Advanced Engineering Mathematics
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Lecture – 46
Binomial Distribution

Hello friends welcome to my lecture on binomial distribution. Binomial distribution is due to James Bernoulli it is a probability discrete probability distribution the Bernoulli process has the following properties. An experiment is repeated n number of times called n trials where n is a fixed integer the outcome of each trial is classified in to two mutually exclusive categories arbitrarily called a success and a failure

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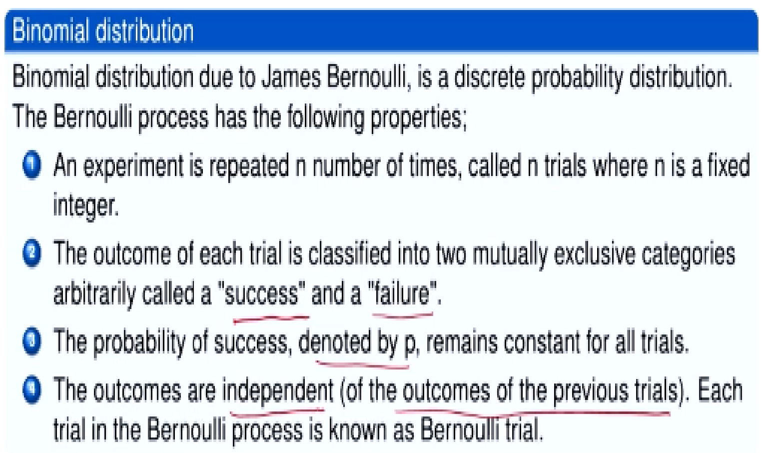

The probability of success is denoted by p and it remains constant for all the trials the outcomes are independent of the outcomes of the previous trials. Each trial in the Bernoulli process is known as a Bernoulli trial.

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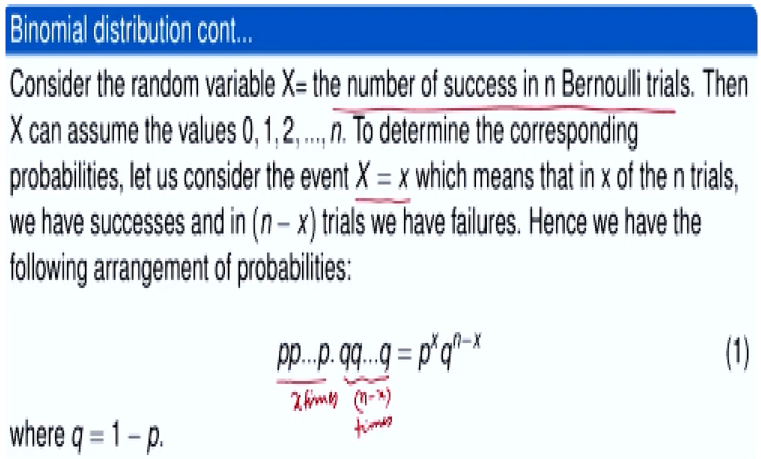

Consider the random variable X = the number of success in n Bernoulli trials then X can assume the values 0, 1, 2 and so on up to n. When you consider n Bernoulli trials and x denotes the number of successes in n Bernoulli trials then x can take value 0, 1, 2 and so on up to n because there could be no successes. In that case x will take the value 0 there could be all successes in all the n trials so then n can take value n then x can take value n.

So, x will take values 0, 1, 2 and so on up to n to determine the corresponding probabilities let us consider the event X=x which means that in x of the n trials we have successes and in n-x trials we have failures. Hence, we have the following arrangement of probabilities p p p x times q q q n-x times. Since the trials are independent so p*p*p that will be p to the power x q*q*q will be q to the power n-x the trials are independent. So, we have multiplied the probabilities, so q here is the probability of failure that is 1-p.

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Binomial distribution cont...

But (1) is merely one order of arranging x successes and $(n-x)$ failures and the probability $P(X=x)$ thus equals $\binom{n}{x}p^x q^{n-x}$, where $x = 0, 1, 2, \ldots, n$ is the number of ways in which we can obtain x successes from n Bernoulli trials. Thus the probability

$$P(X=x) = f(x) = \begin{cases} \binom{n}{x}p^x q^{n-x} & x = 0, 1, 2, \ldots, n \\ 0, & \textit{otherwise}, \end{cases} \qquad (2)$$

The distribution determined by the probability function (2) is called the Binomial distribution or Bernoulli distribution.

Now this is merely 1 arrangement of having x successes in n trials. Total number of base in which we can have x successes in n trials will be given by nc total ncx so probability that X = x ncx*p to the power x*q to the power n-x. Where x takes value 0 1 2 3 and so on up to n and cx here is the number of ways in which we can obtain x successes from n Bernoulli trials. So, probability that X=x = fx= ncx p to the power x q to the power n-x.

And x takes value 0 1 2 so on up to n, fx takes the value 0 otherwise the distribution determine by the probability function 2 is called the binomial distribution or we also call it Bernoulli distribution.

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Binomial distribution cont...

Note that the $(n+1)$ terms of the Binomial expansion

$$(p+q)^n = \binom{n}{0}q^n + \binom{n}{1}pq^{n-1} + \binom{n}{2}p^2q^{n-2} + \ldots + \binom{n}{n}p^n = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}.$$

Since $p+q=1$, we get $\sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x} = 1$.

The Binomial distribution is characterized by the parameter p and the number of trials n. $(q+p)^n$

Mean and variance

Mean $\mu = np$ and variance $\sigma^2 = npq$.

And now it became the note that n+1 terms of binomial expansion q+p to the power n will gives us nc0 q to the power n nc1 pq to the power n-1 and nc2p square q to the power n -2 and so on and ncn p to the power n which can be written as sigma x = 0 to n ncx p to the power x and q to the power n-x. So, ncx p to the power x q to the power n-x is x+1 th term in the expansion of q+p to the power n.

And since p+q=1 so sum of the probabilities= 1 and so fx is the probability function some of the probability fx is >=0 for all values of x and sum of the probability is of fx =1. Now the binomial distribution is characterized by the parameter p okay and the number of trials n and we know the parameter p and the number of trials n we can determine the binomial distribution. Now let us show that the mean of the binomial distribution is mu = np and the variance sigma square = npq.

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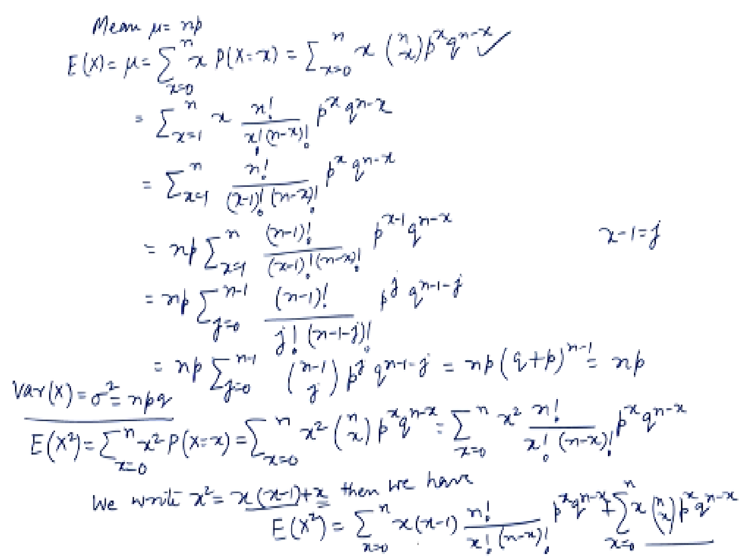

So, let us first show that mean of the binomial distribution mu = np EX = mu this is sigma x=0 to n okay x*px = x. So, this is sigma x=0 to n x times ncx p to the power x q to the power n-x we may write it as sigma x= because first term is 0, I can write it as x=1 to n when x =0 this first term becomes 0 so x times then ncx can be written as n factorial /x factorial n-x factorial p to the power x q to the power n-x now this x.

When we cancel with x factorial becomes x-1 factorial sigma x=1 to n and factorial we can cancel this x here and we get x-1 factorial n-x factorial and p to the power x q to the power n-x now what we do is we take n from here and from n factorial and 1 p from p to the power x so we can write it as n*p and what is left is sigma x=1 to n n-1 factorial /x-1 factorial n-x factorial.

So, this will be p to the power x-1 q to the power n-x now let us replace x-1 by j so we write this as np sigma j=0 to n-1, n-1 factorial/j factorial. And here we have x = j+1 so n-1-j factorial and here we will have p to the power j q to the power n-1-j which is np this is sigma j=0 to n-1 n-1cj okay p to the power j q to the power n-1-j which is = np*q+p raised to the power n-1 q+p = 1. So, it is 1 power n-1 so np okay so now so this is how we show the x =np.

Now let us show variance of x sigma square = npq so for this we will calculate Ex square okay Ex square is in similar manner sigma x=1 to n x square px= x. And what we get is sigma x=1 to n x square and cx p to the power x q to the power n-x. So, this is x =0 to n now this is sigma x =0 to n x square n factorial/x factorial and then we have n-x factorial p to the power x q to the power n-x what we will do is we shall write x square as x* x – 1+x.

Okay if we do that then we shall have then we have EX square = let us break it in to 2 parts sigma x=0 to n 1 part for x*x -1 and 1 part for x. So, x *x-1 n factorial x factorial n-x factorial p to the power xq to the power n-x. Other part is sigma x= 0 to n x ncx p to the power x q to the power n-x here also it is q power n-x now this part sigma x=0 to n x ncx p to the power x q to the power n-1 is same as this.

So, this 2nd term on the right side second sum is just e to the power EX and EX we have already calculated and p. So, this value of this we know so just I have to calculate this 1st sum on the right side okay so let us calculate this one.

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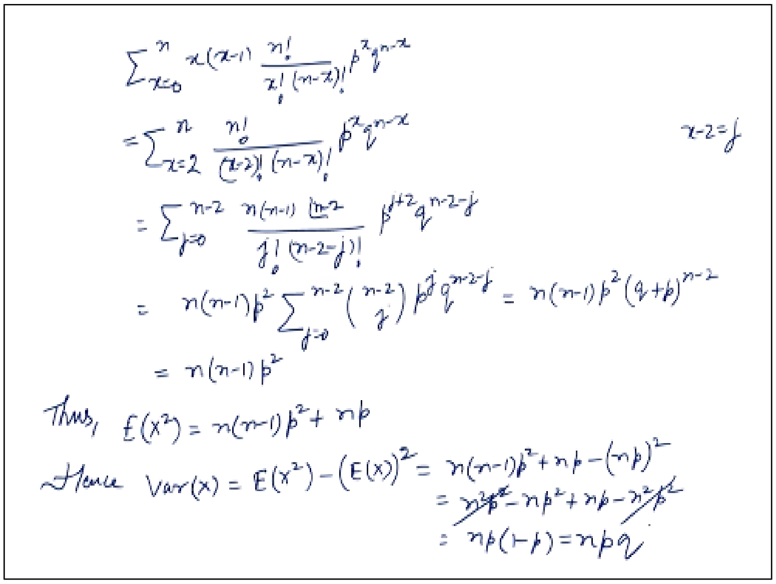

So, we have sigma x = 0 to n and x * x -1 n factorial x factorial n-x factorial p to the power x q to the power n-x, now at x =0 this sum is 0at x =1 also this sum is 0. So, I can write it as sigma x =1 to n okay x=2 to n and we have x*x-1 we can cancel with that x factorial. And x factorial will become x-2 factorial so we get n factorial /x-2 factorial and then n-x factorial p to the power x q to the power n-x.

So, now x-2 let us put at j so we get summation j=0 to n-2 n*n-1 we write outside the n factorial. Then we have n-2 factorial okay and here we have j factorial. This becomes n-2-x factorial replacing n-2-j factorial and here we get x = j+2q to the power n-2-j so I can write it as n*n-1 p square and what is left is that sigma j =0 to n-2 and n-2 cj p to the power j q to the power n-2-j and this is = n*n -1p square q+p raised to the power n-2q+p=1 so this n*n-1 p square.

So, this is n*n-1 p square and this is np, so we have calculated ex square. So, thus EX square = n*n-1p square +np okay and hence variance of x=EX square –EX whole square = n*n-1 p square + np-np whole square so we get n square p square –np square +np-n square p square. So, this n square p square gets cancelled and we get np times 1-p which is npq. So, this is how we calculate the variance of the binomial distribution okay so mean and variance are found like this.

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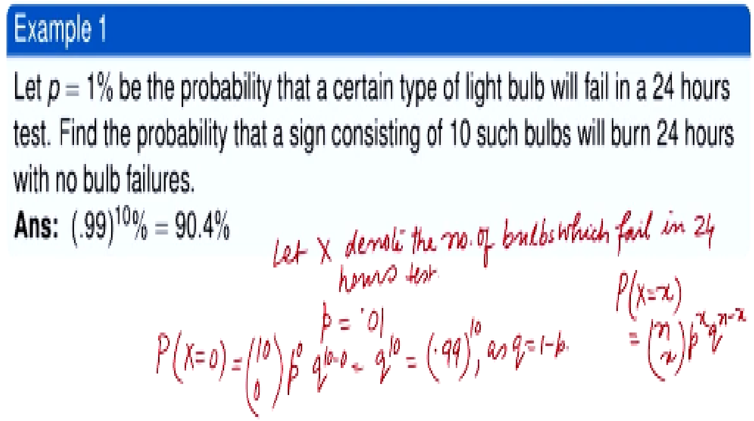

Now let us go to some examples on binomial distribution let p=1% be the probability that a certain type of light bulb will fail in 24 hours. So, let us say let X denote the number of bulbs which fail in 24 hours test then find the probability that is sign and p = 1% that is .01. Now find the probability that sign consisting of 10 such bulbs will burn 24 hours with no bulb failures that means we want the probability that x= 0 there should not be any failure of bulbs.

Now p is the probability of that the certain type of bulbs will fail before 24 hours, so this is = so this gives you by binomial distribution probability that x=x we know probability that x=x is ncx p to the power x q to the power n-x so probability that x =0 no 1failures so ncx n=10 so 10 c 0 p to the power 0 q to the power 10-0 so we get q to the power 10, x denote the number of bulbs which fail in 24 hours.

We do not want any bulb to fail out of the 10 such bulbs so we get q to the power 10 which is .99 to the power 10 because q= as q=1-p so this .99 to the power 10 gives you 90.4% probability.

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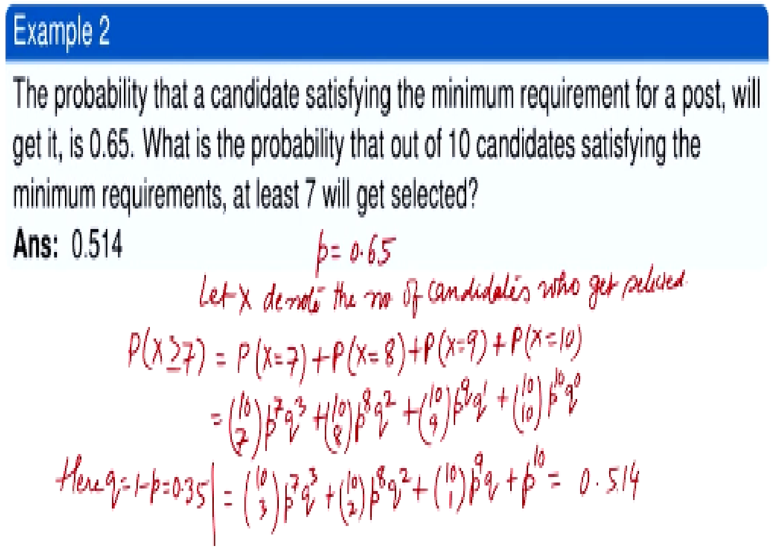

Let us go to 2nd question the probability that a candidate satisfying the minimum requirement for a post will get it is 0.65. So, the probability of the candidate to get the post is 0.65 what is the probability that out of 10 candidates which satisfy the minimum requirements at least 7 will get selected. So, let x denote number of candidates who get selected then we want the probability that x is >=7.

So, this is nothing but straight probability that x = 7+ probability that x =8+probability that x=9+ probability that x = 10. So, probability that x >=7 at least 7 candidates will get selected means we want the probability of having 7 candidates selected and its 8 candidates selected, 9 candidates selected, 10 candidates selected. So, this means that we have this binomial distribution 10 c 7 p to the power 7 q to the power 3.

And here we get 10 c 8 p to the power 8 q to the power 2. And here we have 10 C 9 p to the power 9 q to the power 1 and then here we have 10 c 10 p to the power 10 q to the power 0. So, this is p is given to be 0.65 so here q = 1-p so this comes out to be 0.35. Now we know the value of p is 0.65 we know the value of q is 0.35. The binomial coefficients 10 c7 10 c8 10 c9 10 c 10 will be calculated this is nothing.

But we can also write it as 10 c3 ncr is nc n-r so p 7q to the power 3 and 10 c8 is 10 c2. So,10 c2 p8 q square 10 c9 is 10 c1 p to the power 9q and then 10 c10 is 1 and so we get p to the power 10 there so we can put the values of p and q binomial coefficient and then calculate this value comes out to be 0.514.

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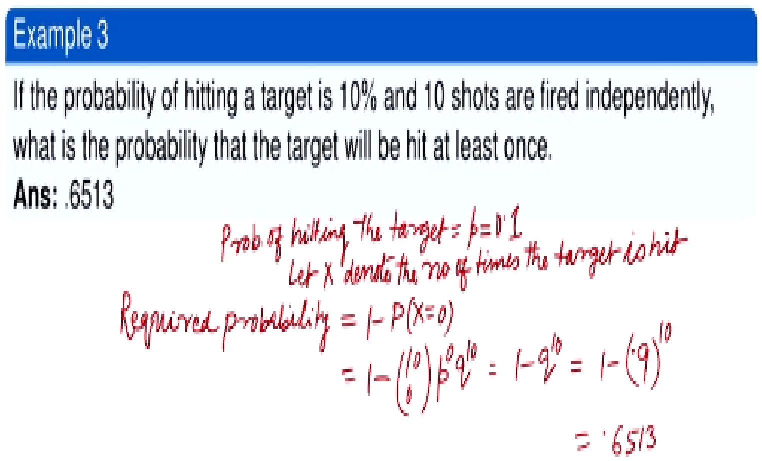

Now if the probability of hitting the target is 10% so probability of hitting let us denote by p it is 10/100 so that means 0.1 p=0.1. Now 10 shots are fired out independently what is the probability that the target will be hit at least once. So, target will be hit at least once we say let x denote the number of times the target is hit let x denote the number of times the target hit the target set is then the required probability is = 1-px = 0, x=0 means the target is not hit in any time.

Out of any shots so that we subtract from 1 so that 1-px=0 will give the probability that the target is hit at least once so 1-px =0 means n=10 so 10 c0 p to the power 0 q to the power 10 so this will be 1- q to the power 10 that means 1-q is .99, no q is.9 so 1-.9 to the power 10 so 1-.9 to the power 10 will come out to be .6513 now let us see problem number 4.

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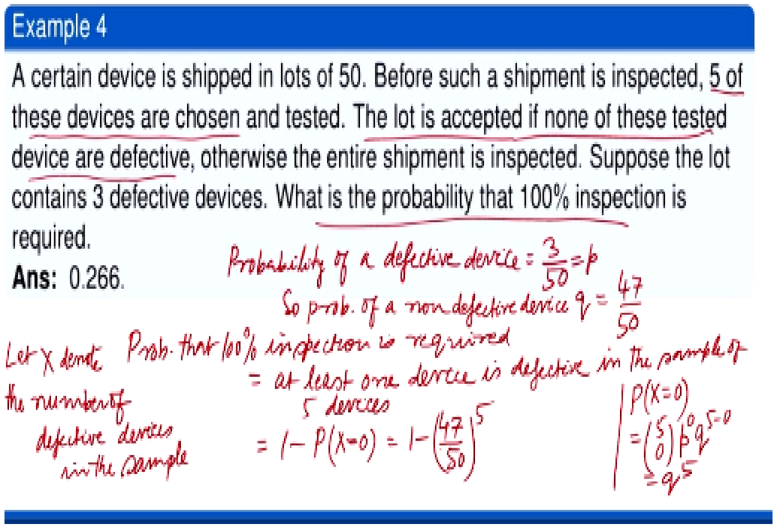

A certain devise is shipped in lots of 50 before such a shipment is inspected 5 out these devices are chosen and tested the lot is accepted if none of these tested device are defective otherwise the entire shipment is inspected. Suppose that the lot contains 3 defective devices what is the probability that 100% inspection is required so probability of defective device let us find first this =3/50 because the lot of 50 contains 3 defective devices.

So, probability of a defective device is 3/50 probability of a non-defective device is therefore so probability of let us say this is p okay so probability of a non-defective device q=47/50. Now we are saying that lot is accepted if none of these tested device are defective, so we have picked up a sample of 5 devices if none of this 5 devices are defective then the lot would be inspected. We want the probability that 100% inspection is required.

100% inspection will be required if out of this 5 devices if at least 1 device is defective okay. So, the required probability that 100% inspection is required it means that at least one defective device is there at least 1 device is defective in lot of in the sample of 5 devices. So, this is nothing but 1-probability that all devices are not defective. So, probability that x= 0 if x gives the number of defective devices okay let x denote the number of defective devices in the sample.

So, if we take x =0 it will mean that there is no defective device in the sample so probability that there is no defective device in the sample if we subtract that from 1 then it will give us the probability that at least one defective device is there so 1-x =0 all devices are non-defective. So, 47/50 raise to the power 5.

Because probability by binomial distribution probability that x= 0 5 c0 p to the power 0 q to the power 5-0 so q to the power 5 so 1- 47/50 to the power 5 this comes out to be 0.266. So, if at least one device is defective in this lot of 5 in the sample of 5 then 100% inspection will be required. So that is the required probability 0.266. That is all in this lecture, thank you very much for your attention.