Advanced Engineering Mathematics
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Lecture – 45
Discrete Probability Distribution

Hello friends welcome to my lecture on discrete probability distribution let us define random variable if we draw 5 bolts from a lot of bolts and measure the diameter we cannot predict how many bolts will be defective that is will not meet the given requirements hence x = number of defectives. Let us define x to be number of defective then x describes the function which depends on chance.

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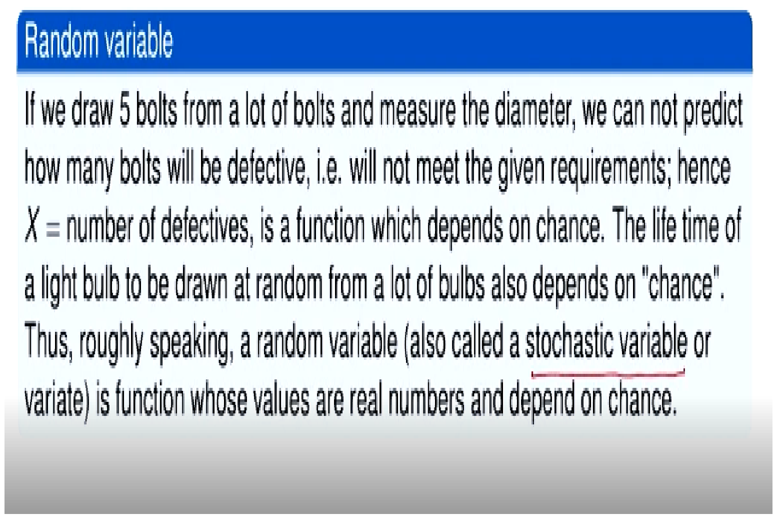

The lifetime of a light bulb to be drawn at random from a lot of bulbs also depends on chance. Thus roughly speaking a random variable also called a stochastic variable or variate is a function whose values are real numbers and depend on chance.

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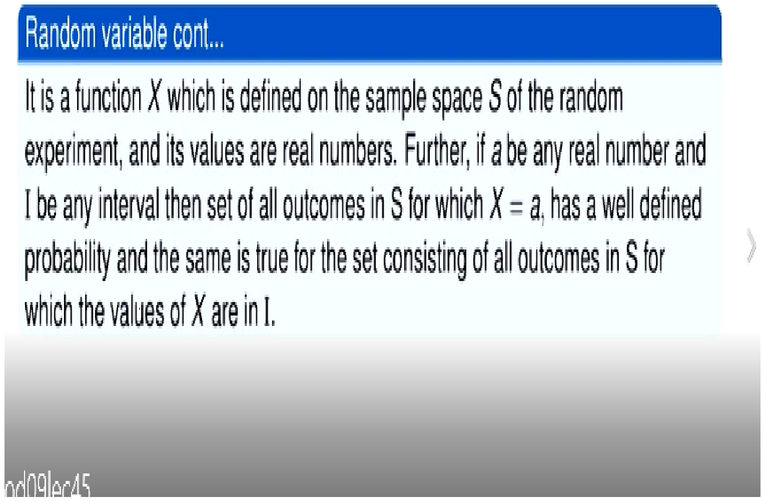

As ideally speaking random variable is a function x which is defined on the sample space S of the random experiment and its values are real numbers. Further if a be any real number and I be any interval then these set of all outcomes in S for which X assumes the value a, has a well defined probability and the same is true for the set consisting of all outcomes in S for which the values of X are in I.

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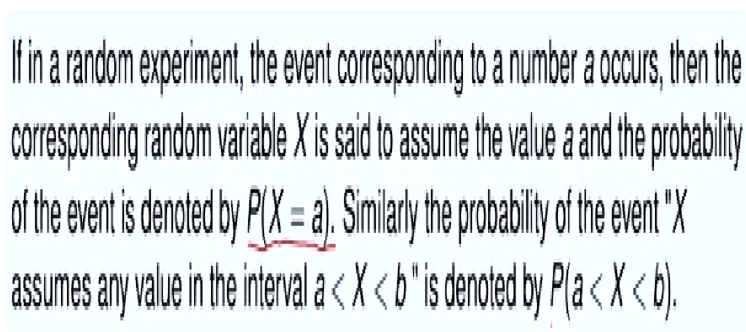

Now if in a random experiment the event corresponding to a number a occurs then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by PX=a okay similarly the probability of the event X assumes any value in that interval a<X<b is denoted by Pa<X<b.

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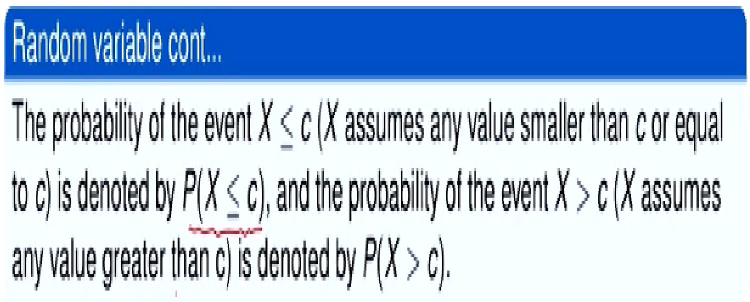

The probability of an event X<=c that is X assumes any value any smaller than c or =c is denoted by PX<=c and the probability of the event X>c that is X assumes any value >c is denoted by PX>c.

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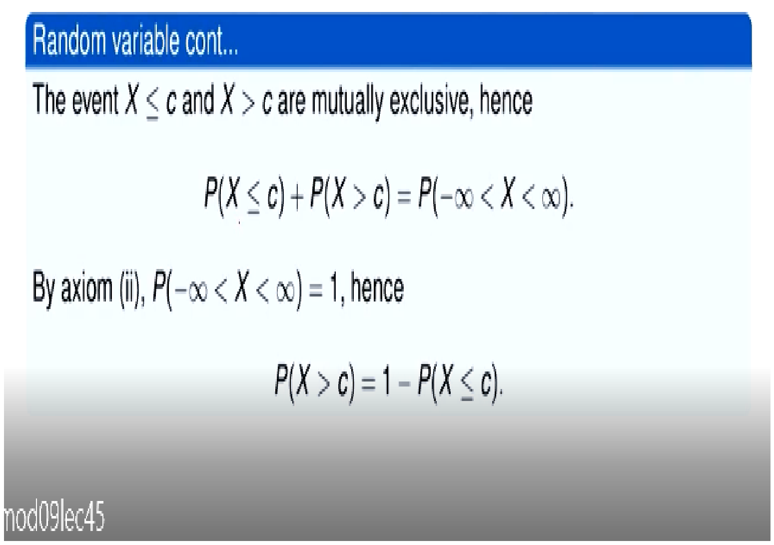

The event X<=c and X<c are mutually exclusive because that intersection is phi hence be PX<=c +PX>c= P- infinity <X< infinity. We know that if A and B are 2 mutually exclusive events then by then by the axiomatic definition of probability p a union b= pa+ pb. So, here if a denotes the event X<= c and b denotes X>c then pa+pb= pa union b and a union b is the real axis that is -infinity to infinity.

Now by axioms 2 of the definition of probability says that probability of S the samples space S=1 so probability of -infinity <X<infinity =1 and therefore probability that X>c =1-probability of x <c from this equation.

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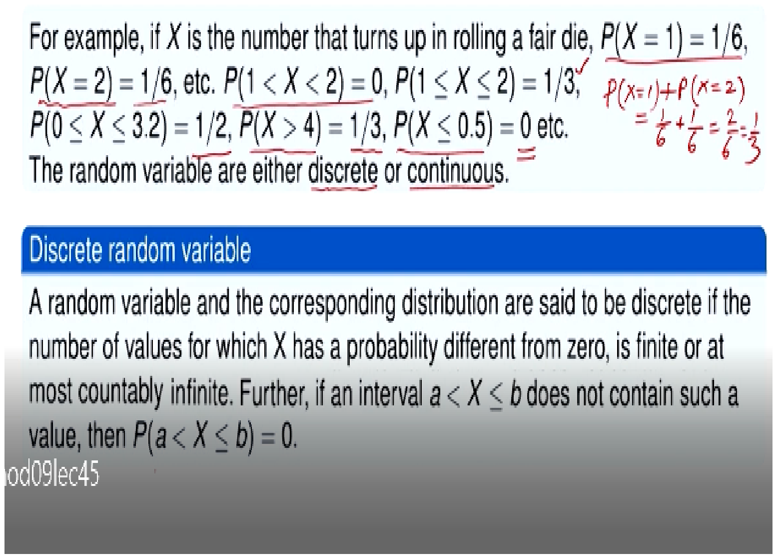

Now for example suppose X is the number that turns up and rolling a fair dice roll a fair dice then the probability of getting a any number say 1 2 3 4 5 6 is = 1/6 okay so probability of X = 1 1/6 probability of X = 2 is also 1/6. But probability of 1<X<2 okay now takes values 1 2 3 4 5 6 so X is not taking any value in between 1 and 2 and therefore probability that 1 is <X<x is 0. Probability that 1 <or =X <or = 2 it includes 2 integers 1 and 2 okay.

So, probability that 1 is <or =X <or = 2 is probability that X=1+probability that X=2 and that is =1/6 +1/6=2/6 and which is =1/3 okay. So, probability that 1 is <or = X <= 2 is 1/3 probability that 0<=X<=.2. Now in this set there are 3 points 1 2 and 3 each 1 has probability 1/6 so probability of X= + probability of X = 2 + probability of X = 3 is years 3/6 that is 1/2 and then probability X>4 okay X>4 means X can take values 5 and 6.

So, probability of getting 5 is 1/6 priority of getting 6 is 1/6 so probability of X>4 is sum of the 2 probability that is 1/6+1/6 by 6 which is = 2/6 and so 1/3 and probability that X is <or = 0.5 no value of X satisfies X <or = 0.5 because X is taking values 1 2 3 4 5 6. So, probability that X<or

= 0.5 is 0. Okay the random variables are either discrete or they are continuous okay now let us define a discrete random variable.

A random variable and the corresponding distribution are set to be discrete if the number of values for which X has a probability different from 0 is finite are at most finite or at most countably infinite. Okay further if an interval a<X<=b does not contain such a value then probability that a<X<=b is +0.

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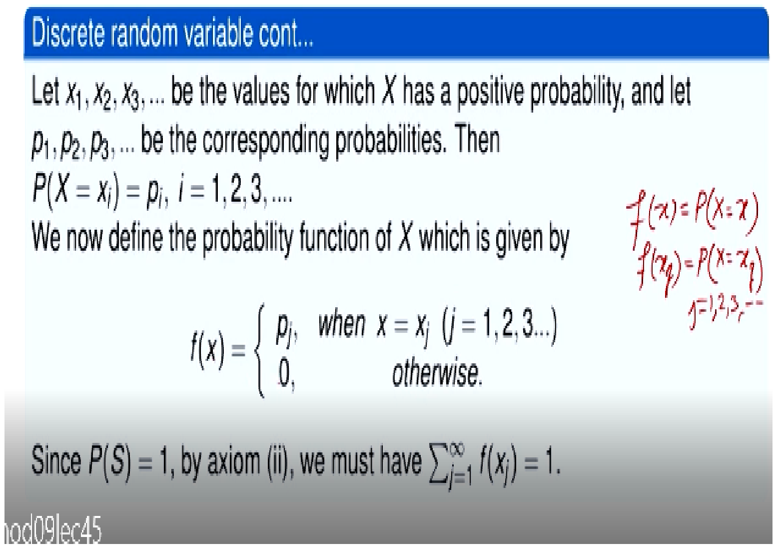

Now let x1 x2 x3 the values for which X be the values for which X has a positive probability and let p1 p2 p3 denote the corresponding probability that is probability of x=xi=pi I takes 1 2 3 and so on. Then we define the probability function of X given by fx takes the value pj when x takes the value xj so fxj=pj for j=1 2 3 and so on. so that means fx is nothing but probability that X takes the value x okay.

So, fxj= probability that x takes the value xj and j=1 2 3 and so on. Okay so this pxj=xj is pj so fx j=pj and when x does not take the value xj then okay fx takes the value 0. So, now PS=1 probability of the sample space S=1 by axiom 3of the axiomatic definition of probability. So, we must have sum of the probability =1 that is sigma j=1 to infinity fxj= 1.

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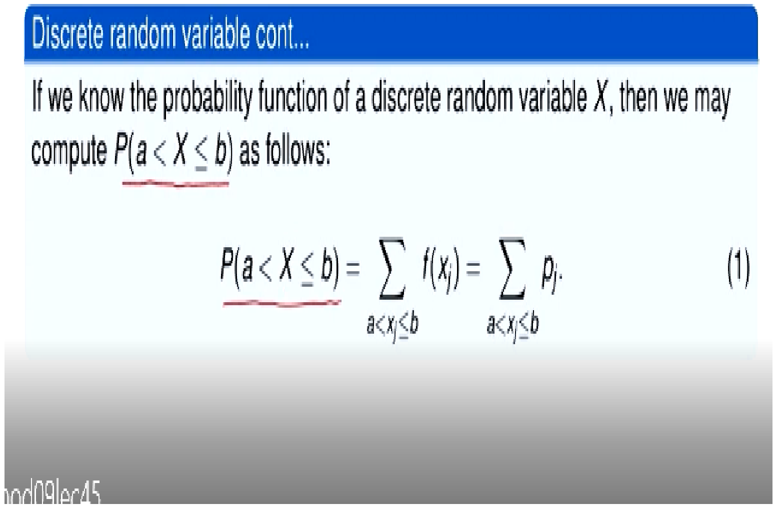

Now if we know the probability function of a discrete random variable X then we may compute probability that a<X<=b okay as follows. Probability that a<X<=b is sigma fXj a<xj<=b. We will take the sum of the probabilities of the points xj which lie in the interval a to b okay that is sigma bj a<xj<=b.

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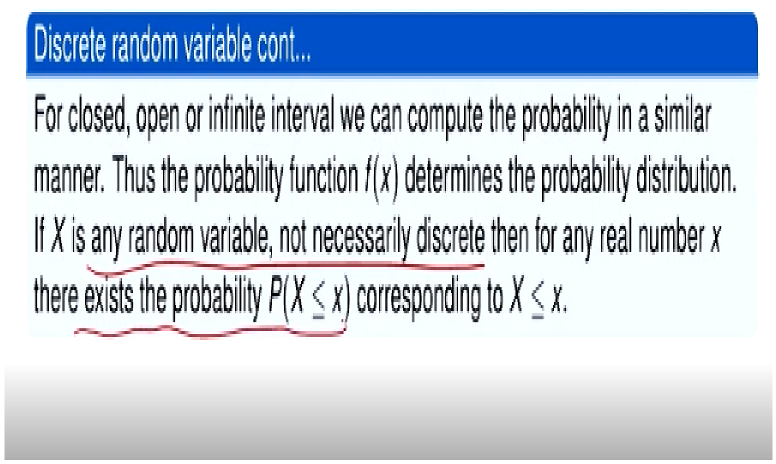

Now for closed open or infinite interval we can compute the probability in a similar manner here you can take a <or = X<or = b that then you can take the closed interval ab okay? you may also take the open from write on the side b that is you can consider a<=X<b okay you may also take a=- infinity to b=+ infinity so you may also consider infinite interval you may also consider open intervals that is a<X<b.

Then we will calculate the probability that a<X<b or a<=X<=b are probability of manage infinity <X<=b in a similar manner okay like we have shown here okay. We will take the sum of the probabilities with all those points xj which lie in a given interval okay. Now if X is any random variable this is true for any random variable not necessarily discrete. If X is any random variable not necessarily discrete.

Then for any real number of X there exist the probability P X<=x corresponding to X<=x okay so we call it as the cumulative distribution function of X.

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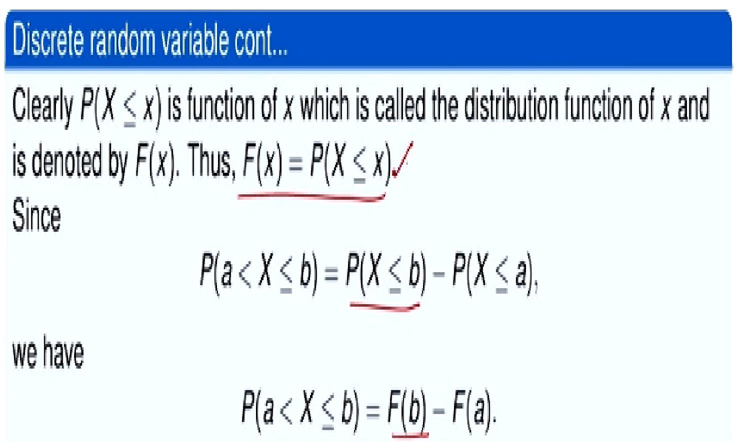

And denote it by fx okay PX<=x is a function of X which is called the distribution function of x or cumulative distribution function of X and it is denoted by FX. So, FX=PX<=x. Now probability of a<x<=b can be written as Probability of X<=b-Probability of X<=a. So, probability that A<X<=b can be expressed in terms of the cumulative distribution function this gives you Fb okay using this definition and this gives you Fa okay by the same definition.

So, the distribution function determines the distribution of X uniquely okay and it can be used for computing probabilities. We can compute the probabilities by using the distribution function you can see Fb-Fa gives the probability that a<X<=b.

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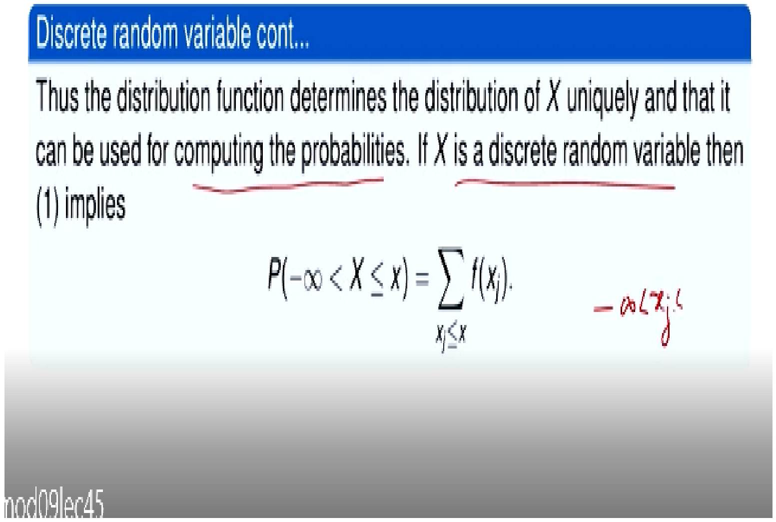

Now if X is a discrete random variable then from 1 okay from the definition 1 from the definition probability that -infinity<x<=x okay =sigma xj<=x. We will take the sum of the probability is all goes to xj which lie in the interval - infinity to X that is xj<=x okay.

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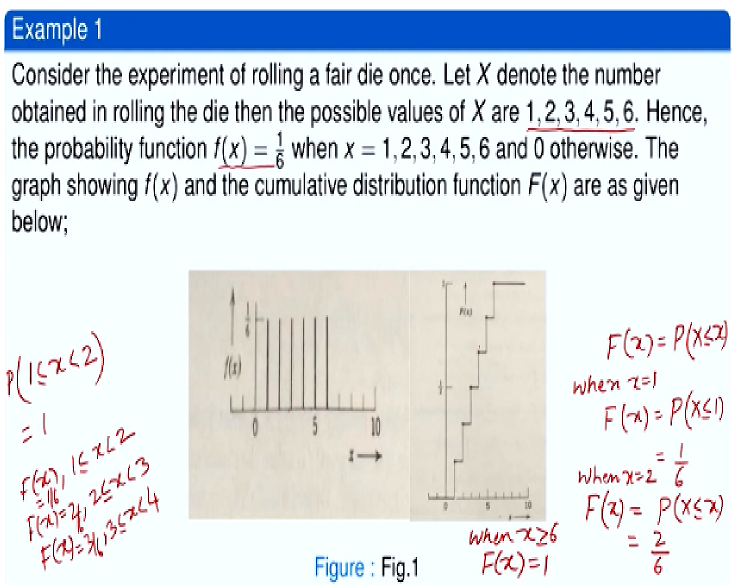

Now let us consider the experiment of rolling a fair dice once. Let x denote the number obtained in rolling the dice then the possible values of X are 1 2 3 4 5 6 hence the probability function f it takes the value 1/6 when x takes the value 1 2 3 4 5 6 and Fx= 0 otherwise. Now the graphs showing the function Fx and the cumulative distribution function Fx are as given below. You can see here this is our x axis okay and this is the y axis okay.

So, Fx takes the value 1/6 when X is = 1 2 3 4 5 6 okay at all the values of X 1 2 3 4 5 6 Fx takes the value 1/6 and it is 0 at all other values of X. Now this is cumulative distribution function of the function cumulative distribution function of the probability function Fx okay so here when X is<1 okay probability that X is <1 okay this is Fx=probability that X is <or = X okay so Fx= let me write like this Fx= probability that X<=X.

Now when X=1 okay Fx=PX<=1 okay and PX<=1 is 1/6 so F1=1/6 to the left of 1 okay Fx is 0 Fx=0 because the probability Fx=0 okay at X=1 F1=1/6 and then Fx okay when X takes the value 2 okay Fx=PX<=X gives you 2/6 because X<= 2 means we will get the probability of X =1 and X=2 so we get here we write here Fx=2/6 and then when you take X=3 okay we reach here okay Fx=PX<=3 so we get 3/6 that is 1/2.

And when you take X=4 you get Fx=4/6 and we come here then at 5 we come here 5/6 and at 6 we come here and then after that Fx becomes = 1 but guess so if X is > 6 okay > or = 6 when X is > or = 6 okay Fx is = 1 okay now in between okay if 1 is <or = X <2 okay if 1 is <or = X <or if 1 is <or = X <or<2 then probability remains 1 okay. Because in this interval 1 to 2 only X = 1 lies.

So, here 1 to 2 interval you too take 1 to 2 then the cumulative distribution function remains 1 okay so Fx=1 for the interval 1 <or = X<2 similarly Fx=2 when 2 is <or = X <3 like this. Fx=1/6 okay here it is 2/6 and then 3/6 when 3<X<4 like this okay. So, this is how we draw the graph of the cumulative distribution function in the case of rolling a fair dice.

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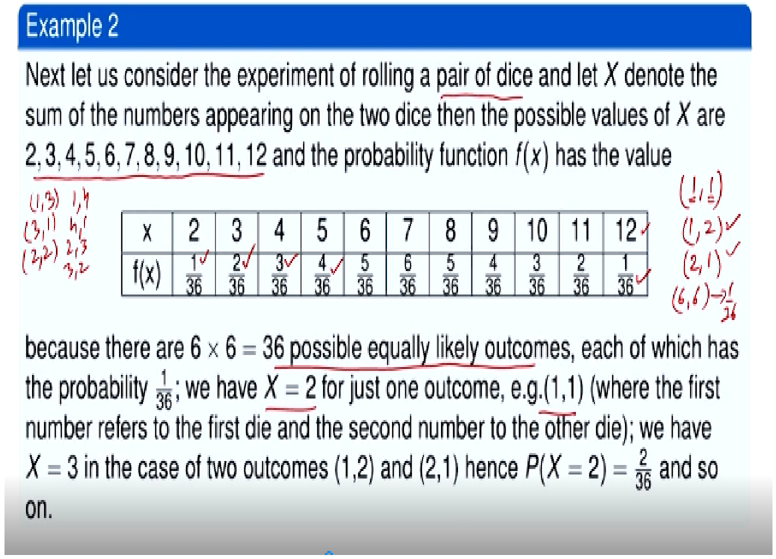

Now let us take the experiment of rolling a pair of dice okay? we have 2 dice we roll them together okay then X denote the sum of the numbers appearing on the 2 dice then the possible values of X are 2 3 4 5 6 7 8 9 10 11 12 okay the probability function Fx has the value. Now when we will get the number 2 sum of the numbers on the 2 dice will be =2 when each one has each one shows 1 on.

Okay so that means 1 1 okay on the dice 1 dice 1 the first number shows dice 1 second number shows dice 2 okay so first number on the dice 1 we get 1 on the dice 2 we get 1. Then the sum will be 2 and each one has probability 1/6. So, having 1 1 that is sum 2 will turn up will be the sum of X=2 when we will have 1 1 on both the dice okay and probability for that is 1/6*1/6 that is 1/36.

Now when the sum will be 3 the sum will be 3 if we have on dice 1 we have 1 on dice 2 we have 2 on dice 1 we have 2 on dice 2 we have 1 okay now there are 2 possibilities when the sum will be 3 okay? now this has probability 1/6 this has probability 1/6 so having 1 2 is having probability 1/36 and 2 1 will also have a probability 1/36. So, total probability is 2/36 and we get 2/36. Now sum will be 4 sum of the two dice will be 4 if we have 1 3 3.

Okay 2 2 there are 3 possibilities 1 3 3 1 2 2 okay each 1 has probability 1/36 so we have 3/36 similarly 5 5 can come when we have 1 4 4 1 2 3 3 2 and then so we have 4 possibilities each one has probability 1/36. So, we get the probability of having some 5 as 4/36 so in a similar manner we can compute the probabilities for having X= 6 7 8 9 10 11 now 12 will come 1 each 1 each dice shows 6.

Okay so that means 6 6 so this is 1/6 probability of this is having 6 on 1 dice is 1/6 so 1/6*1/6 so this has probability 1/36 so we have 1/36 now you can sum these probabilities okay 1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 this comes out to be 36/36 and so sum of probabilities is = 1 now there are 36 possible outcomes equally likely outcomes each of which has the probability 1/36 X=2 comes for 1 1 we have discussed okay we are in the first dice number refers to the first dice.

Second number to the other dice we have X=3 in the case of 2 outcomes 1 2 and 2 1 okay so probability that X=2 is 2/36 and so on.

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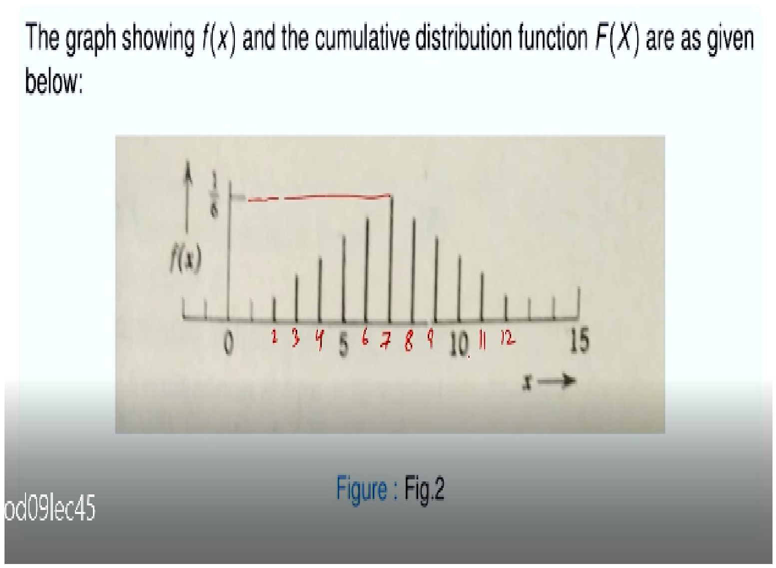

Now let us look at the graph okay so you can see here this is your 1/6 okay now there is 1 2 3 4 5 6 7 8 9 10 11 12 okay so we can have sum 2 okay sum can come sum cannot be 1 okay sum can be 2 so this is 2 3 4 5 6 7 8 9 10 11 and 12 okay now 2 2 2 has probability 1/36 okay so this is 1/36 there is 36 probability that X has a value 3 is a 2/36 so this is double of this okay so 2/36 then this is 3/36 at 4 3/36 at 5 4/36 this is 4/36 okay at 6 5/36.

So, this is 5/36 and 7 is 6/36 6/36 means 1/6 okay so you can see this is 1/6 okay at 7 the probability is 6/36 that is a 1/6 okay so and then at 8 the probability is 5/36 same as the probability at 6 okay so you have these same hike here at 8 okay then 9 has the probability 4/36 4/36 means the probability for having the sum 5 okay and then 10 10 has a probability 3/36 so it is of the same height as height at 4.

And then 10 11 11 has 2/36 so it has the same height as the height at 3 okay and then the probability that X=sum=12 has the same height as height at X=2 okay. So, this is the graph of the probability distribution of X where X denote the sum of the numbers on the 2 dice when we roll them okay.

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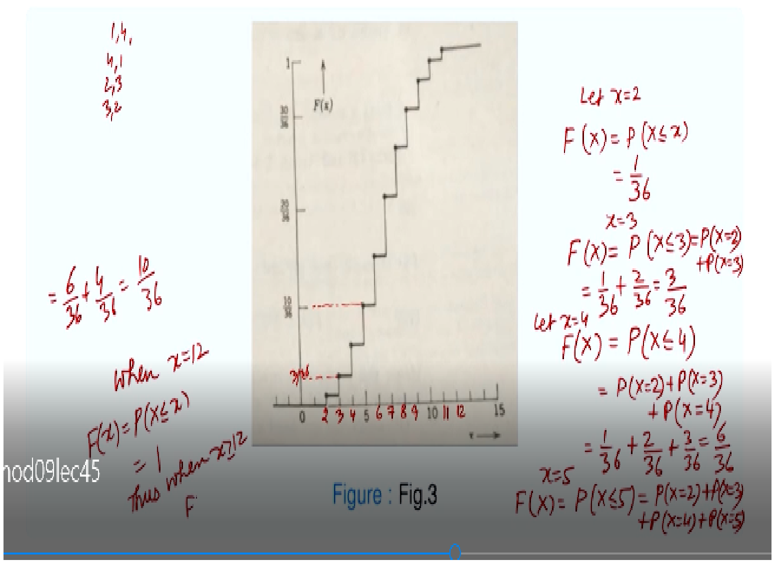

Now this is cumulative distribution function we can see here that this is 2 3 4 5 this is 2 3 4 5 6 7 8 9 10 11 12 okay so probability that Fx Probability that X<=x okay so if you take X to be 2 okay so let X be 2 okay then this is probability that actually <or = 2 is a probability of having X = 2 X = 2 means 1 1 that means 1/36. Okay so you can see this is 1/36 then at x = 3 we have to 2/36 okay.

So X=3 has probability 2/36 X= 1 has probability 1/36 so 1/36+2/36 we have to add the probability at X=1/ see this is = probability that X=1+ probability that X=2. So, this is =1/36+2/36 so we get 3/36 okay so this is 3/36 this one is 3/36 okay and then you find the Fx for

X=4 okay. So, Fx this will be Fx okay so Fx where X<4 PX<=4 okay let X be 4. So, then this is probability that X=1 X=2 X=3 X=2 not X=1 23 and 4.

And here we have to write 2 and 3 okay 2 and 3 so this will be 1/36 2/36 and 4 comes when we take 1 3 3 1 and 22 okay we get 3/36 so this comes =6/36. Okay 6/3 and when you take X=5 here okay X=5 we will see it comes to be 10/36 so we have Fx=PX<=5 and we get Px=2 PX=3 X=4 X=5 okay which is =1/36 2/36 3/36 so probability that X=2+ probability that X=3+ probability of X=4 is 6/36 we have to add the probability of X=5.

When X=5 comes when we have 1 4 4 1 2 3 2 okay that means we have 4 cases. So, 4/36 okay so 6/36+4/36 so we get 10/36 so you can see here when X=5 it is 10/36 okay and similarly we can calculate and when X is 12 okay when X is 12 the sum of the probability becomes =1 so when X=12 okay Fx=PX<=X will be=1 and when X is >12 so thus when X is >=12 Fx=1 so this is like this. So, this is the cumulative distribution function of Fx.

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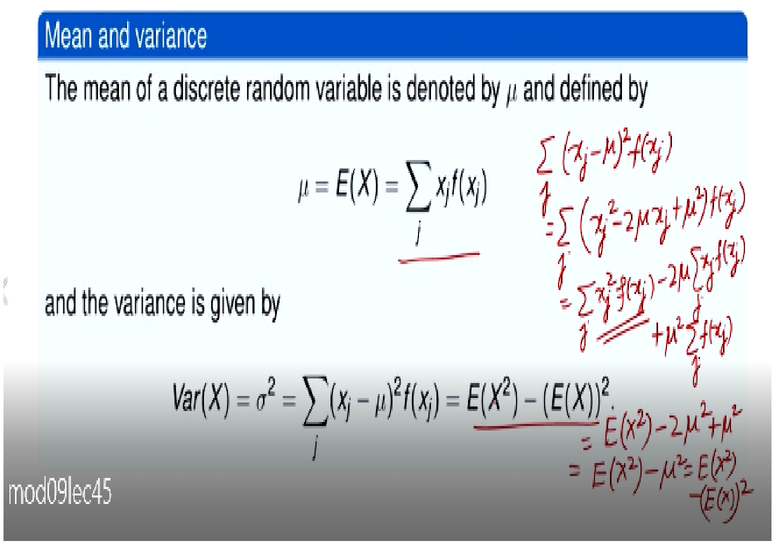

Now the mean of a discrete random variable when you have a discrete random variable the mean of the discrete random variable is given by sigma /j xj fxj where xj are the values taken by the random variable X okay. We multiply xj by the corresponding probabilities and take the sum/j the variance is given by variance of X = sigma square which is sigma/j xj-mu whole square*Fxj that is we multiply the deviation of xj from the mean square it.

And then multiply by the probabilities of a corresponding probabilities and take the sum/ j. Now this expression comes out to be expectation of X square - expectation of X whole square we can easily see this. Let us write sigma/j xj-mu whole square*Fxj= sigma/j xj square -2 mu xj+mu square*Fxj then this will be sigma/j xj square*Fxj- 2 mu is a constant sigma/j xj*Fxj+mu square sigma Fxj sum/j okay.

Now this is =sigma/j xj square Fxj is expectation of X square okay the first term first sum is E x square-2 mu sigma/j xj Fxj over here is mu. So,-2 mu square and then +mu square - sigma/Fxj=1 sum of the probability =1 this is Ex square-mu square okay which is Ex square- Ex while square okay we have this result okay. So, variation of x can be expressed in terms of the expectation it is expectation of x square -expectation of x whole square.

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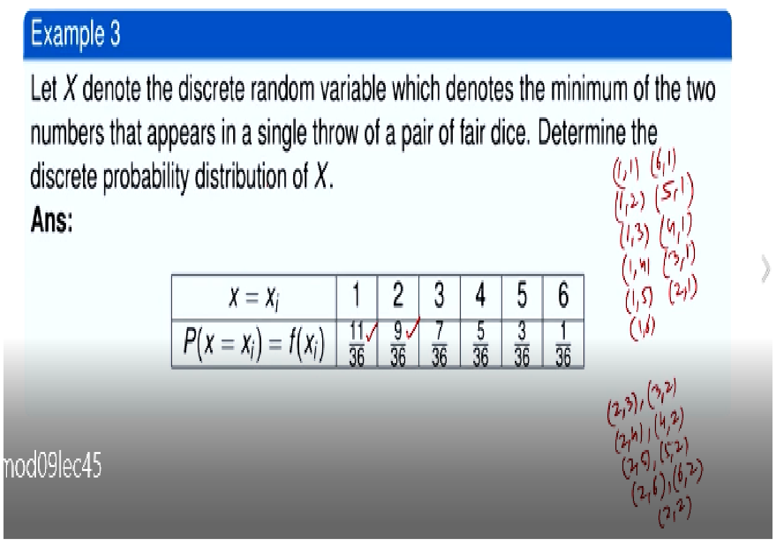

Now let us consider this problem let X denote the discrete the minimum of the two numbers that appears in a single throw of a pair of fair dice okay? Then determine the discrete probability distribution of X okay so now if the minimum of the numbers on both the dice is 1 okay this can happen how many in how many cases are there were minimum will be 1 1 1 okay 1 2 1 3 1 4 1 5 1 6 okay.

And then we have 6 1 5 1 4 1 3 1 2 1 okay 11 we have already taken so there are 11 cases 1 2 3 4 5 6 7 8 9 10 11 each one has a probability 1/36 so we have 11/36 probability okay similarly if you want the minimum on the of the 2 numbers to be 2 then the cases are 1 2 2 1 2 3 3 2 2 4 4 2 2 5 5 2 2 6 6 2 and 2 2 okay so we have one pair here one pair here one pair here no this will not come because the minimum be 1 to be 2 okay.

So, this will not come minimum we want to be 2 there minimum will be 1 okay. So, here so we have two cases here two cases here two here okay 8+19. So, we have 9/36 so similarly if we want the minimum to be 3 there will be 7 cases okay each one will have probability 1/36. So, we will have 7/36 okay so this is the probability distribution of the discrete random variable which is the minimum of the 2 numbers that appears in a single throw of a pair of fair dice.

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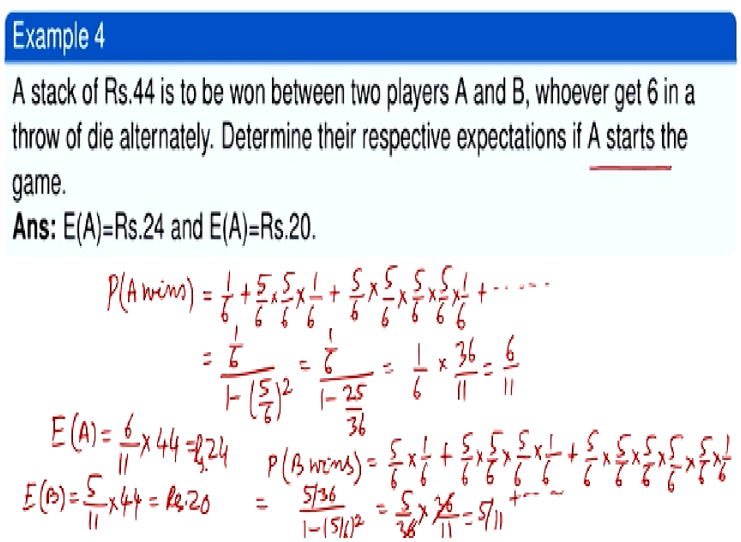

Now let us take the example for a stack of rupees 44 is to be won between two players and be whoever gets 6 in it throw of dice alternatively. Now determine their respective expectations if A starts the game okay? So let us find the probability that A wins so probability that A wins okay if he if he gets 6 in the first throw he will win. Okay so probability of A wins will be 1/6 okay suppose he does not get 6 in the first throw then what will happen? B will get a chance okay?.

So, then B should not win because we want A to okay so B should not win so in the next case what will happen? 5/6 okay because if the if in the first throw he does not get 1/6 that means the probability will be 5/6 and then we will get a chance we should also not get 1/6 so the probability 5/6 and then A gets the chance A should if we want A to win now then the probability will be 1/6. Because he should throw 6.

Now if he does not throw 6 then again the probability will be 5/6 it will go to the player B again so 5/6*5/6*5/6 okay and then 5/6*1/6 and he does not get in the first chance he does not get 6 the probability will be 5/6. Then B also does not get 1/6 the probability is 5/6 then A gets the chance the probability will be 1/6 if he wins okay if he does not win then 5/6*5/6*5/6 then we will get a chance we should not throw 6 okay.

So 5/6 then A will win if it throw 6 that will be probability 1/6 so and so on okay. So, this is a geometric progression and the sum of this series A/1-R s 1/6-5/ whole square so this is 1/6/1-25/36 and this comes out to be 1/6*36/11 so we get 6/11 okay. So, determine the their respective expectations so expectation of A will be = probability that A wins so that is 6/11*the amount that is 44 so rupees 24 okay.

So, determine the respective expectation of A okay will be 24 similarly probability that B wins because A starts the game okay? So B will win because if A does not get 1/6 so that means 5/6* B will win if he throws 6 so 5/6*1/6. Now if B does not throw 1/6 then 5/6*5/6 then A should not throw 6 5/6 then B will throw 1/6 B will throw 6 so 1/6 okay? And then we can similarly have so 5/6*5/6*5/6 now B does not throw 6 so 5/6 A does not throw 1/6 so 5/6*1/6 and so on.

Okay now the sum of the series is 5/36/1-5/6 whole square so this time it is5/36*36/11 okay so we get 5/11. So EB the expectation of B will be =5/11/*44so that means rupees 20. So, expectation of A is 24 rupees expectation of B is 20 rupees.

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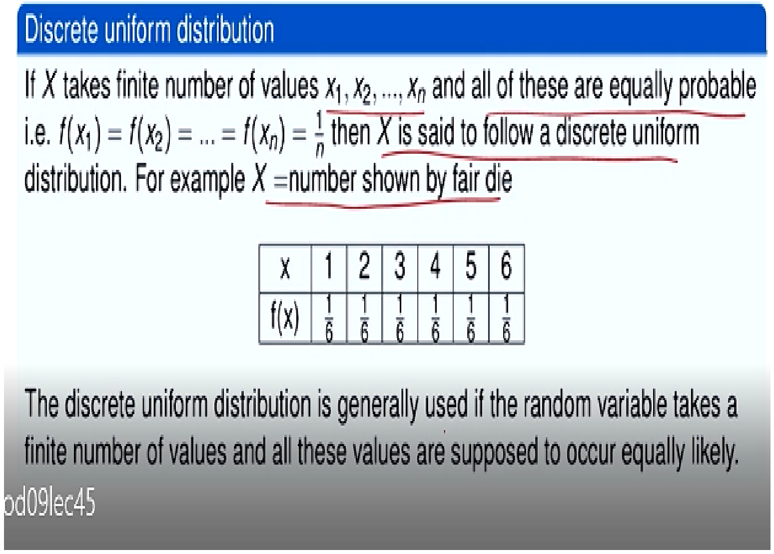

Now let us discuss discrete uniform distribution if X takes finite number of values x1 x2 xn because we have said when we defined the discrete random variable we have said that X either takes discrete number of values or countably infinite number of values discrete in the case of discrete uniform distribution X will take only a finite number of values okay? So let us say the values are x1 x2 and xn and all of these are equally probable.

Because each one has equal probability so that means probability of having x=x1 x=x2 x=xn each one will be `1/n because they are all equal okay? And total probability is 1 now X is said to follow a discrete uniform distribution okay? If x takes the finite number of values okay and all of these values are equally probable okay? Now let us say for example x is = number shown by a fair dice okay when we have a fair dice the number shown by the dice when it is thrown okay.

The numbers are 1 2 3 4 5 6 and they are all equally probable okay? So each one has the probability 1/6. So, the discrete uniform distribution is generally used if the random variable takes a finite number of values and all these values are supposed to occur equally likely. So, number shown by a fair dice is an example of a uniform discrete distribution. Now let us suppose let us consider this problem.

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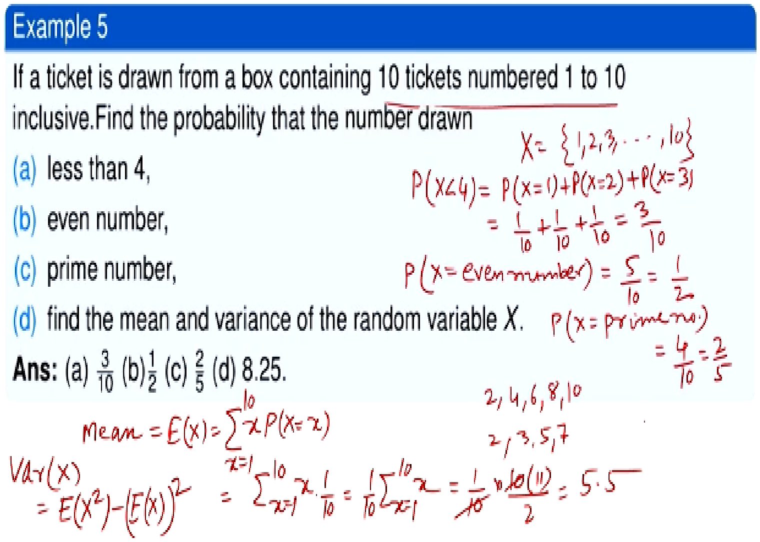

If a ticket is drawn from a box containing 10 tickets number 1 to 10 inclusive okay? Find the probability that the number is <4 okay? So when we draw a ticket from the box okay? all the tickets x takes values 1 2 3 4 5 and so on up to 10. Okay x takes the value 1 2 3 4 5 6 so on and so on up to 10 okay find the and each number okay has the same probability they are all equally probable of being drawn.

It is the case of a discrete uniform distribution we want the probability that the number drawn is <4 okay? so p x <4 we want px<4 means 1 2 3 okay. So, each 1 has probability 1/10 so PX=1+PX=2+PX=3 so it could be any number PX=1 or X=2 or X=3 each number has an equal probability 1/10 so 1/10+1/10+1/10 when you take it from the 10 tickets which are numbered from 1 to 10 okay?.

The probability of trying a ticket will be having any number say X where X takes value 1 to 10 will be 1/10 even number okay so probability then X is an even number now even numbers are 2 4 6 8 and 10 so there are 5 possibilities okay so 5/10 1/2 prime number prime numbers are 2 3 5 7 okay each 1 has probability 1/10 okay so probability that X is prime. It is =4/10 or 2/5 so okay find the mean and variants of the random variable X okay.

Now mean will be = expectation of X so sigma PX=x takes values from 1 to 10 again now what we have sigma x=1 to 10 XPX=x takes value 1 to 10 each number has probability 1/10 so

X*1/10 so /10 sigma X=1 to 10 which is 1/10* 10*10+11 11/2 because sigma n =n*n+1/2. So, this is this term will cancel and you get 5.5 okay so expectation is 5.5 okay? Then variance=expectation of X square-expectation of x whole square okay?

So expectation of X square let us find okay? Just find expectation of X square.

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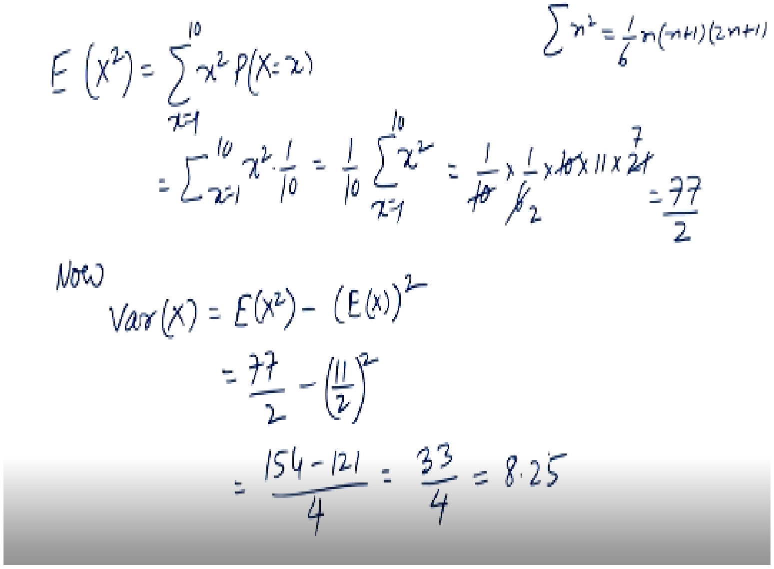

This is sigma X square*PX=x so x varies from 1 to 10 so this is sigma x = 1 to 10 X square and *1/10 okay so 1/10 sigma X square x varies from 1 to 10 we know sigma n square okay it is 1/6 n*n+1*2n+1. So, let us apply this formula so 1/10 n is 10 here okay so 10*10+1 that is 11*20+121 okay so this cancels with this and what we have here 3 okay 77/2. Okay now variance of X=expectation of X square - expectation of x whole square.

So, this is 77/2 and EX came out to be 11/2 okay so 11/2 whole square. So, this is 154-121 so we get here 33/4 so that means 8.25 okay. So, variance comes out to be 8.25 okay now so with this I would like to end my lecture thank you very much for your attention.