Advanced Engineering Mathematics
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Lecture – 47
Negative Binomial Distribution and Poisson Distribution

Hello friends. Welcome to my lecture on Negative Binomial Distribution and Poisson Distribution. First we discuss negative binomial distribution. Suppose we perform Bernoulli trials where p is the probability of success in every trial.

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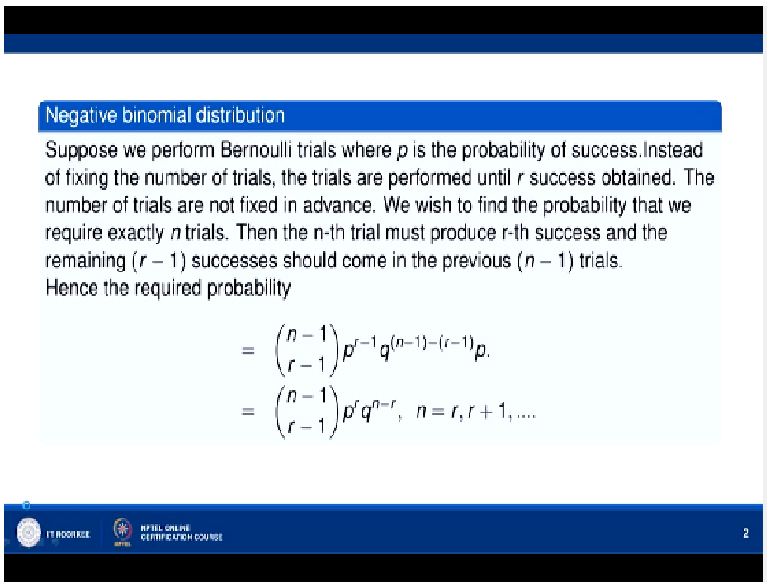

Instead of fixing the number of trials, let us assume that the trials are performed until r success are obtained, okay. The number of trials are not fixed in advance. We wish to find the probability that we require exactly n trials. Then the n-th trial must produce r-th success and the remaining r-1 successes should come in the previous n-1 trials. This means that we go on performing trials until we get the r-th success.

As soon as we get the r-th success, we stop the trials. So let us say n number of trials are performed to get the exactly r successes. Then r-th success will come in the n-th trial and r-1 successes should come in the previous n-1 trials. Hence the required probability is that n-1Cr-1p to the power r-1q to the power n-1-r-1 by binomial distribution because r-1 success are obtained from n-1 trials.

So the probability for that will be n-1Cr-1p to the power r-1q to the power n-1-r-1. And r-th success comes in the n-th trial. The probability for that is p, okay. So the probability that exactly r successes are obtained in n trials where r-th success comes in the n-th trial is given by this expression. And this can be written as n-1Cr-1p to the power rq to the power n-r. n will take values from r onwards, r, r+1 and so on.

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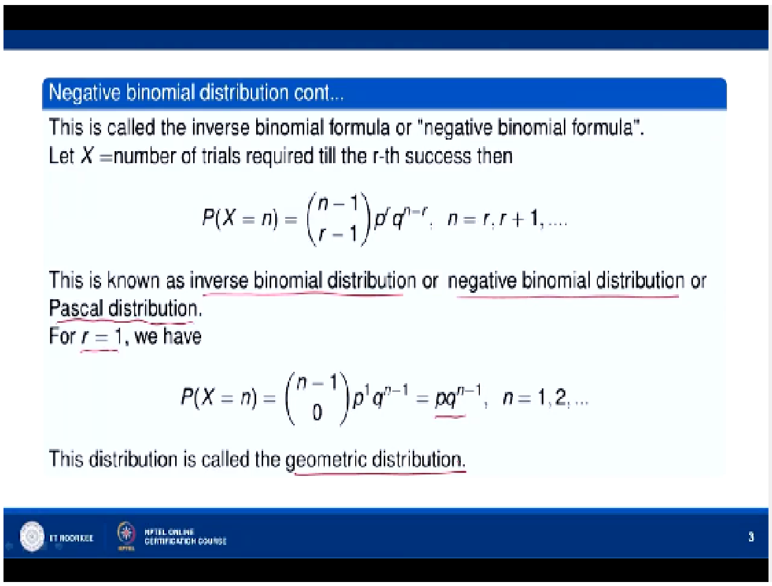

Now this is called the inverse binomial formula, okay. The formula that we have obtained, okay, this formula is called as the inverse binomial formula or we also call it as negative binomial formula. Let X be the number of trials required till the r-th success, okay. Let the number variable X we denote the number of trials required till the r-th success, then probability that X=n is n-1Cr-1p to the power rq to the power n-r as we have just now seen.

And minimum value of n will be of course r because to get r successes in n trials, n cannot be less than r, okay. So n will be greater than or equal to r. So n=r, r+1 and so on. This is known as inverse binomial distribution or we also call it as negative binomial distribution. It is also known as Pascal distribution, okay. In particular, if you take r=1, then you can see pX=n=n-1C0-p to the power 1q to the power n-1.

n-1C0=1, so we get pq to the power n-1, that means in n-1 trials, we do not get any success. We get failures every time. So the probability is q to the power n-1. In the n-th trial, we get the success, so the probability for that is p. And therefore, it is pq to the power n-1. This distribution is called as geometric distribution.

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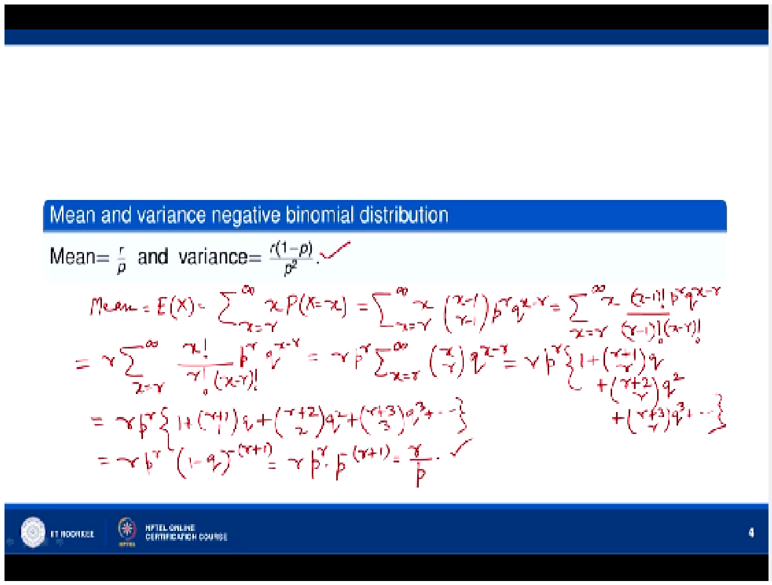

Now let us discuss the mean and variance of the negative binomial distribution, okay. So mean or you can say expected value of the random variable X, okay=, let us say, sigma x=r to infinity, because suppose x trials are performed, okay, to get the r successes where r-th success comes in the x-th trial, okay. Then we have sigma x=r to infinity, okay. x*probability that X=x, okay. So this is sigma x=r to infinity x*probability that X=x, okay.

Instead of n here, I am taking x. So a probability X=x is x-1Cr-1p to the power rq to the power x-r, okay. So we have x-1Cr-1p to the power rq to the power n-r and then, okay, x is there, right. So we can write it as, this is x-r, okay, yes. So we can write it as now, let us multiply and divide by r, okay. Then we can write it as, this x can be absorbed in x-1 factorial. We can write it as x factorial.

We have multiplied by r and divided by r. So r*r-1 factorial gives r factorial. Then we have x-r factorial and we get p to the power rq to the power x-r. Now this is equal to r*, p to the power r is independent of x, so we can write it outside. Sigma x=r to infinity xCrq to the power x-r, okay. Now this is what? r*p to the power r, if you expand this, when you put x=r, rCr is 1, okay. Then q to the power r-r, so q to the power 0 that is 1.

So 1, then we have x=r+1, so r+1Cr, that means r+1Cr*q to the power r+1-r, so q. Then we have next term r+2, the next is r+2. We get r+2Cr, okay, q to the power 2, then r+3, okay, Crq to the power 3 and so on, okay. Or we can also write it as r*p to the power r1+r+1C1q, then r+2C2q square r+3C3q cube and so on, okay. Now this is nothing but r*p to the power r1-q to the power -r+1, okay.

This is the binomial series, okay, for the expression 1-q to the power -r+1, okay. And this is valid because q is lying between 0 and 1. p>0, okay. We are taking p to be greater than 0 here. So q is between 0 and 1, okay. And therefore, this binomial expansion can be written as 1-q to the power -r+1. Now 1-q=p, okay. So we have r*p to the power r*p to the power -r+1 which is equal to r/p, okay. So mean of the negative binomial distribution is given by r/p. Now let us find the variance, okay. So to get the variance, we determine EX square, okay.

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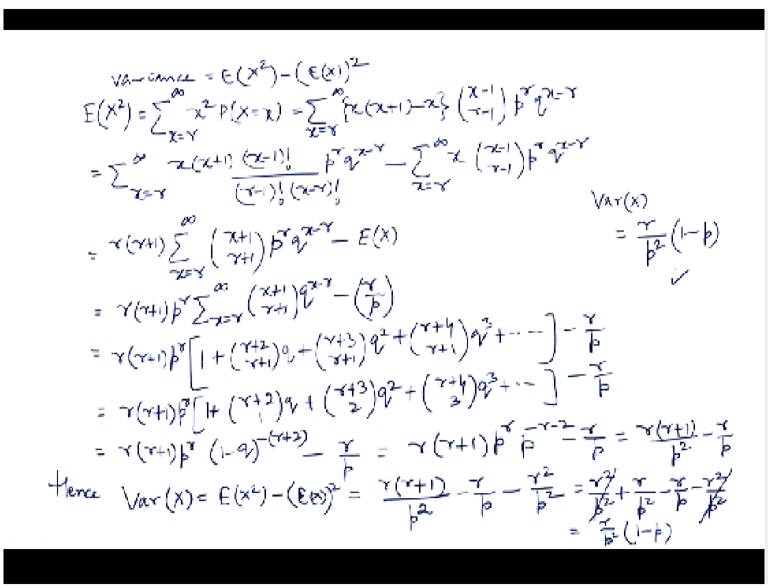

So variance=EX square-EX whole square, okay. So EX, we have already found. It is r/p. We just need to find EX square, okay. So EX square similarly can be written as sigma x=r to infinity x square pX=x, okay. So this is sigma x=r to infinity and x square, let us write as x*x-1+x, okay. And then we have, okay, we should write it as x*x+1-x, okay. Now pX=x is n-1C, this is instead of n, it is x, so x-1Cr-1p to the power rq to the power x-r, okay.

Now let us write it as sum of 2 terms. Sigma x=r to infinity x*x+1 x-1 factorial/r-1 factorial x-r factorial, then we have p to the power r1 to the power x-r. And the second term is for this x, so sigma x=r to infinity xx-1Cr-1p to the power rq to the power x-r. Now what will happen is, this x+1*x can be absorbed in x-1 factorial and we will have x+1 factorial, okay. And this r-1 factorial, we can make it as r+1 factorial by multiplying and divide by r+1 and r, okay.

So we multiply by r and r+1 and divide by r and r+1 and then I write it as sigma x=r to infinity, x*x+1 when you multiply to x-1 factorial, it becomes x+1 factorial, r*r+1 you multiply in the denominator, it becomes r+1 factorial. And this is x-r factorial, so I will write it as x+1Cr+1, okay. It can be written like this. And then p to the power rq to the power x-r, okay. This is nothing but expectation of X, okay.

So I can write it as expectation of X. We do not have to calculate this. This we have already determined. Its value is r/p, okay. Now here what do we get? p to the power r again is independent of x. So I write it r*r+1p to the power r, okay. And then we have sigma x=r to infinity x+1Cr+1q to the power x-r-r/p the value of EX, okay. Now this is r*r+1p to the power r, let us say open this, expand this sum.

So put x=r. What we get? r+1Cr+1, that is 1 and then q to the power r-r, that is equal to 1. So we have 1. Then qx=r+1. So you get r+2Cr+1 and we get q to the power r+1-r, so q. Then r+3Cr+1, okay, q square. Then r+4Cr+1q cube and so on, okay, -r/p. Now and Cr we know and Cn-r. So r+2Cr+1 will be r+2C1. This is r+3C1. This is r+4C1. No, this is sorry, this is rr+1p to the power r1+r+2C1q, then r+3C2q square, then r+4C3q cube and so on, okay, -r/p.

So this is r*r+1p to the power r, and this is 1-q to the power -r+2, okay, -r/p. This is binomial expansion of 1-q to the power -r+2 because q is lying between 0 and 1. So it is valid. Now this is r*r+1p to the power rp to the power -r-2-r/p, okay. p is 1-q. So p to the power r gets cancelled with p to the power -4 and what we get? r*r+1/p square-r/p. So this is the value of EX square, okay.

Hence variance of X=EX square-EX whole square=r*r+1/p square-r/p, that is the value of EX square, -, EX is r/p, so r/p whole square, so r square/p square, okay. Now we multiply r to r+1, we get r square+r. So r square/p square+r/p square-r/p-r square/p square. So this gets cancelled with this and what we get here? Let us take out r/p square. Then we get 1-p, okay, yes. So variance of X=, okay, r/p square*1-p, okay. So this is the variance, okay. r*1-p/p square.

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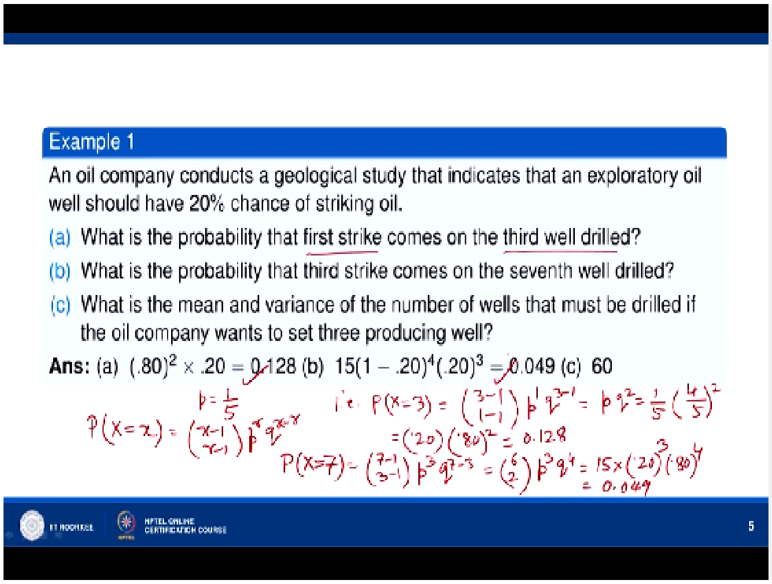

Now let us take some examples. An oil company conducts a geological study that indicates that an exploratory oil well should have 20% chance of striking oil. So p is here 1/5, 20% means 1/5. What is the probability that first strike comes on the third well drilled, okay? So we know that probability that X=x=x-1Cr-1p to the power rq to the power x-r, okay, where x is the number of wells drilled, okay.

So we want that the first strike should come in the third well drilled, that means we need to find the probability of PX=3, okay, in the third well, we should get the first success. So we get 3-1, r=1, so 1-1, p to the power 1q to the power 3-1, okay. So this is 2C0, that means 1, we get p*q square, okay. p=1/5, 1=4/5, so 4/5 square, okay. So what we get is, you can say this is 0.20, okay, 1/5 is 0.20 and 4/5 is 0.80, okay.

So 0.80 square*0.20 is 0.128, okay. This is the first part. What is the probability that third strike comes on the seventh well drilled? That means we get the third success at the seventh well, okay.

So we get probability that X=7 and r=3, okay. So we get 7-1 3-1, okay, p to the power 3q to the power 7-3, okay. So this is 6C2p cube q to the power 4, okay. 6C2 is 15. p cube means 0.20 raise to the power 3, okay. q means 0.80 raise to the power 4, okay.

So 15*0.20 ttp0 3 and 0.80 to the power 4. So this is 0.049, okay. So this is the answer, okay. Now what is the mean and variance of the number of the wells that must be drilled if the oil company wants to set 3 producing well, okay? So here we are given r=3, 3 producing well we want, okay. And for the 3 producing well, what is the mean and variance of the number of wells that must be drilled, okay?

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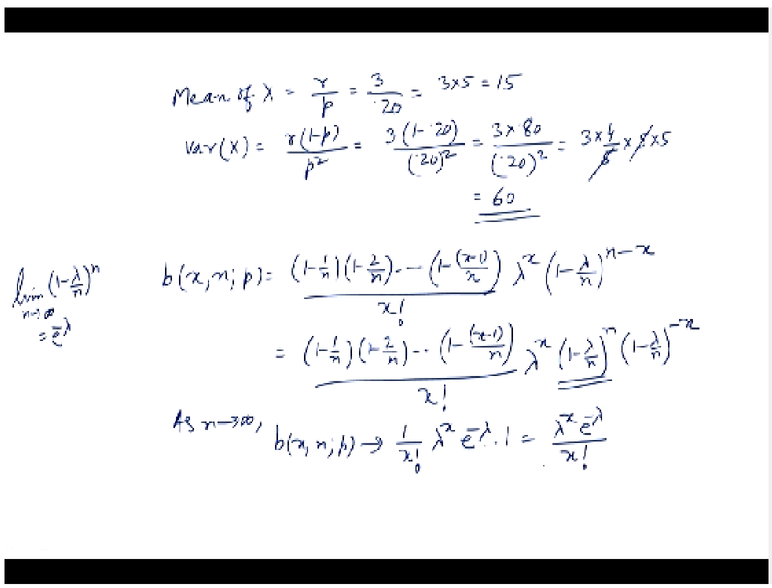

So we have just now found mean, mean of X, mean of the random variable X, okay. This is equal to r/p, okay. r=3 because we want 3 producing well, 3/0.20, that means 3*1/5, okay. This is 3/1/5, that is 3*5, okay. So this is 15, okay. So mean is 15 and variance of X=r*1-p/p square, okay. So we get 3 1-0.20/0.20 raise to the power 3. So this is 3*0.80/0.20 raise to the power 3, okay. We can write it as 3*, 0.80 is 4/5, okay, and 0.20 is 1/5, so 5 cube. So 5*5*5, okay.

So what we get here? 5*5 is 25, 25*4 is 100, okay, means 3, multiplied by 3, okay. So this is r*1-p/p square. What is, sorry it is not cube, it is square, okay. So we have to delete one 5, okay and then this is 60, okay. So r*1/p/p square is 3*1-0.20/0.20 raise to the power 2 and we get 3*0.80/0.20 to the power 2=3*4/5*5/5, that is 60, okay. So mean is 15, okay and variance is 60.

That means on an average, 15 wells need to be drilled, okay, in order to have 3 producing wells, okay. So this is the question number 1.

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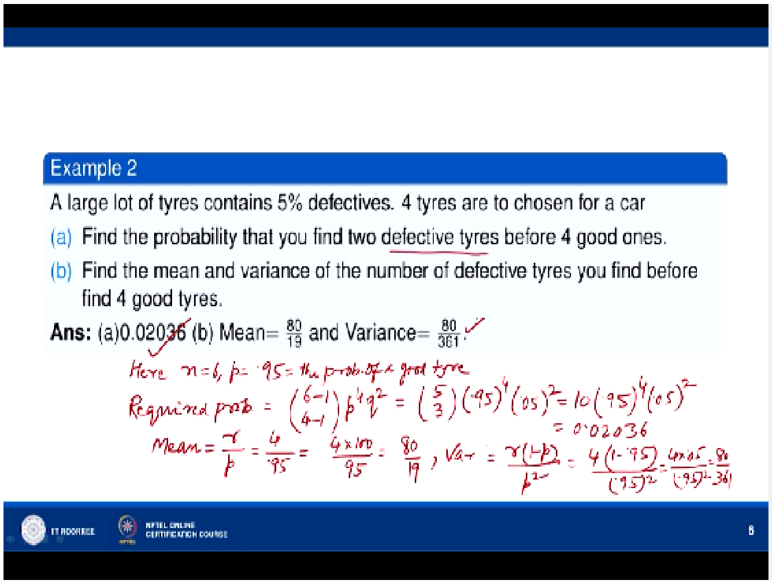

Now let us go to question number 2. A large lot of tyres contains 5% defectives, okay. So 5% defectives are there. 4 tyres are to be chosen for a car. Find the probability that we find 2 defective tyres before 4 good ones. That means the fourth goon tyre must come in the 6th trial, okay. So we have here n=6 and p=0.95, okay because it contains 5% defectives. So the probability of a defective tyre is 0.05 and probability of a good tyre is 0.95.

This is the probability of a good tyre. And we need to have 2 defective tyres before we get 4 good ones. That means when we pick the tyres, okay, we should have 2 defectives before having the fourth good one, okay. That means 2 defective and 4 good tyres means 6. So at the sixth trial, we shall have the good tyre. So this is probability, required probability is 6-1, okay, x-1Cr-1, r-1 means we have 4 good tyres, okay.

So 4-1, okay. And then p to the power 4q to the power 2, okay. So we have 5C3, okay. And p to the power 4 means 0.95 to the power 4, q to the power 2 means 0.05 to the power 2, okay. 5C3 is equal to 5C2 and %C2 is 5*4/2, so that is 10, okay. So 10*0.95 raise to the power 4*0.05 raise to the power 2. When you multiply this, it comes out to be 0.02036, okay. Now mean. Mean=r/p, okay.

So find the mean and variance of the number of defective tyres before you find 4 good tyres, okay. So this is 4/p, okay. p=0.95, okay. So 4/0.95 means 4*100/95, so that is 80/19, okay. And then we have variance. Variance=r*1-p/p square. So this is r=4, 1-p is 1-0.95, /0.95 square. So this comes out to be 4*0.5/0.95 square. So this is equal to 80/361, okay. Now let us go to Poisson distribution.

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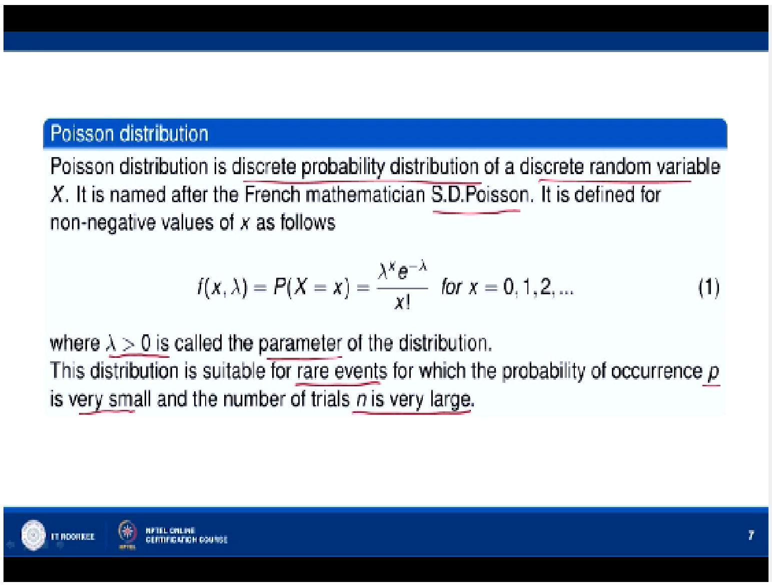

Poisson distribution is a discrete probability distribution of a discrete random variable, okay, say X. It is named after the French mathematician S. D. Poisson. It is defined for non-negative values of x as follows. The probability that X takes the value x is lambda to the power x*e to the power -lambda/x factorial and x takes the values 0,1,2,3 and so on. Lambda>0 is a parameter of the distribution, okay.

So for the distribution, we need to know the value of X and lambda, okay. Then we get the probability of X=x. This distribution is suitable for rare events for which the probability of success are the probability of occurrence, p is very small and the number of trials n is very large. So we apply it in those cases, it is very useful in those cases where number of trials n is very large and p is very small. In those cases, if you apply the binomial distribution to calculate the probability, there is a lot of difficulty.

It becomes very cumbersome calculation because n is large and p is small. So calculating the probability is very laborious. So we apply this Poisson distribution. It is an approximation to the binomial distribution and it works very nicely when n is very large and p is very small. So the example of rare events are.

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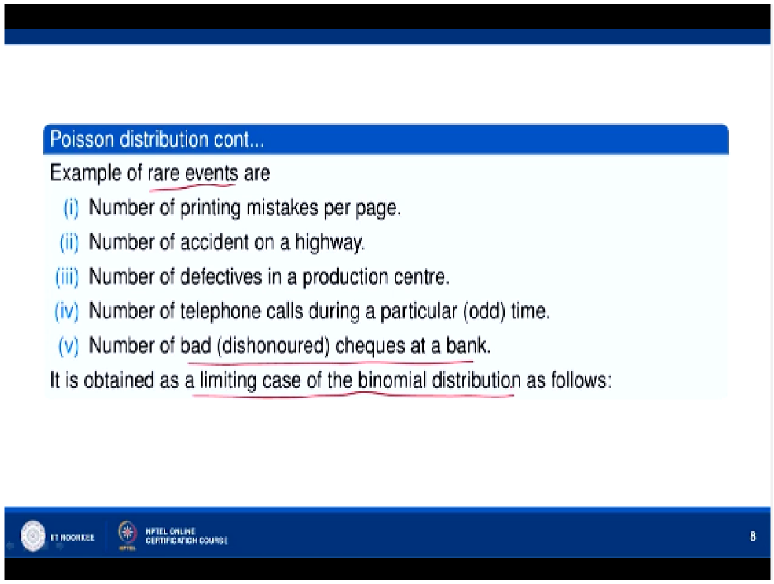

Number of printing mistakes per page made by a good typist, okay. Number of accidents on a highway. Number of defectives in a production center, okay. Number of telephone calls during a particular odd time, okay. During the night like for example or early morning, the number of telephone calls are very less in comparison during the day. So number of telephone calls during a particular odd time.

Number of bad dishonoured cheques, okay, at a bank. So dishonoured cheques is a very rare event, okay. And while the number of honoured cheques is very large. So n is very large and p is very small. It is obtained as a limiting case. The Poisson distribution we shall see now is obtained as a limiting case of the binomial distribution. Let us see how we get it, okay.

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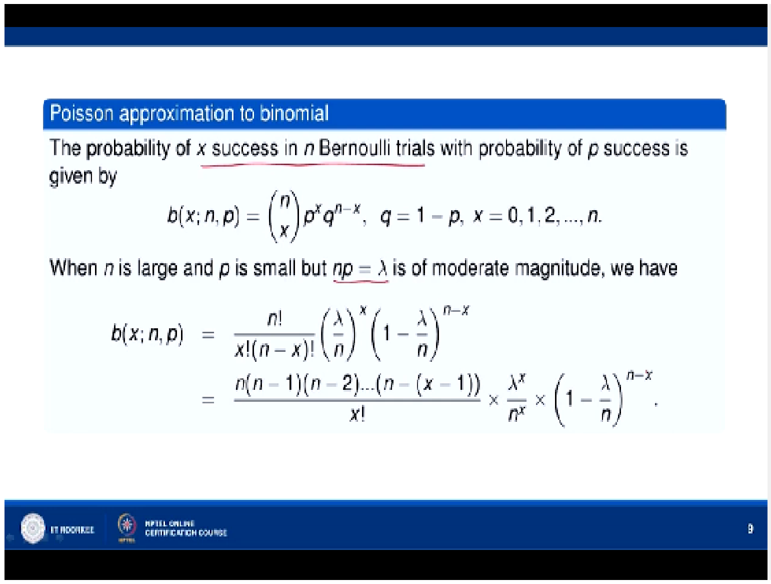

The probability of x successes in n Bernoulli trials with probability of success as p, okay, is given by bxnp. bxnp denote the probability of success, probability that X=x, probability of X successes in n trials. So nCxp to the power xq to the power n-x where q=1-p and x takes the values 0,1,2,3 and so on up to n, okay. Now when n is large and p is small such that np=lambda is of a moderate size, okay.

Then let us see what happens to this bxnp. bxnp can be written as n factorial over x factorial n-x factorial and p can be written as lambda/n, okay. np=lambda gives p=lambda/n. So lambda/n to the power x and q is 1-p. So 1-lambda/n to the power n-x. And this n factorial/n-x factorial, this gives you n*n-1 n-2 n-x-1, remaining factors get cancelled with n-x factorial. So n*n-1 n-2 n-x-1/x factorial, okay, *lambda to the power x/n to the power x 1-lambda/n to the power n-x.

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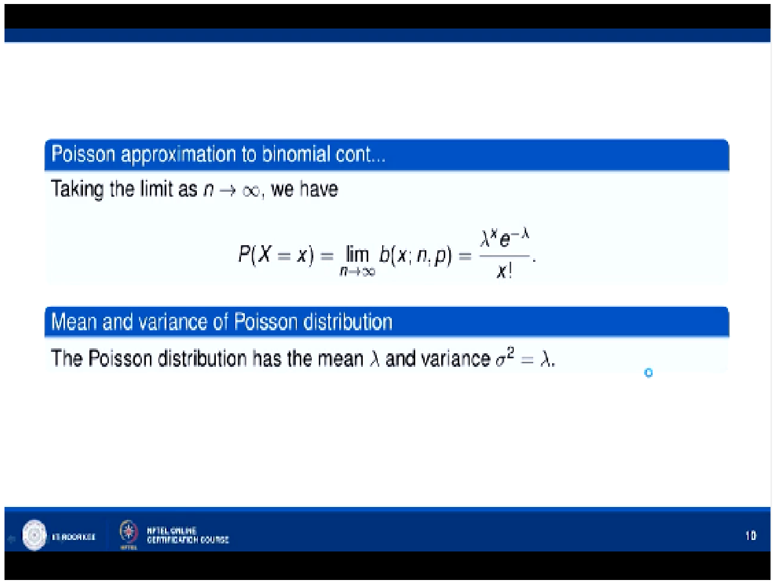

Now let us see when we take the limit as n goes to infinity, this bxnp tends to lambda to the power x*e to the power -lambda/x factorial. How we get that? So this n to the power x you can divide, these are n factors, okay, 1, 2, 3, 4 and so on up to x factor. So n to the power x we can divide here.

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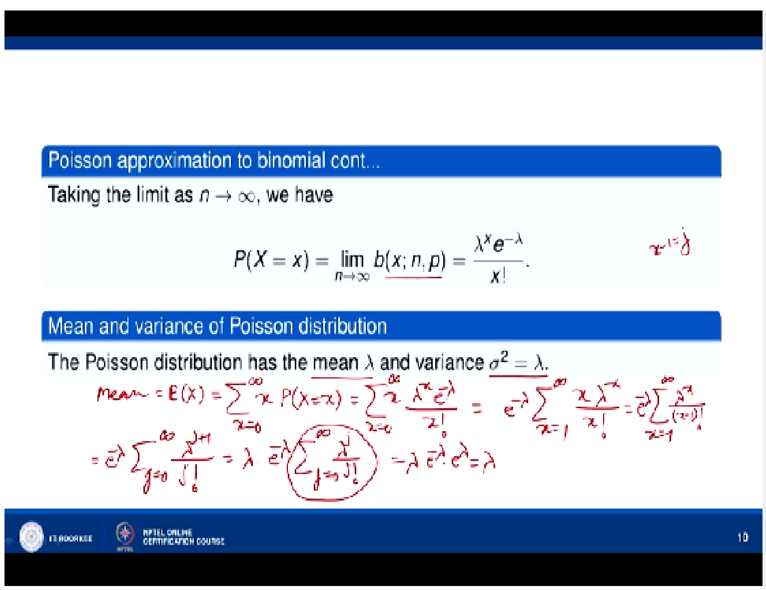

And this is equal to, we can do it here. bxnp, this is equal to, we can write as 1-1/n 1-2/n 1-x-1/n/x factorial, then lambda to the power x and then 1-lambda/n to the power n-x, okay. Now this is equal to, this can be written further as. Now as goes to infinity, okay, bxnp goes to, this will go to 1, this will go to 1, this will go to 1. So we will have 1/x factorial, then lambda to the power x. It is independent of n. Then 1-lambda/n to the power n will go to e to the power -lambda.

We know that limit of 1-lambda/n raise to the power n as n goes to infinity is e to the power -lambda. So this will go to e to the power -lambda and 1/lambda/n to the power -x will go to 1, okay. So we get lambda to the power xe to the power -lambda/x factorial, okay. So as n goes to infinity, bxnp goes to, this goes to lambda to the power x*e to the power -lambda/x factorial. So when n is very large, np is very small, the Poisson distribution is a good approximation of binomial distribution.

Now it has mean lambda. Let us show that it has mean lambda and its variance is also lambda. So in the case of Poisson distribution, it turns out that mean and variance are same, okay. So mean=EX=sigma x varies from 0 to infinity, okay, x here takes values from 0 to infinity. So x*probability that X=x, okay. The random variable x here which is Poisson distributed takes values from; in the case of binomial distribution, x takes the value 0, 1, 2, and so on up to n.

Now n is going to infinity. So x takes the values 0, 1, 2, 3 and so on up to infinity. So x*lambda to the power xe to the power -lambda/x factorial and x varies from 0 to infinity. e to the power -lambda we can write outside, okay, because it is independent of x. And then here first term becomes 0 because when x is 0, this first term gives us 0. So x varies from 1 to infinity, okay. x*lambda to the power x/x factorial.

And this x I can cancel with 1x here in the x factorial and I get e to the power -lambda sigma x=1 to infinity lambda to the power x/x-1 factorial. Now let us take x-1=j, okay. So then it is equal to e to the power -lambda sigma/j=0 to infinity, okay, lambda to the power, x is equal to now j+1, /j factorial, okay. When x is 1, j is 0. And when x is infinity, j is infinity. So lambda to power j+1, we can take 1 lambda from here.

So lambda*e to the power -lambda sigma j=0 to infinity lambda to the power j/j factorial. Now this quantity, this quantity is e to the power lambda, okay. So lambda e to the power -lambda*e to the power lambda and we get lambda, okay. So mean comes out to be lambda like this, okay. Now let us show that variance is also lambda, okay.

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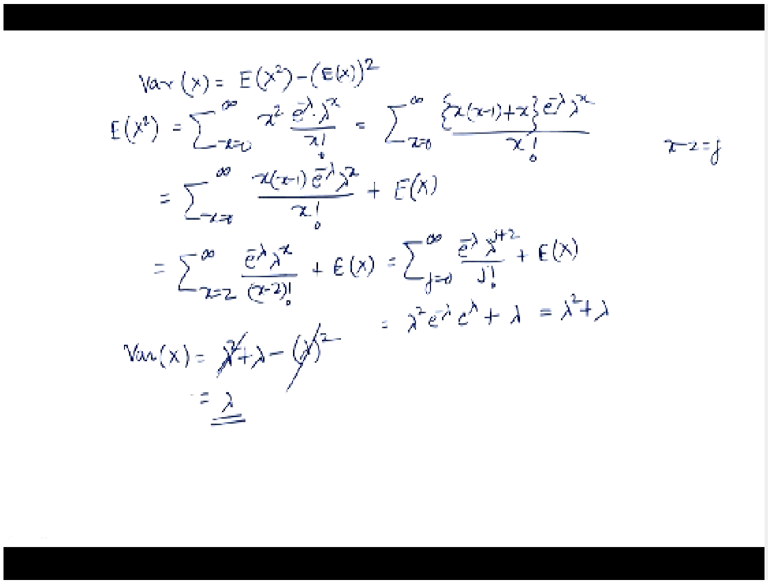

So to show that variance is lambda, let us write variance of X=EX square-EX whole square, okay. So we need to, EX we have already found. It is lambda. We need to find EX square. So EX square=sigma x=0 to infinity x square*e to the power -lambda lambda to the power x/x factorial. We can write it as, first term is 0, okay. We can write it as sigma x=0 to infinity x*x-1+x, okay. x square let us write as x*x-1+xe to the power -lambda lambda to the power x/x factorial.

Then we can write it as sum of 2 terms. So sigma x=0 to infinity x*x-1e to the power -lambda*lambda to the power x/x factorial. And the second term sigma x=0 to infinity x*e to the power -lambda lambda to the power x/x factorial is expectation of X which we have already got, okay. Now x*x-1 we can cancel with x*x-1 in x factorial and then we get here x runs from 2 to infinity because when x=0 or x=1, this term becomes 0, okay.

So e to the power -lambda lambda to the power x/x-2 factorial we get, okay. Now putting x-2=j. Let us put x-2=j. Then this is sigma j=0 to infinity e to the power -lambda lambda to the power j+2/j factorial and we get this as; j square lambda square, let us take common. Then lambda square e to the power -lambda*sigma j=0 to infinity lambda to the power j/j factorial is e to the power lambda and here Ex is, we have already found lambda, okay.

So what do we get is? Lambda square+lambda we get, okay. Now variance of X we can find. So EX square is lambda square+lambda. EX whole square is lambda whole square and we get variance as lambda. This cancels with this, okay. So in the case of Poisson distribution, the mean and variance are same.

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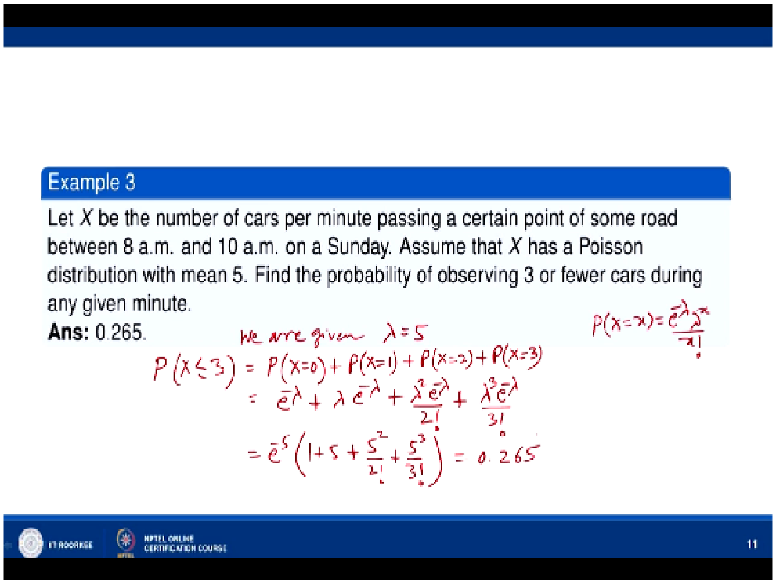

Now let us go to this problem. Let X be the number of cars per minute passing a certain point of some road between 8 a.m. and 10 a.m. on a Sunday. Assume that X has a Poisson distribution with mean 5, okay. So mean is lambda in the case of Poisson distribution. So we are given lambda=5. Find the probability of observing 3 or fewer cards during any given time, okay, during the given minute.

So X denotes the number of cars observed, okay any given minute. So this is X is 3 or fewer, less than or equal to 3, okay. So we need the probability that X is less than or equal to 3 which is probability that X takes the value 0+probability that X takes the value 1+probability that X takes the value 2+probability that X takes the value 3, okay. And probability that X takes the value x=e to the power -lambda lambda to the power x/x factorial we know, okay.

So when x is 0, it is e to the power -lambda. When x=1, it is lambda e to the power -lambda. Then x=2 means lambda square e to the power -lambda/2 factorial. Then x=3 means lambda cube e to the power -lambda/3 factorial. Lambda=5. So we get e to the power -5 1+5+5 square/2 factorial+5 cube/3 factorial. When we calculate this, it comes out to be 0.265, okay.

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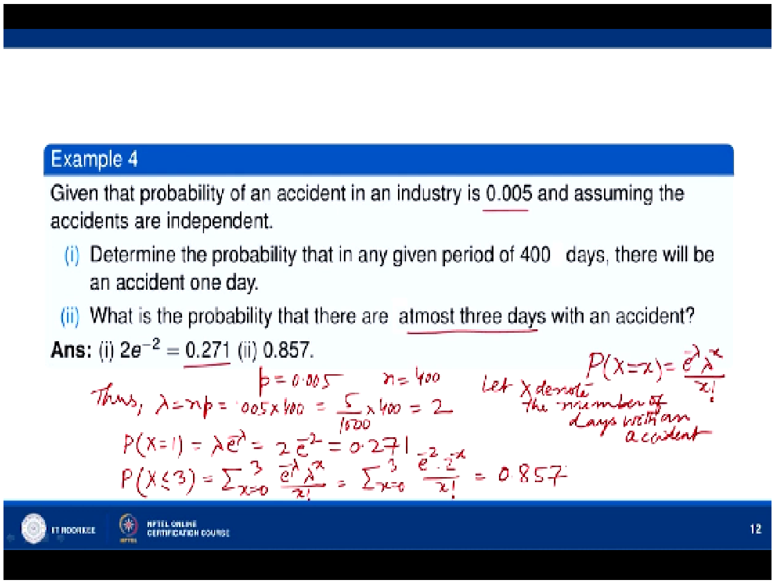

Now let us go to next question. Given that probability of an accident in an industry is 0.005, okay. So p here is 0.005. Assuming that the accidents are independent, determine the probability that in any given period of 400 days, there will be an accident one day, okay. So number of days, okay, n=400, okay. So n=400 gives lambda=, thus lambda we know lambda=np, okay. So 0.005*400 which means that this is 5/1000*400.

So it is 2, okay. So now what is the probability that there are atmost 3 days, okay, before that determine the probability that in any given period of 4000 days, there will be an accident one day, okay. So probability that X=1, okay. There will be an accident one day. Now lambda=2, so probability that X=1, probability that X=x is e to the power -lambda lambda to the power x/x factorial. x here denotes the number of accidents, okay.

So the probability that X=1 means lambda*e to the power -lambda. Lambda=2, so we get 2*e to the power -2. And this is 0.271. Now what is the probability that there are atmost 3 days with an accident, okay? So probability that X is less than or equal to 3, okay. Number of days with an accident, X denotes number of days. Here let X denote the number of days with an accident, okay.

So here there must be atmost 3 days with an accident. So this is PX less than or equal to 3. So PX=, this is sigma s=0 to 3 e to the power -lambda lambda to the power x/x factorial, okay. So lambda=2, okay. So sigma x=0 to 3 e to the power -2 2 to the power x/x factorial. And when you put the values of x 0, 1, 2, 3, okay and find this sum, it is 0.857 we get, okay.

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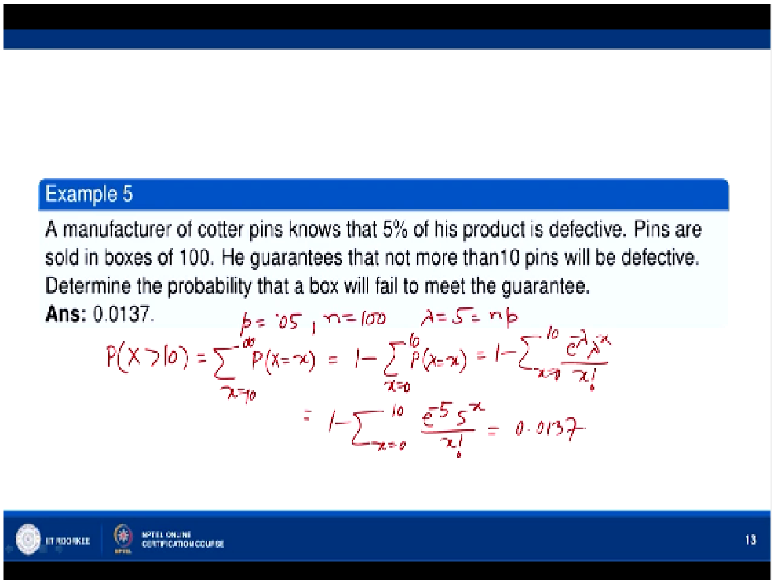

Now we go to this question. A manufacturer of cotter pins knows that 5% of his product is defective. So p is 5/100 that is 0.05, okay. Pins are sold in boxes of 100. So n=100 and which means lambda=5, okay, n*p. he guarantees that not more than 10 pins will be defective. Determine the probability that a box will fail to meet the guarantee. The box will fail to meet the guarantee if the number of pins are more than 10, okay.

So let X denote the number of pins in a box, then probability that X>10, okay, required probability is that X must be greater than 10 and this is nothing but sigma x=10 to infinity PX=x, okay. So this can also be written as 1-sigma x=0 to 10 PX=x because the total probability is 1, okay. So 1-sigma x=0 to 10, okay, PX=x is e to the power -lambda lambda to the power x/x factorial. And this is equal to 1-sigma x=0 to 10 e to the power -lambda, so e to the power -5, okay, 5 to the power x/x factorial. When we calculate this, it comes out to be 0.0137, okay.

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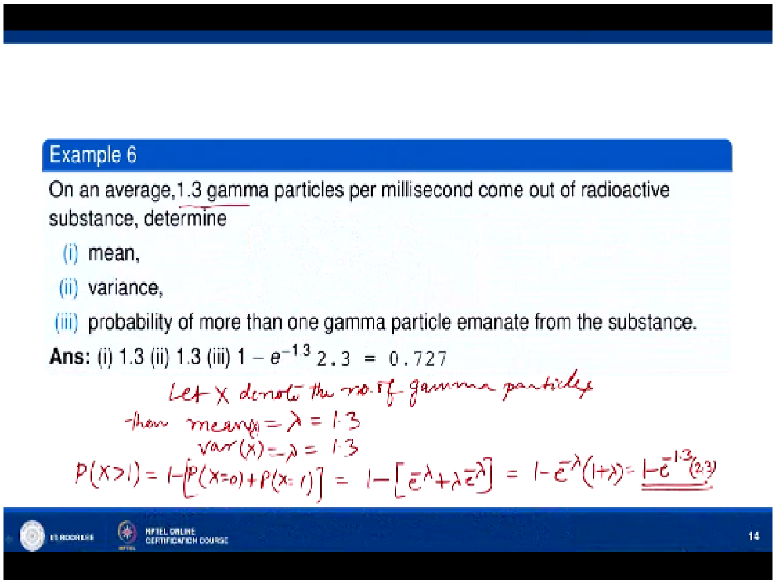

Now this is our last example. On an average, 1.3 gamma particles per millisecond come out of radioactive substance, determine mean, variance, probability of more than 1 gamma particle emanate, okay. So let X denote the number of gamma particles emanating from the substance, okay. Particles, then mean, mean of the Poisson distribution is equal to lambda. So man=lambda and we are given that on average, 1.3 gamma particles emanate per millisecond.

So lambda=1.3 and variance is also equal to lambda, okay. So variance is also 1.3, okay in the case of Poisson distribution. Now probability of more than 1 gamma particle emanating from the substance we have to find. So probability that X>1 which is what we need to find. This is 1-probability that X=0+probability that x=, okay, x is more than 1. So X=0 and X=1, okay. We need to subtract. Probability of more than 1 gamma particle, okay. So this is P, 1-, PX=0, so e to the power -lambda and then lambda e to the power -lambda, okay.

So 1-e to the power -lambda*1+lambda we will get. This is 1-e to the power -1.3*2.3, okay. So the answer will be here, for the probability of more than 1 gamma particle emanating from the radioactive substance, probability should be, of X>1. And for that we need to subtract the probability of X=0 and probability of X=1 from 1 which means that it has 1- -1.3*2.3. So this value is the answer of the third part of this question, okay. So with that I would like to end my lecture. Thank you very much for your attention.