Advanced Engineering Mathematics
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Lecture - 44
Bayes Theorem and Probability Networks

Hello friends. Welcome to my lecture on Bayes theorem and probability networks. We know that the conditional probability of A given B is=P A intersection B/PB. This formula can be put in a different form by using the Bayes theorem. The formula is known as Bayes formula or Bayes theorem.

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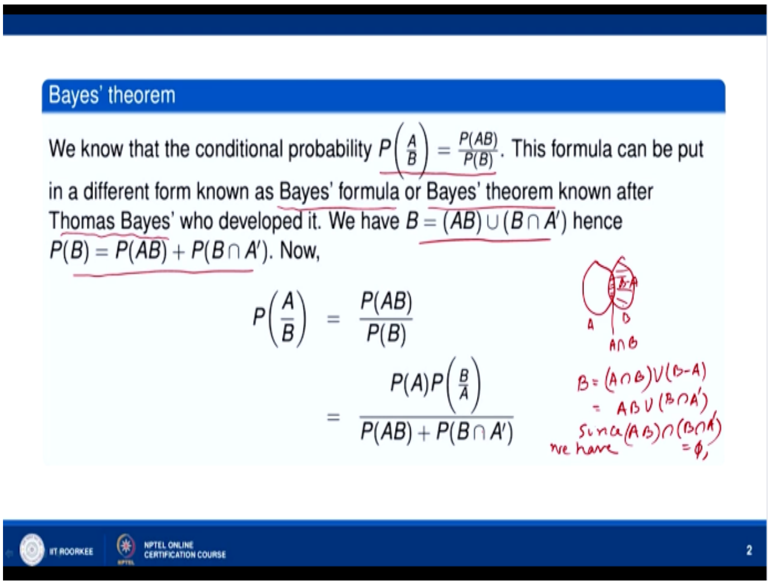

It is known after Thomas Bayes who actually developed this formula. Now we can write B= if you have the set A and the set B, then this is A intersection B and this part is B-A this part okay. So B can be written as A intersection B union B-A which is=AB union B intersection A dash, A intersection B we also write as AB. So B is=AB union B intersection A dash and therefore now AB and B intersection A dash are mutually exclusive okay.

So since AB intersection B intersection A dash is=phi okay, we have probability of B=probability of AB+probability of B intersection A dash and now probability of A given B. Probability of A given B is PA intersection B/PB we can write it as P AB=by multiplication theorem P AB=PA*probability of B given and probability of B is=probability of AB that is probability of A intersection B+probability of B intersection A dash okay.

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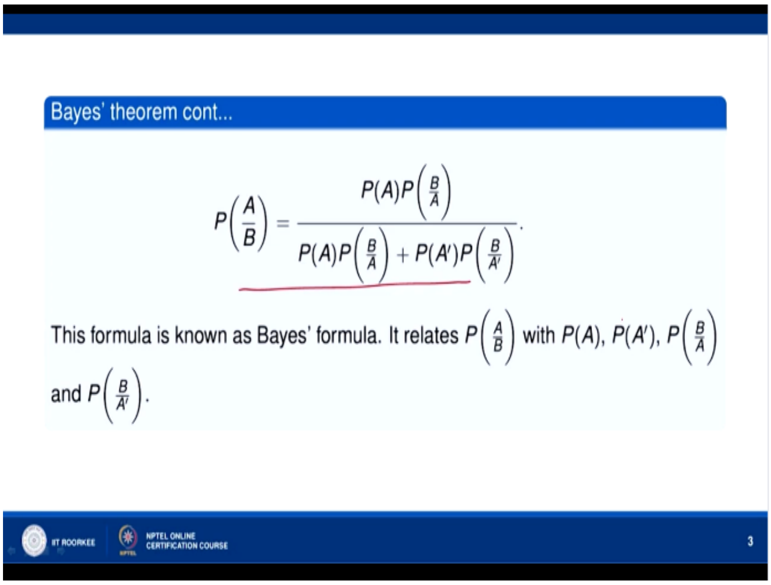

Or we can put is as probability of A given B is=probability of A*probability of B given A and then probability of AB is=probability of A*probability of B given A dash and probability of B intersection A dash again by multiplication theorem can be written as probability of A dash*probability of B given A dash okay. This formula is known as Bayes formula. It relates the probability of A given B with the probability of A, probability of A dash, probability of B given A and probability of B given A dash.

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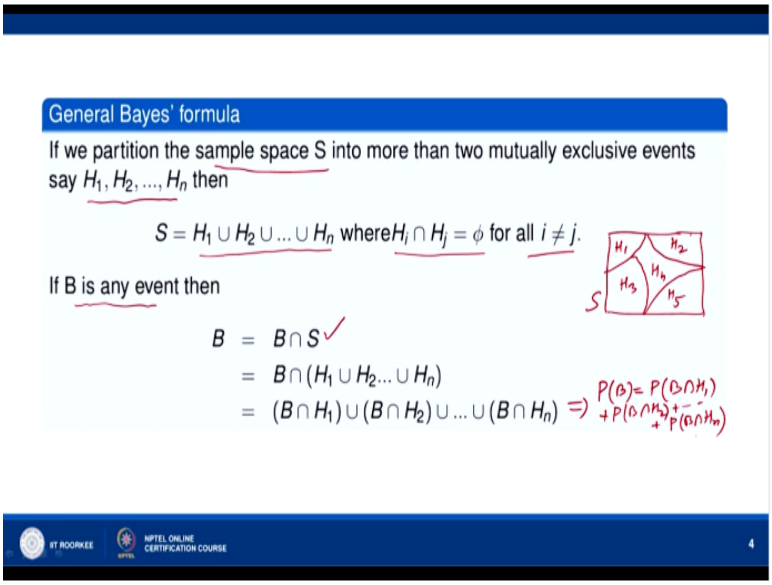

If we partition the sample space S into more than two mutually exclusive and suppose you have this sample space. Let us partition this okay. So this is your H1, H2, H3, H4, H5 and so on okay. So if we partition the sample space S into this is your S into more than two mutually exclusive events H1, H2, Hn then S is=H1 union H2 union H3 and so on union Hn where Hi intersection Hj is=phi.

You can see here that H1, H2, H3, H4, H5 they are all mutually exclusive that means Hi intersection Hj is=phi for all i not equal to j. Now if B is any event then B can be written as B intersection S because S is the sample space okay and S is=union of H1 H2 Hn, so B=B intersection H1 union H2 union H3 and so on union Hn. Now using the formula of the said theory intersection is distributive over union.

So we have B intersection H1 union, B intersection H2 union, B intersection Hn and since H1, H2, Hn are mutually exclusive B intersection H1, B intersection H2 and B intersection Hn are also mutually exclusive and therefore probability of B will be=probability of B intersection H1+probability of B intersection H2 and so on probability of B intersection H1 but probability of B intersection H so we can write here.

So this implies that probability of B is=probability of B intersection H1+probability of B intersection H2 and so on probability of B intersection Hn okay.

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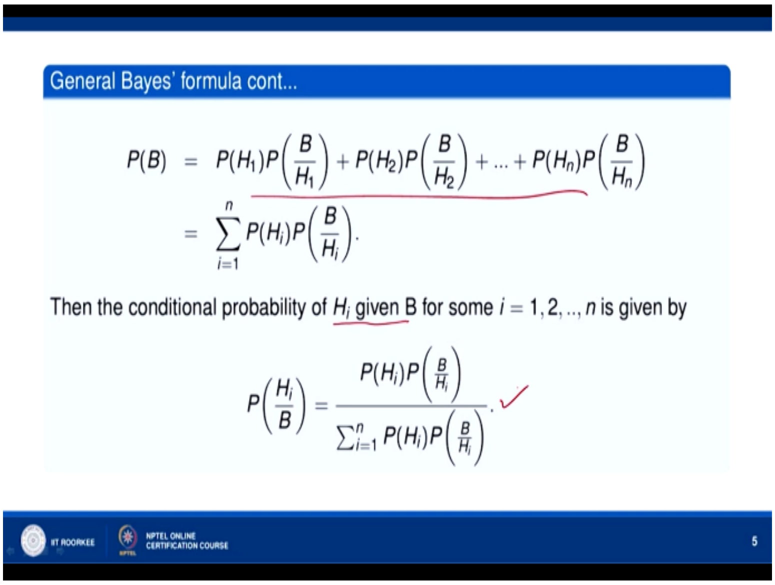

Now we can use multiplication theorem. The probability of B intersection H1 will be probability of H1*probability of B intersection H1+probability of H2*probability of B intersection H2 probability of Hn*probability of B intersection Hn. B given Hn probability of B intersection Hn is probability of Hn*probability of Hn* probability of B given Hn. So this is=this right hand sum can be written as sigma i=1 to n P Hi P B given Hi.

Then, the conditional probability of Hi given B okay, the probability of Hi given B is=P Hi*P given Hi/sigma i=1 to n P Hi P of probability of B given Hi okay. So this is known as Bayes theorem, Bayes formula.

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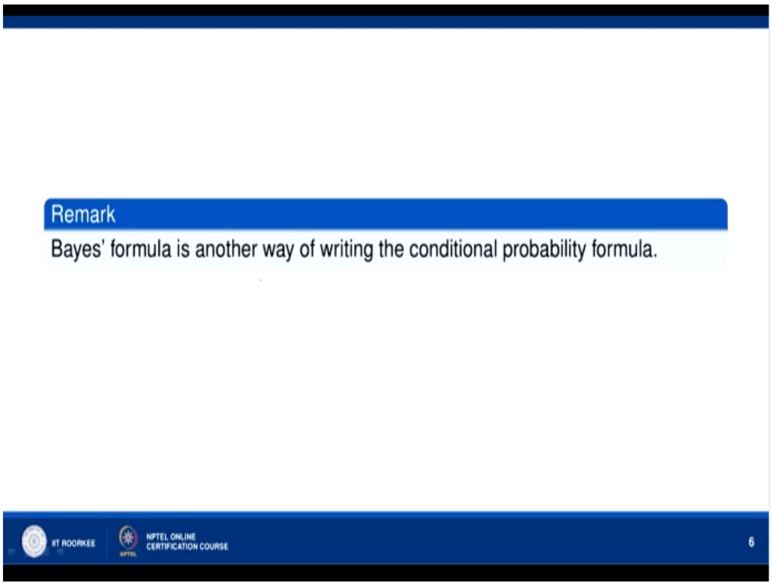

Now Bayes formula is another way of writing the conditional probability formula.

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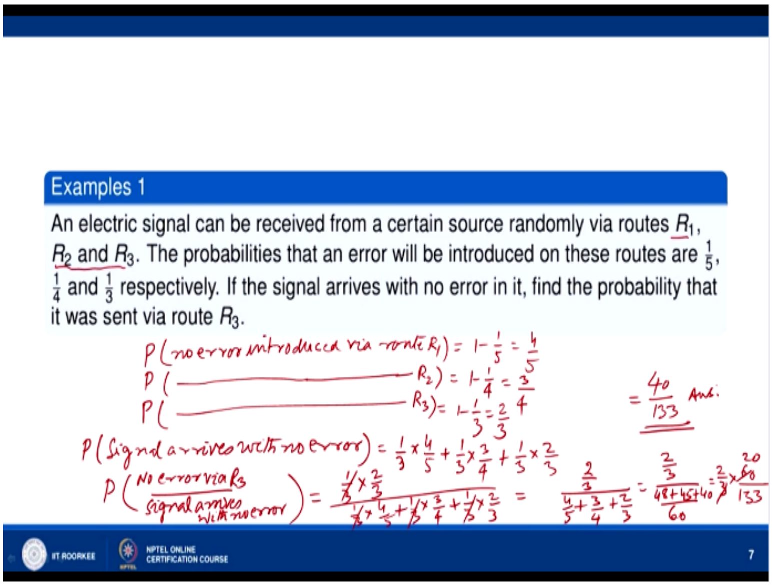

Let us consider a problem on this Bayes theorem. An electric signal can be received from a certain source randomly via routes R1, R2 and R3. The probability that an error will be introduced on these routes are 1/5, 1/4 and 1/3 respectively. If the signal arrives with no error in it okay, then we have to find the probability that it was sent via route R3. So 1/5, 1/4, 1/3 are the probabilities that error occurs via route R1, via route R2, via route R3.

So probability that no error is introduced via route R1 will be=1-1/5 that is=4/5 okay. Similarly, probability that no error is introduced via route R2, this will be=1-1/4=3/4 okay and similarly probability that no error is introduced via route R3, this will be=1-1/3=2/3 okay. Now we want the probability if the signal arrives with no error in it. So we are given that signal arrives with no error in it.

Probability that signal arrives with no error will be equal to now there are 3 routes R1, R2, R3 so each route has probability 1/3 okay. So 1/3 okay, let us say we are taking the route R1. So R1 has route 1/3, R1 has the probability 1/3 multiplied by no error so 4/5. Similarly, R2 route has the probability 1/3*no error in it, there is no error in the signal via route R2, so it is 3/4, then 1/3, the probability that the signal comes via route R3 with no error in it, so 1/3*2/3 okay.

Then, this is the probability that the signal arrives with no error in it. Now then we need to find the probability that it was sent by route R3. So no error via R3 given that signal arrives with no error okay, signal arrives with no error. So this is probability that there is no error via route R3, so 1/3*there is no error via route R3. So 1/3*2/3/1/3*4/5+1/3*3/4+1/3*2/3, 1/3 we can cancel and then we have 2/3/4/5+3/4+2/3.

And this is 2/3/here we have LCM 60 and we have 12*4=48+15*3=45+20*2=40 okay. So this comes out to be 2/3*60/133 okay that means 40/133 okay. So this is the answer okay 40/133. The probability that the signal was sent via route R3 with no error in it.

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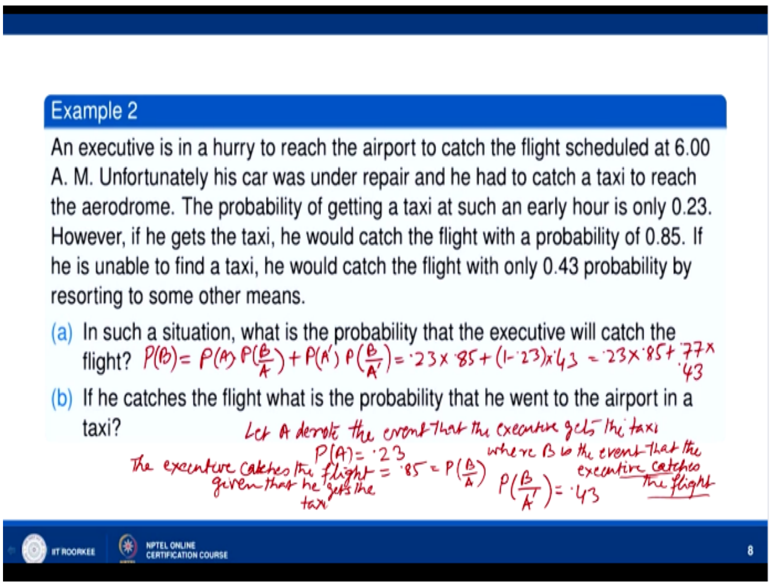

Now let us go to second question. An executive is in a hurry to reach the airport to catch the flight scheduled at 6 A.M. Unfortunately, his car was under repair and he had to catch a taxi to reach the aerodrome. The probability of getting a taxi at such an early hour is only 0.23 okay. So suppose let us say A denote the event that the executive gets the taxi okay, so probability of A is=0.23 okay.

If he gets the taxi, he would catch the flight with probability 0.85 okay. So let us say the executive catches the flight okay given that he got the taxi okay. Executive catches the flight, this is 0.85 okay. So this is if given that he gets the taxi, so this is probability of B given A okay. This is probability of B given A where B is the event that he gets the flight okay, so where B is the event that the executive catches the flight okay.

So probability that the executive catches the flight given that he gets the taxi is probability of B given A which is 0.85. If he is unable to find a taxi, he would catch the flight with 0.43 probability by resorting to some other means okay. So here let say then he gets the 0.43 probability that he does not get the taxi okay. So probability of B given A dash is=0.43 okay.

He catches the flight, when he does not get the taxi he resorted to some other means. So probability of B given A dash is 0.43. In such a situation, what is the probability that the executive will catch the flight? So probability of B okay, probability B is the event that the executive catches the flight okay. So probability of B is=probability of A*probability of B given A+probability of A dash*probability of B given A dash okay.

So probability of A is=0.23*probability of B given A is 0.85+probability of A dash is=1-0.23 okay, probability of A dash is 1-0.23*probability of B given A dash which is 0.43 okay. So this is=0.23*0.85+0.77*0.43 okay. So that is the probability okay, 0.23*0.85+0.77*0.43 okay. That is the probability that the executive catches the flight. Now if he catches the flight what is the probability that he went to the airport in a taxi okay.

So let us find this. If he catches the flight, so we are given that he catches the flight, now we have to simply find the probability that he went to the airport in a taxi okay.

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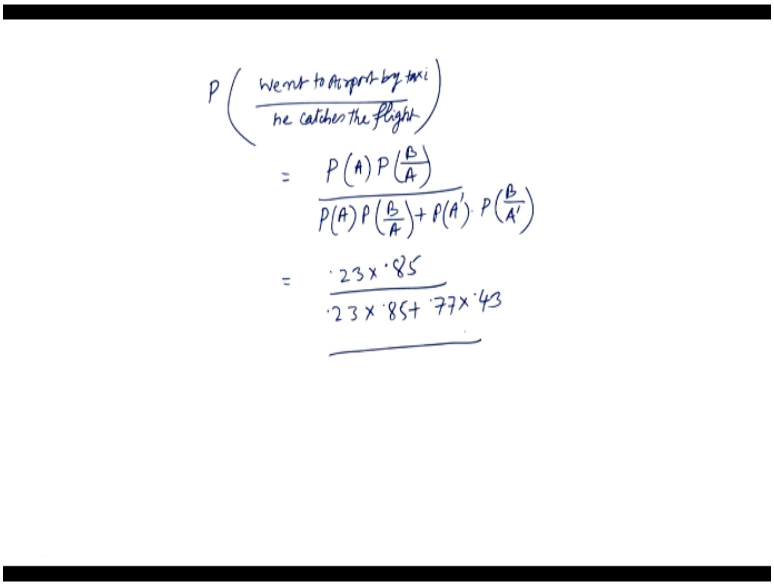

So we have to find the probability that he catches the flight, it is given, he catches the flight and he went to the airport by taxi okay. So let us see. PA*P B/A. PA is the probability, A denote the event that he gets the taxi and P B/A is the probability that he catches the flight given that he gets the taxi okay and the probability that he catches the flight is 0.23*0.5+0.77*0.43 okay.

So we have to find the probability he went to the airport by taxi so PA*P B/A/he went to the airport by taxi that is PB okay, PB is PA*P B/A+PA dash*P B/A dash okay. The probability that he went to the airport by taxi given that he catches the flight okay, so PA is 0.23 okay and P B/A is 0.85 okay, PA is 0.23, P B/A is 0.85 and divided by PB, PB is 0.23*0.85+0.77*B/A probability of B/A is 0.43 okay. So this is the required probability.

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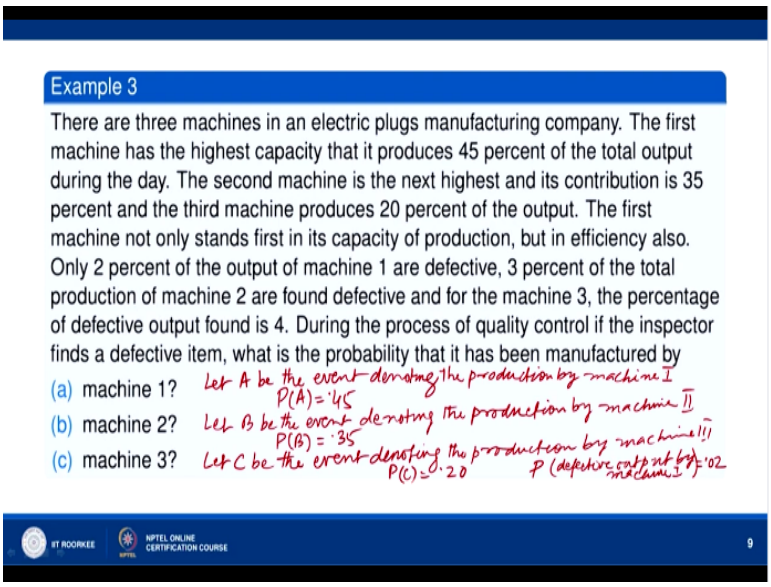

Now let us go to question number 3. There are 3 machines in an electric plugs manufacturing company. The first machine has the highest capacity that it produces 45% of the total output during the day okay. So let A denote the event that the first machine has the highest capacity that it produces production by machine A okay. So let A be the event denoting the production by machine A by machine 1 okay.

There are 3 machines, machine 1, machine 2, machine 3 okay by first machine. So this is so probability of A then is given by 0.45 okay. Similarly, let B be the event denoting the production by machine 2 okay. Then, PB is=0.35 okay. Similarly, let C be the event denoting the production by machine 3 okay. Then, PC is=0.20 okay. The production by machine 1 is 0.45 45%, probability is 0.45. Here the probability for machine 2 is 0.35.

For machine 3, it is 0.20. The first machine not only stands first in its capacity of production, but in efficiency also. Only 2% of the output of machine 1 is defective okay. So probability of defective output/machine 1 is=0.02 okay. Only 2% of the output of machine 1 is defective, 3% of the total production of machine 2 is found to be defective. So this is let me write okay, this is 2, then we have 3 then we have 4 okay.

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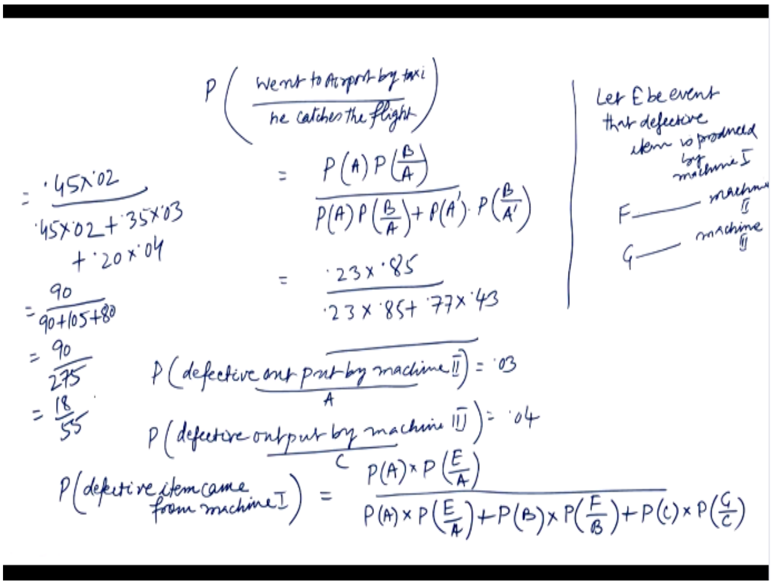

So similarly probability of defective output by machine 2 okay given A okay is=0.03 and probability of defective output by machine 3 okay given C okay is=0.04 okay. So this is probability of defective output by machine 1 given A okay, A is the event denoting the production by machine 1. So this is 0.02 and then this probability of defective output by machine 2 given B is=0.03 and probability of defective output by machine 3 given C is=0.04.

Now we have to find during the process of quality control if the inspector finds a defective item, what is the probability that it has been manufactured by machine 1 okay? So probability that the defective item came from machine 1 okay. So it is=probability of a defective item from machine 1/probability of defective item okay. So probability of A*probability of a defective item given A okay from machine 1.

Probability of defective item from machine 1 given A/PA*probability of defective item from machine 1 given A, so we can call them as E, F, G, probability of defective item from machine 1 is the event E okay, probability of defective let us say E denotes the defective item from machine 1 okay. You can say like that let E be the event that defective item is produced by machine 1.

Similarly, F be the event that the defective items produced by machine 2 and G be the event that it is produced by machine 3 okay. So then you can write here. In short, we can write like this. Probability of E given A okay, probability of E given A+probability of B*probability of F given B and then probability of C*probability of G given C okay. So this will be equal to then PA this will be=0.45*0.02 okay, 0.45*0.02/0.45*0.02 okay for the first term.

Then, 0.35*0.03+0.20 okay 20% 0.20*0.04. So this will be equal to we can simplify this, this 45*2 that is 90/90+105+80. So we get 90/275 which will be=18/55 okay. So this is the probability that defective item came from machine 1. Similarly, we can find the probability that the defective item has been manufactured by machine 2 okay.

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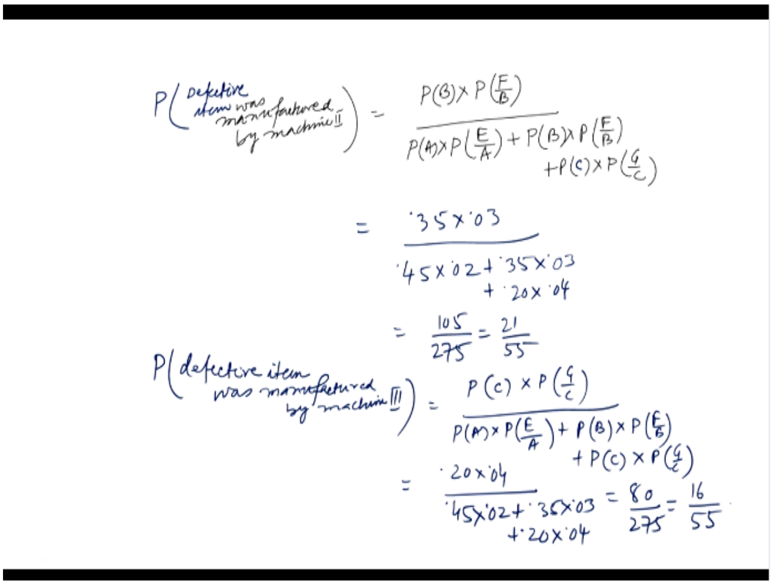

So that we can also find probability that the defective item came from machine 2 okay. So we have probability that defective item machine 2. So this will be=PB*P F/B/PA*P E/A+PB*P F/B+PC*P G/C. So this is=PB, PB is=0.35 and the probability of that it produces a defective item in the machine 2 that is 0.03 okay. So 35 0.35*0.03/0.45*0.02+0.35*0.03+0.20*0.04 and this is=105/275.

So we get 21/55 and similarly probability that defective item was manufactured by machine 3 can be found is=PC*P G/C/PA*P E/A+PB*P F/B+PC*P G/C. So this is=0.20 okay, PC is 0.20*P G/C is 0.04/0.45*0.02+0.35*0.03+0.20*0.04. So this is 80/275. So this will be 16/55 okay. So this is how we find the probabilities in this example.

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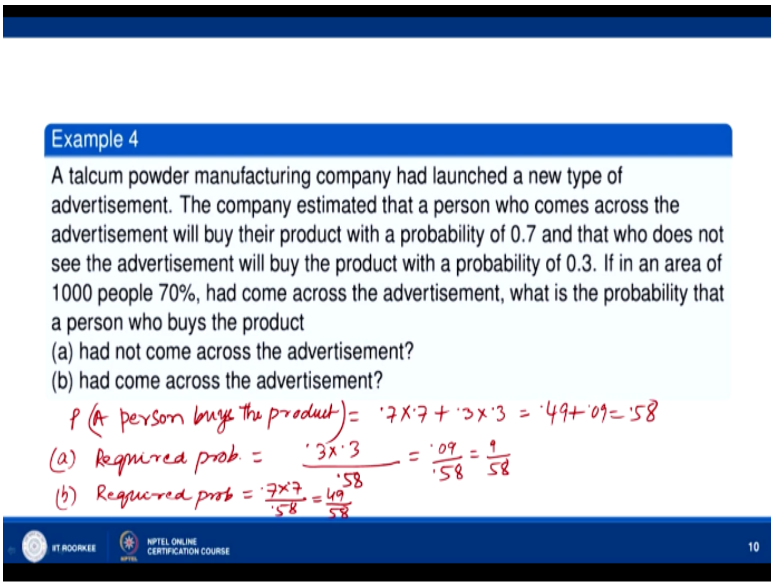

Now let us say a talcum powder manufacturing company had launched a new type of advertisement. The company estimated that a person who comes across the advertisement will buy their product with a probability of 0.7 okay. So the person who has come across the advertisement buys their product with a probability of 0.7 and that who does not see the advertisement will buy the product with a probability of 0.3.

If in an area of 1000 people, 70% had come across the advertisement, what is the probability that a person who buys the product had not come across the advertisement okay, so what is the probability that a person who buys the product okay had not come across the advertisement, had come across the advertisement. So first we find the probability that a person buys the product okay.

A person buys the product, probability that a person buys the product, so the company estimated that a person who comes across the advertisement will buy their product with the probability of 0.7 okay and that who does not see the advertisement will buy the product with the probability of 0.3, 70% had come across the advertisement, so first we assume that the 70% have seen the advertisement.

So the probability that the person has seen the advertisement is 0.7 okay*if he has seen the advertisement he buys the product with the probability of 0.7 okay+the person has not seen the advertisement that probability is 0.3 okay*he buys the product with probability 0.3 okay. So this is 0.49+0.09, so we have 0.58 okay. So probability that a person buys the product okay is 0.7*0.7 where 0.7 is a probability that he has seen the advertisement.

And he has seen the advertisement then he buys the product is the probability again so that probability is also 0.7 and then he does not see the advertisement that probability is 0.3 and when he does not see the advertisement and he buys the product that probability is 0.3, so 0.7*0.7 is 0.49, 0.3*0.3 is 0.09, so it is 0.58 okay. Now it is given that he buys the product and then if he had not come across the advertisement, so for the part a, the required probability is okay.

He had not come across the advertisement that means 0.3*0.3 still he buys the product okay. So 0.3*0.3/0.58, so this will be 0.09/0.58 that is 9/58 okay and in the part b, had come across the advertisement so that probability is 0.7*0.7/0.58, so that will be 49/58.

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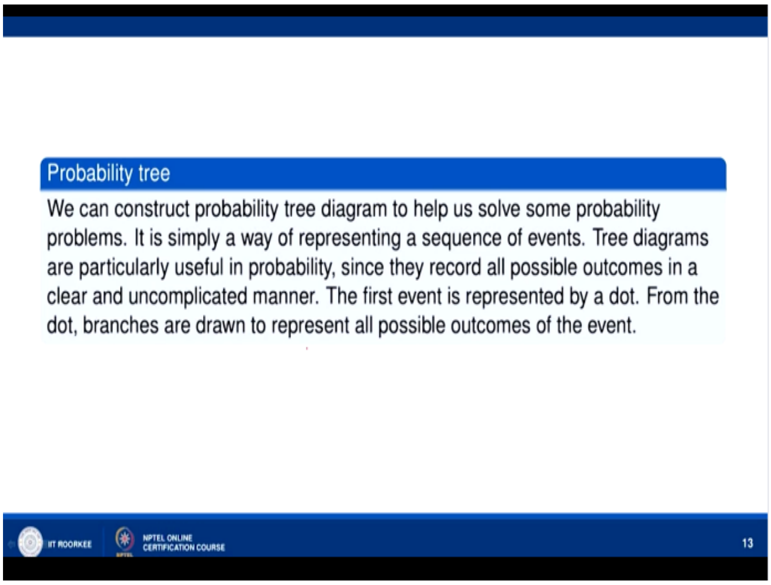

Now we discuss the probability networks or we also call it probability tree. We can construct probability tree diagram to help us solve some probability problems. It is simply a way of representing a sequence of events. Tree diagrams are particularly useful in probability since they record all possible outcomes in a clear and uncomplicated manner. The first event is represented by a dot. From the dot, branches are drawn to represent all possible outcomes of the event.

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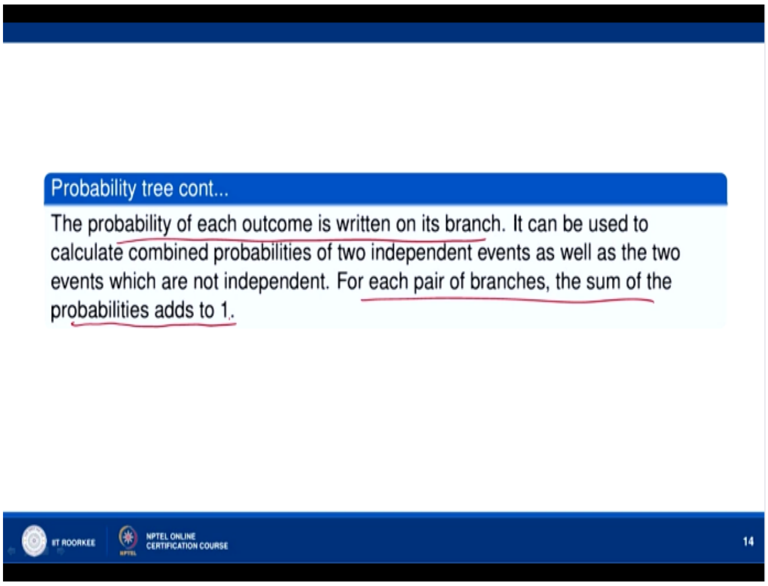

The probability of each outcome is written on its branch okay. It can be used to calculate combined probabilities of two independent events as well as the two events which are not independent. For each pair of branches, the total of the probabilities adds to 1. For each pair of branches, the sum of the probabilities must be=1.

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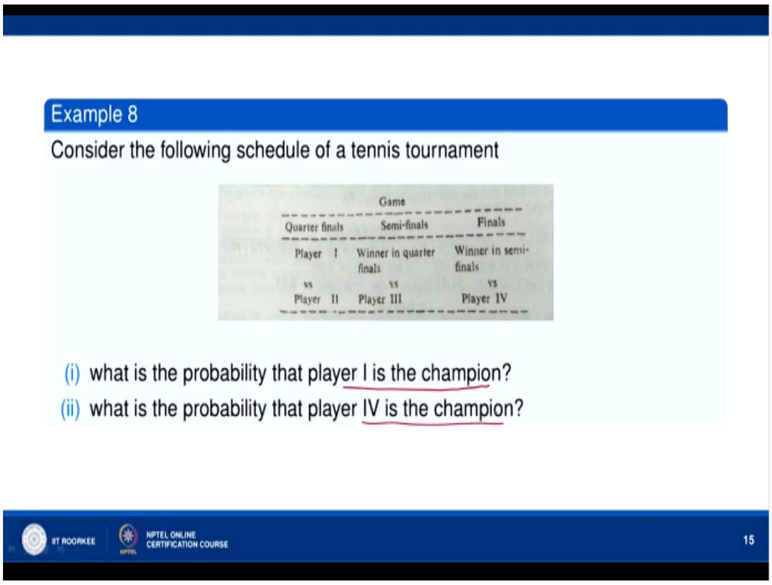

Now let us consider the following schedule of a tennis tournament. In the quarter finals, player I plays player II okay whosever is the winner he plays player III in the semifinals and then whosever is winner he plays with player IV in the finals okay. Now we have to find the probability that player I is the champion in part a and then we have to find the probability that player IV is the champion okay. So let us write probability diagram okay probability tree.

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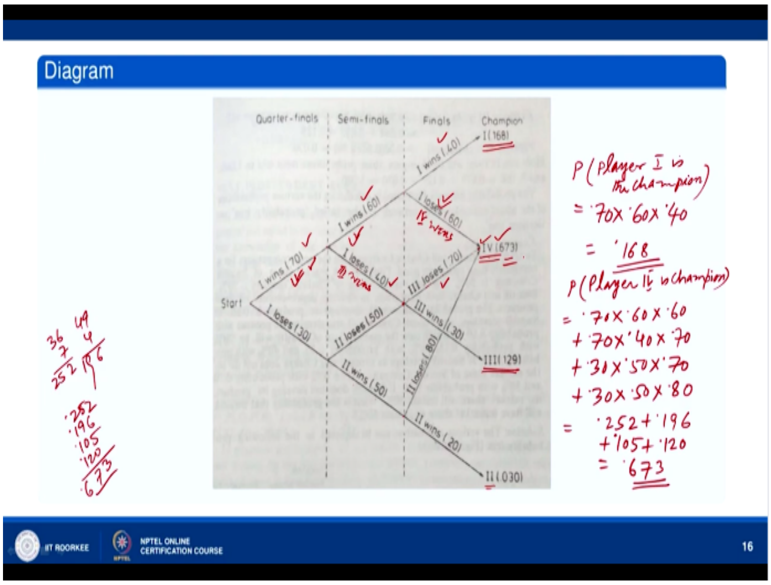

So this is the probability tree. I plays in the quarter finals with the player II. If I wins okay then he plays the player III okay. If I wins okay he plays the player IV okay. Again, if he wins, he becomes the champion okay. So player I plays with player II he wins with probability 0.70 okay and then he wins again if he wins okay he wins with probability 0.60 and again when he plays with player IV, he wins with probability 40.

Now if I loses okay then II wins the quarter final okay. So if I loses then II plays with player III in the semifinals. Now if II loses okay then he loses with probability 0.5, he wins with probability 0.5. You can see the total of both the branches, the sum of the both the probabilities is 0.70+0.30 is 1 okay for each player of branches. So it is 0.50 here, 0.50 there.

Now II player after he wins in the semifinals, he plays with player IV in the final okay. If he wins, he wins with probability 0.20 and he then becomes the champion okay. Now if II loses in the semifinals okay if II loses in the semifinals, then III wins okay, III when plays with player IV in the finals okay. If III loses IV becomes the champion okay. So III loses means IV becomes the champion.

Now here if the II wins okay, then he plays with player IV okay. At this point, he plays with player IV. If he loses, the IV becomes the champion okay. So IV becomes the champion. Now here if I wins okay then he plays with player II okay in the semifinals. If he loses, then II wins the semifinals, then II plays player IV okay, II plays player IV in the finals okay if II loses okay then here I wins.

Then, in the quarter finals if I wins then he plays with the player III okay, he plays with the player III in the semifinals. So here I loses means III wins. Now here player number III plays with player number IV in the finals okay. Now if III loses, IV becomes the champion and we reach here okay. If III wins against the player number IV then III becomes the champion okay.

Here let us look at this branch. If I wins against player II okay, here if I wins against player III, here I plays with player IV okay. If I loses IV wins okay. So IV becomes the champion okay. Now let us find the probability that I is the champion okay. So probability that player I is the champion, so let us go to take this branch okay. So this is 0.70*0.60*0.40 okay and it is 7*6=42, 42*4=168, so this is 168 okay.

So player I is the champion and the probability of player I becoming the champion is 0.168. Now player IV is the champion, so let us find this probability. Let us follow all the paths that lead to IV becoming champion okay. First path is this one, this path, this and this okay. So this path so along this path, we have 0.70*0.60*0.60 okay. So we have considered this path, I wins, I wins, I loses okay.

Now I wins, I loses, III loses, look at this path so 0.70 we are considering this path, this path, this path, this path. So 0.70, 0.40, 0.70, 0.70*0.40*0.70 okay, so I have considered this path okay, this path now I have considered this path, I will now consider this path okay. So 0.30*0.50*0.70 okay. Now you consider this path okay, so 0.30, 0.50 and then 0.80. So there are 4 paths which lead to IV being the champion okay.

So the probabilities are 6*6=36, 36*7=0.252 okay first one is 6*6=36, 36*7 is 252. Then, 7*7=49, 49*4=196 okay. Then, we have 7*5=35, 35*3 is 105, so 0.105 and then we have 5*8=40, 40*3 is 0.120 okay. So we have 0.252, 0.196, 0.105 and 0.120. So we have 0.673 okay, so this is the answer, 0.673 okay.

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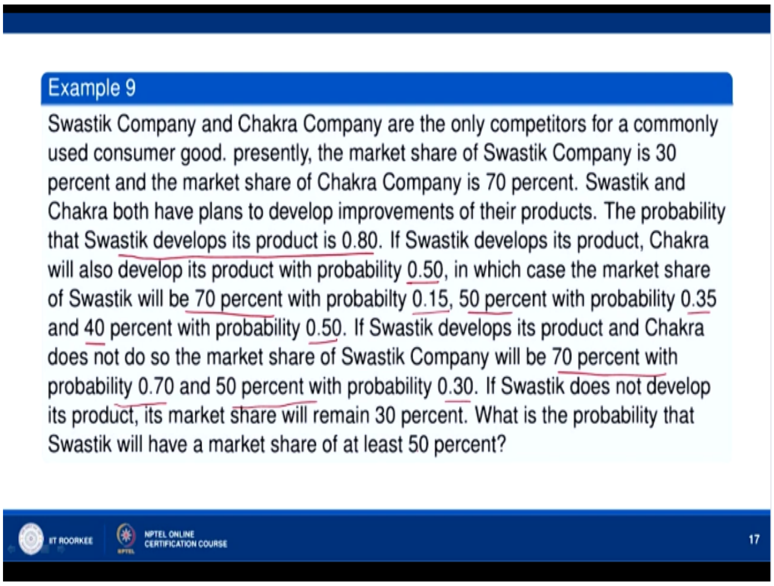

Now let us go to one more question on probability networks. Swastik Company and Chakra Company are the only competitors for a commonly used consumer good. Presently, the market share of Swastik Company is 30% and the market share of Chakra Company is 70%. Swastik and Chakra both have plans to develop improvements of their products. The probability that Swastik develops its product is 0.80 okay.

If Swastik develops its product, Chakra will also develop its product with probability 0.50 in which case the market share of Swastik will be 70% with probability 0.15, 50% with probability 0.35 and 40% with probability 0.50. If Swastik develops its product and Chakra does not do so the market share of Swastik Company will be 70% with probability 0.70, 50% with probability 0.30.

If Swastik does not develop its product, its market share will remain 30%. What is the probability that Swastik will have a market share of at least 50%?

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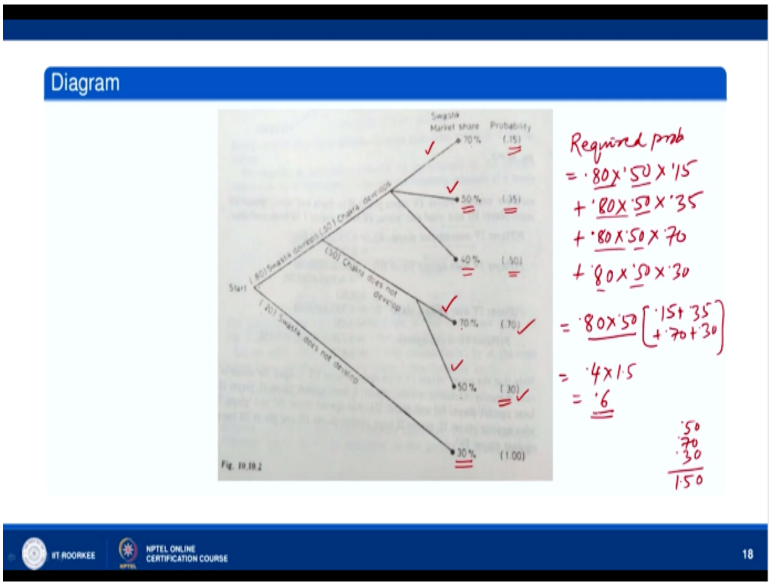

So let us see. We have drawn the tree diagram here. So start, Swastik develops its product okay Swastik develops its product, the probability of Swastik developing its product is 0.80 okay, so this is 0.80 and Swastik does not develop its product that probability is 0.20 okay. Now Swastik developed its product, suppose Chakra also develops okay. Chakra develops with probability 0.50 okay.

Chakra develops with probability 0.50, so suppose Chakra also develops this product then the market share of Swastik Company will be 70% with probability 0.5, 50% with probability 0.35, 40% with probability 0.50. If Chakra does not develop okay, here Chakra develops then the probabilities of Swastik are given. If Chakra does not develop its product, the probability of Chakra not developing its product is again 0.50.

Then, the market share if Chakra does not develop its product Chakra in which case we have this okay. So if Swastik develops its product and Chakra does not do so, the market share of Swastik Company will be 70% with probability 0.70, 50% with probability 0.30. So 70% with probability 70 and 50% with probability 30, that is the market share of Swastik okay and if Swastik does not develop its market share remains 30%.

So we have this probability tree diagram. Now let us see what we have to find. What is the probability that Swastik will have a market share of at least 50% okay? So let us see which route give us the market share of Swastik as>or=50% okay. So one route is this, we move directly and reach here, it is 70%. We go this route and come here okay, this is 50% and when you follow this path, it is 40% so we will not consider this.

Here we can consider this path, this is 70% okay and again this path this is 50%. So this path, this path, this path and this okay, 4 paths give us the market share of Swastik as >or=50%. So let us first consider this path okay, path 1 okay. So the probability, required probability is 0.80*0.50*0.15. Then, we again go this way, this path, this way then 0.80*0.50*0.35. Then, we come this way 0.80 0.50 and then 0.70 okay, 0.80, 0.50, 0.70.

And then 0.80*0.50*0.30 okay because this is having probability 0.30 okay. So how much we get here? 0.80, 0.50 is common okay in all the 4 terms, so we can take that as common and then we have 0.15+0.35+0.70+0.30 okay. So how much is that? This is how much? 8*5=40 okay, so we have 0.4 okay. This is 0.80*0.50 which means it is 0.4 okay and here how much we have 0.15+0.35 is 0.50, 0.50+0.70 is 0.50+0.70+0.30 okay, so this is 1.5 okay.

So this is 1.5, so we get this is 60, so that means 0.6. So the required probability is 0.6. So this is how we can find the probability. You can see that these are very complex questions but when we draw the probability tree diagram or we consider the probability network, it becomes simple to calculate the required probability. So that is the benefit of the probability tree diagram. With this I would like to end my lecture. Thank you very much for your attention.