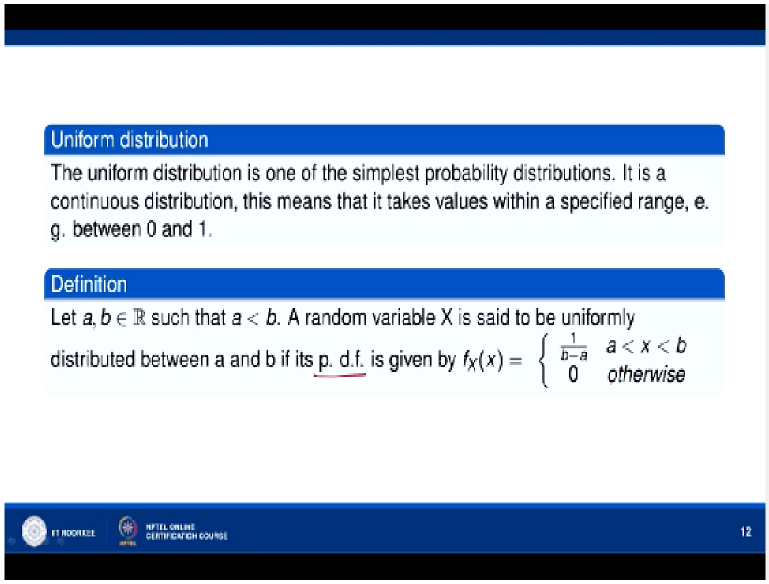

Now let us go to uniform distribution. The uniform distribution is one of the simplest probability distributions. It is a cumulative distribution. This means that it takes values within a specified range, that is between 0 and 1, okay. So let us take a and b to be any 2 real numbers such that a<b, okay. As a random variable X is called uniformly distributed between a and b if its probability density function is given by 1/b-a a<x<b and 0 otherwise. We can draw its graph.

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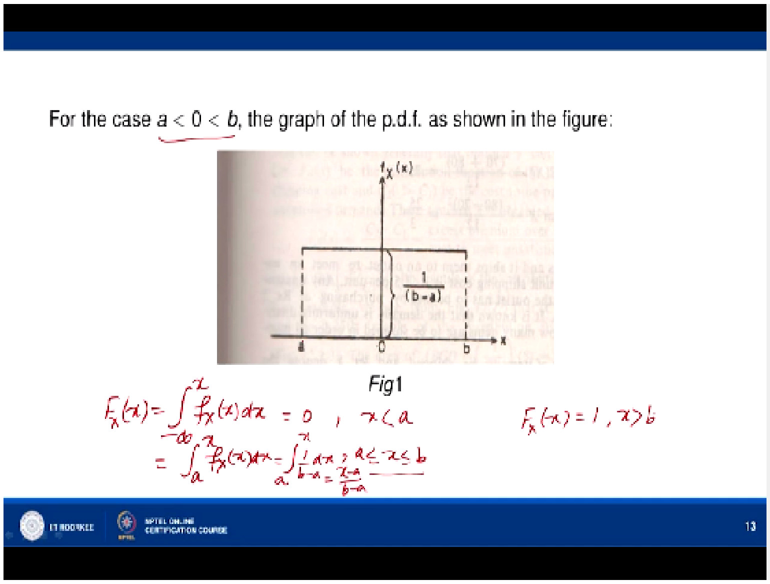

Suppose a<0<b, then the graph of the probability density function of the normal uniform distribution is given by this, okay. Between a to b, it takes values 1/b-a, elsewhere it takes values 0. The area under the graph of fx, you can see is 1/b-a*b-a. So equal to 1. The area bounded by the rectangle is 1, okay.

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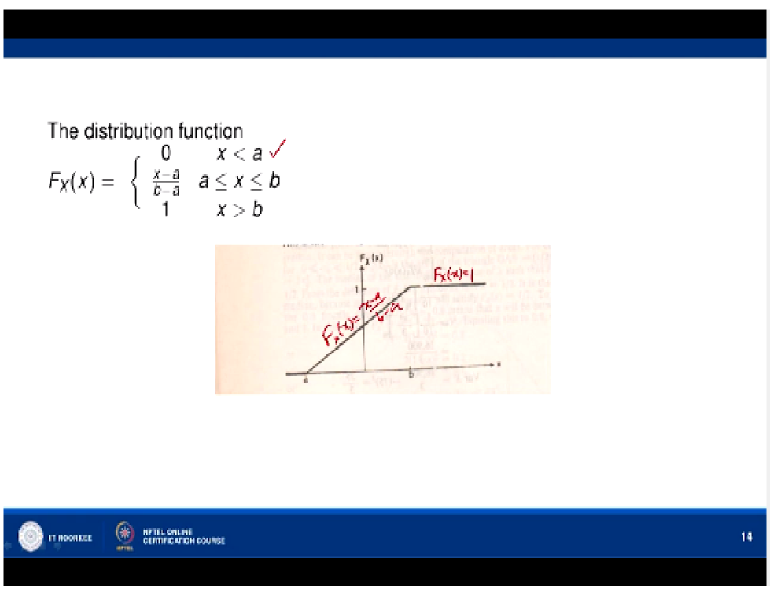

Now the distribution function, we can see here, the distribution function fxx. We know its definition is -infinity to xfxxdx integral over -infinity to xfxxdx and this will be equal to 0 if x<a, okay. If x<a, okay, so this will be 0. If z is less than or equal to x less than or equal to b, so this will be equal to integral over a to xfxxdx. Because over -infinity to a, fxx is 0. So this will be equal to integral over a to x, okay, 1/b-adx and this will be equal to x-a/b-a, okay.

So over the interval a less than or equal to x less than or equal to b it is x-a/b-a. And when x>b, okay, fxx=1, okay. So we have this cumulative distribution function and we can see its graph when x<a, it is 0. When x lies between a and b, it is given by this straight line x-a/b-a. So this is fxx=x-a/b-a. And here, fxx=1, okay. So this is the graph of cumulative distribution function of an uniform distribution.

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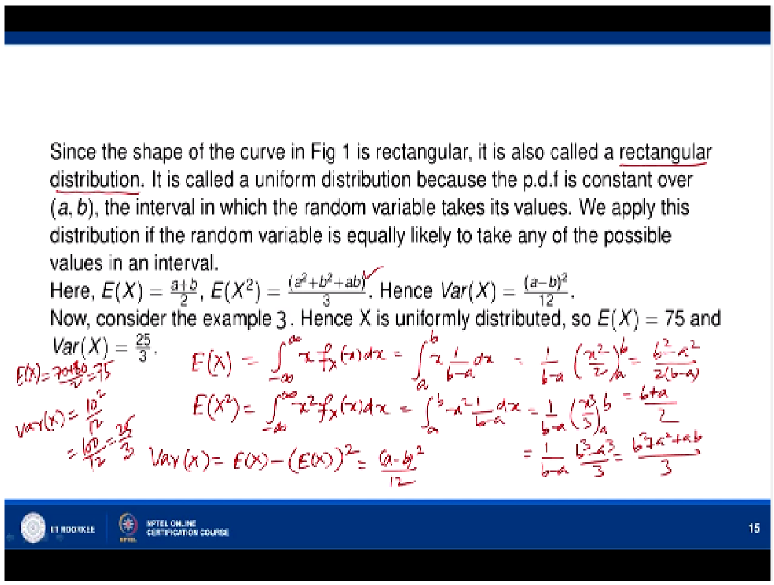

Now you can see the shape of the curve in this one is rectangular, okay. This curve is said, here it is rectangular shape. So that is why rectangular, this uniform distribution is also called a rectangular distribution, okay. It is called uniform distribution because probability density function is constant over interval ab the interval in which the random variable takes its values.

We apply this distribution if the random variable is equally likely to take any of the possible values in an interval. EX=a+b/2 as we have just now calculated EX in this case of this example. In the case of this example. Similarly, we can calculate the EX for the case of uniform distribution. So EX will be equal to integral over -infinity to infinity x*fxxdx. So this will be equal to integral over a to b, okay, x*fxx integral over a to b fxx is 1/b-adx, okay.

So this will be equal to 1/b-ax square/2 a to b and this will be b square-a square/2*b-a. So this is b+a/2. Similarly, if we calculate EX square. EX square will be integral over -infinity to infinity x square fxxdx. This is equal to integral over a to bx square 1/b-adx. And this will be equal to 1/b-ax cube/3. So it is a to b, so this is 1/b-ab cube-a cube/3. So this is b square+a square+ab/3, okay. We get this formula.

And then we can calculate variance of X from this formula, EX-EX whole square, okay. So we can apply this formula and we will get variance of X to be equal to a-b whole square/12 okay. Now we can consider the example a, okay, this example, this one, okay. So we can consider in this example, we have been given the values of a and b, a is 70 b is 80, okay. So we can put these values.

a=70 b=80, then EX is 75 and when you put here a=70 b=80, then 80-70, 10, 10, 100 square, 100 square/12, so variance of X=, EX=70+80/2, meaning that it is 75. And variance of X will be equal to 70-80, that is 100 square/12. So 100/12, which means that it is 25/3, okay. So that is the variance of X, okay. Now let us go to this example.

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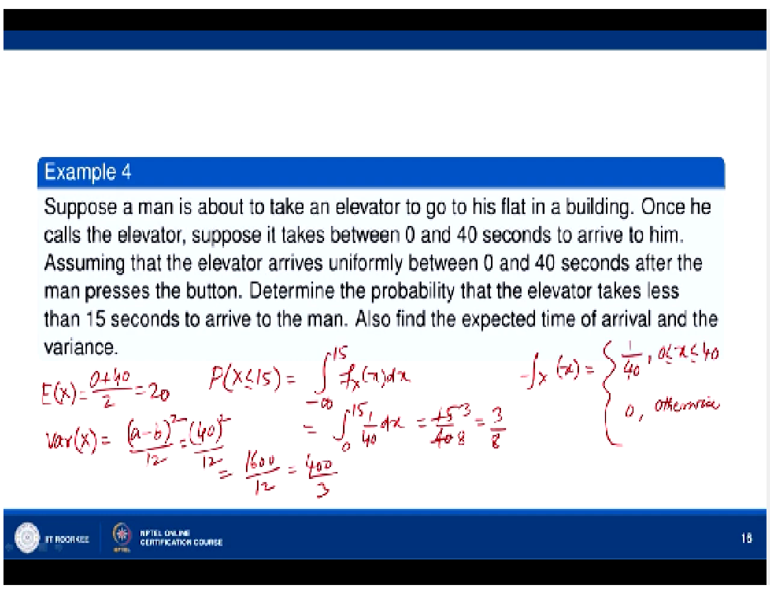

Suppose a man is about to take an elevator to go to his flat in a building. Once he calls the elevator, suppose it takes between 0 and 40 seconds to arrive to him, okay. Assuming that the elevator arrives uniformly between 0 and 40 seconds after the man presses the button. Determine the probability that the elevator takes less than 15 seconds to arrive to the man, okay. So this is uniform distribution here.

So we have to find the probability that X is less than or equal to 15, okay. The fxx is given by 1/b-a. So 1/b-a means 1/40-0. So 1/40 when 0<x<40, okay and 0 otherwise, okay. 0<x less than or equal to 40, it is 1/40 and 0 otherwise, so this will be equal to integral over -infinity to 15, okay, fxxdx. So this will be equal to integral over 0 to 15 fxx, that is 1/40dx, okay. So 15/40 which means that 3/8, okay.

Also find the expected time of arrival and variance. Expectation of X will be equal to as we have just now seen, expectation of X=a+b/2. a+b/2 means 0+40/2, okay. that means 20. And variance of X we have just now seen. Variance of X=a-b whole square/12, okay. So a is 0, b is 40. So 40 square/12. So this means 1600/12, okay. So this is 400/3, okay. That is the variance.

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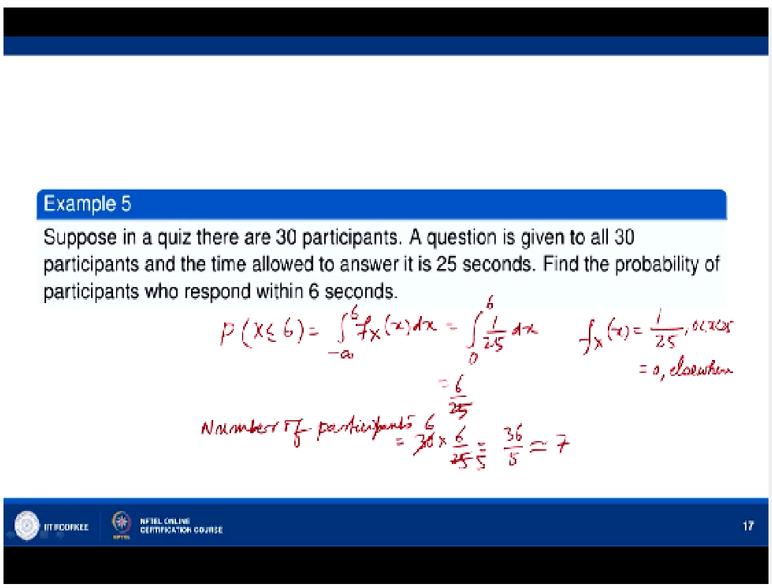

Now let us suppose in a quiz there are 30 participants. A question is given to all 30 participants and the time allowed is 25 seconds. Find the probability of participants who respond within 6 seconds, okay. So we have to find the probability that X is less than or equal to 6, okay. So this is integral over -infinity to 6, okay, fxxdx, okay. So this is integral over 0 to 6, okay, fxx is 1/25, okay, dx.

So this is 6/25, okay. So here the fxx is given by 1/b-a that is 25, okay, 0<x<25 and 0 elsewhere. The number of participants, okay, find the number of participants who respond within 6 seconds.

So number of participants=30*6/25, that means 36/5. So that is approximately 7. So number of participants who respond within 6 seconds is 7, okay.

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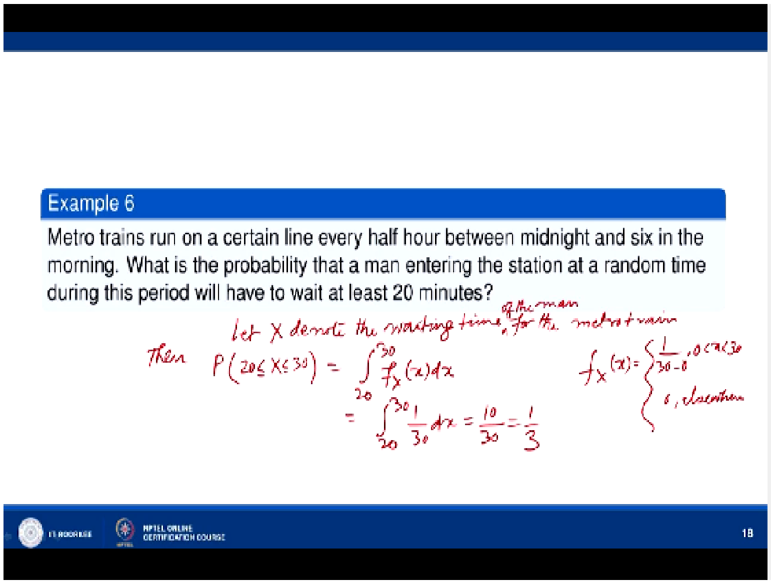

Now let us see metro trains run on a certain line every half hour between midnight and 6 in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least 20 minutes? So let us say let X denote the waiting time for the metro train, okay. Let X denote the waiting time of the man for the metro train, okay. Then probability that he has to wait at least 20 minutes.

So X varies between 20 less than or equal to x less than or equal to 30 because every half hour there is a train. So integral over 20 to 30, okay, fxxdx, okay. Now fxx=1/b-a. 1/30-0, because every half hour there is a train, so 30-0, 0<x<30 and 0 elsewhere. So this will be 20 to 30 1/30dx and it will be equal to 10/30, that means 1/3. So probability that the man will have to wait at least 20 minutes is 1/3. So this is where I would like to end my lecture. Thank you very much for your attention.