Advanced Engineering Mathematics
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 42
Basic Concepts of Probability

Hello friends. Welcome to my lecture on basic concepts of probability. Modern probability theory is based on the axiomatic definition of probability. We define a probability model and state the axioms which any reasonable definition probability must satisfy. Let us consider a random experiment with the number of possible outcomes. The number of possible outcomes is either finite or infinite. Let N denote the number of possible outcomes

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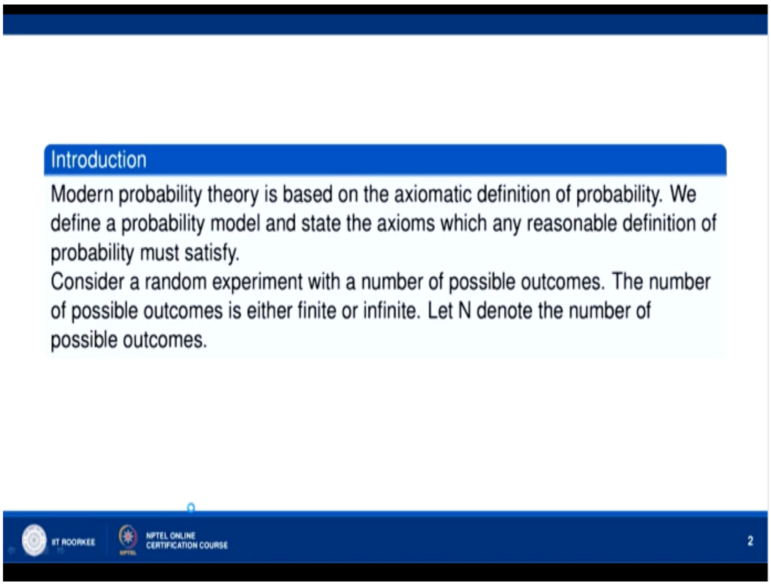

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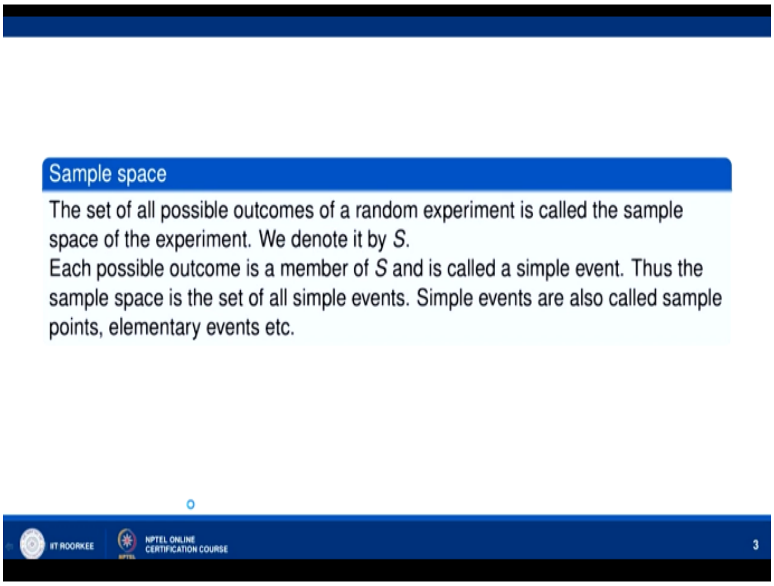

The set of all possible outcomes of a random experiment is called the sample space of the experiment. We shall denote it by S. Each possible outcome is a member of S and is called a simple event. Thus, the sample space is the set of all simple events. Simple events are also called sample points or elementary events etc.

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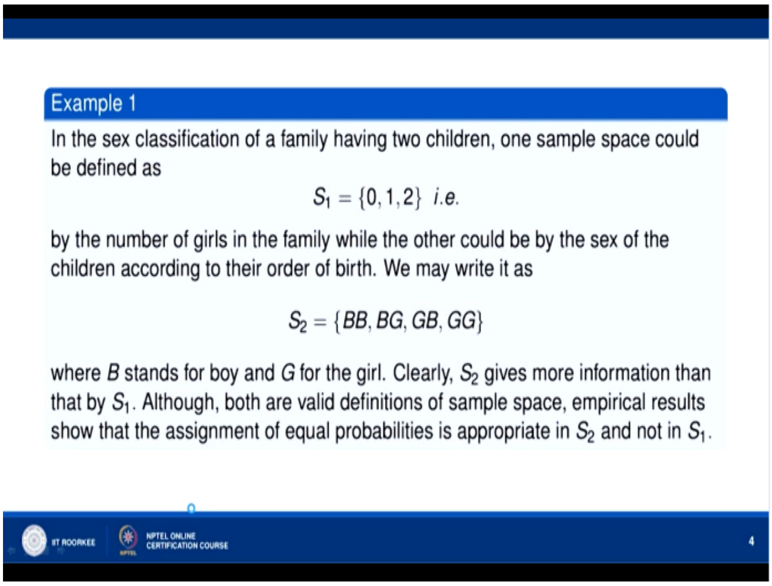

In the sex classification of a family having two children, one sample space could be defined as S1=the set of elements 0, 1, 2 that is by the number of girls in the family while the other could be by the sex of the children according to their order of birth. We may write it as S2=the set consisting BB, BG, GB, GG where B stands for boy and G for the girl. Clearly, S2 gives more information than that by S1.

Although, both are valid definitions of sample space, empirical results show that the assignment of equal probabilities is appropriate in S2 and not in S1.

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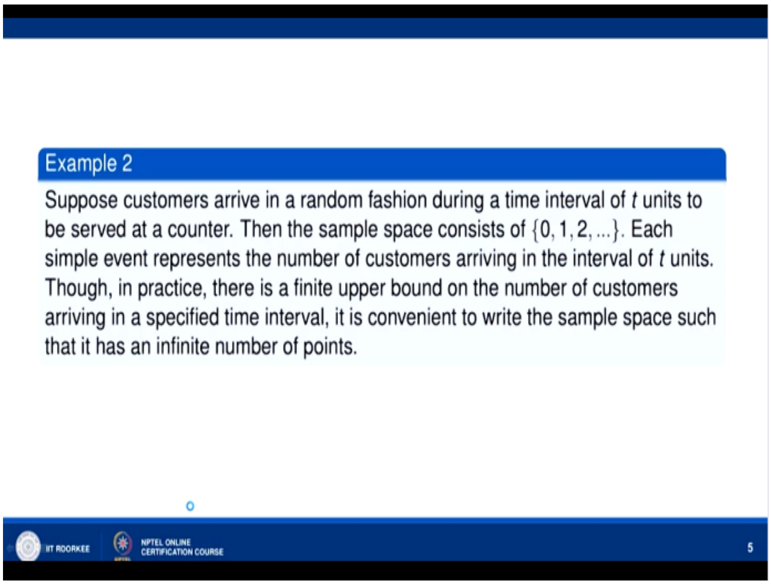

Suppose customers arrive in a random fashion during a time interval of t units to be served at a counter. Then, the sample space consists of 0, 1, 2 and so on. Each simple event represents the number of customers arriving in the interval of t units. Though, in practice, there is a finite upper bound on the number of customers arriving in a specified time interval, it is convenient to write the sample space such that it has an infinite number of points.

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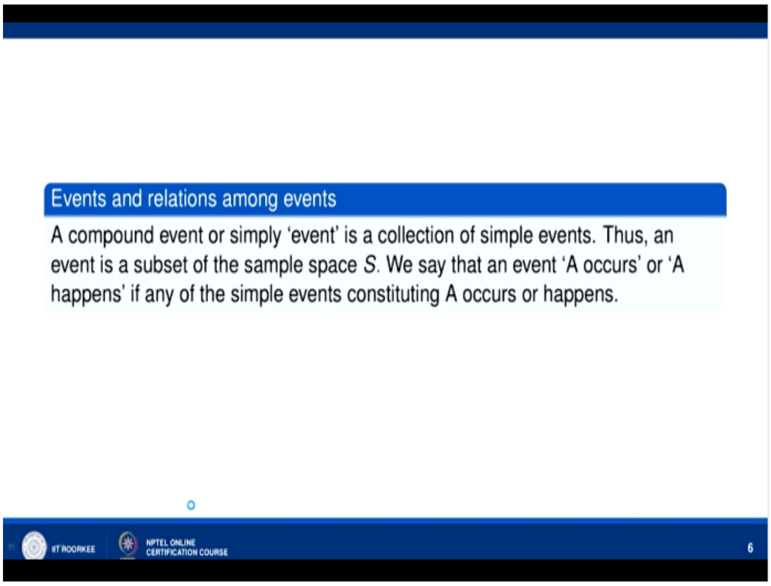

A compound event or simply event is a collection of simple events. Thus, an event is a subset of the sample space S. We say that an event A occurs or A happens if any one of the simple events constituting A occurs or happens.

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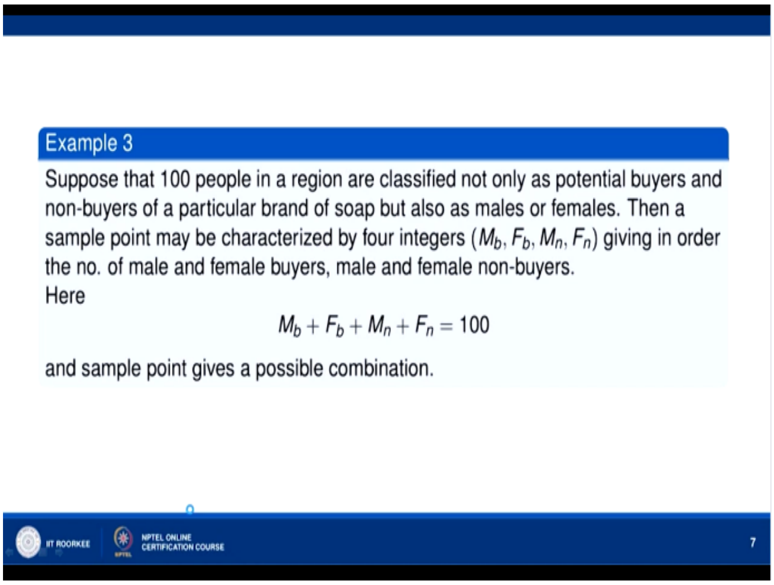

Suppose that 100 people in a region are classified not only as potential buyers and non-buyers of a particular brand of soap but also as males or females. Then, a sample point may be characterized by 4 integers Mb, Fb, Mn, Fn giving in order the number of male and female buyers, male and female non-buyers. Here Mb+Fb+Mn+Fn is=100 and sample point gives a possible combination.

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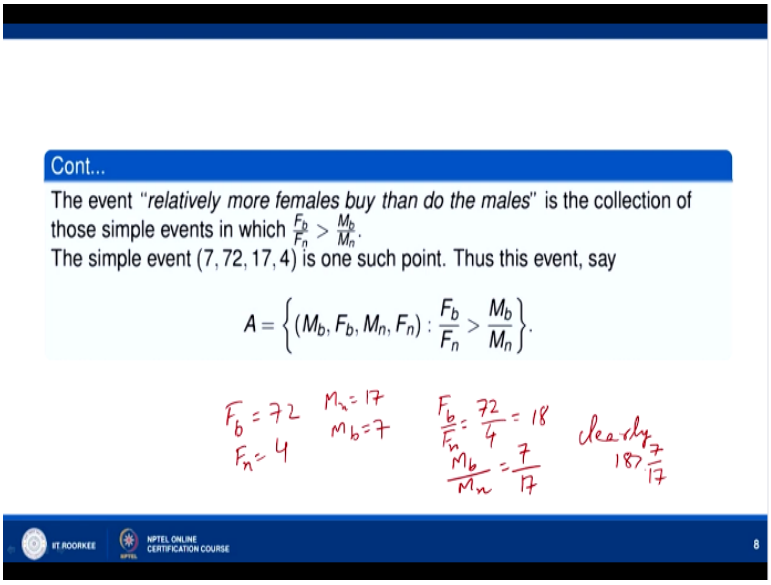

The event relatively more females buy than do the males is the collection of those sample points in which Fb/Fn is>Mb/Mn. Fb/Fn gives the relative frequency of the female buyers, so Fb/Fn>Mb/Mn. The simple event 7, 72, 17, 4 is one such point because here the value of Fb=72, Fn=4, Mn=17 and Mb=7. So you can see that Fb/Fn is=72/4=18 while Mb/Mn is=7/Mb/Mn okay Mb is=7/Mn, Mn is=17.

So clearly 18 is>7/17. That is Fb/Fn is>Mb/Mn okay. So 7, 72, 17, 4 is a one such sample point.

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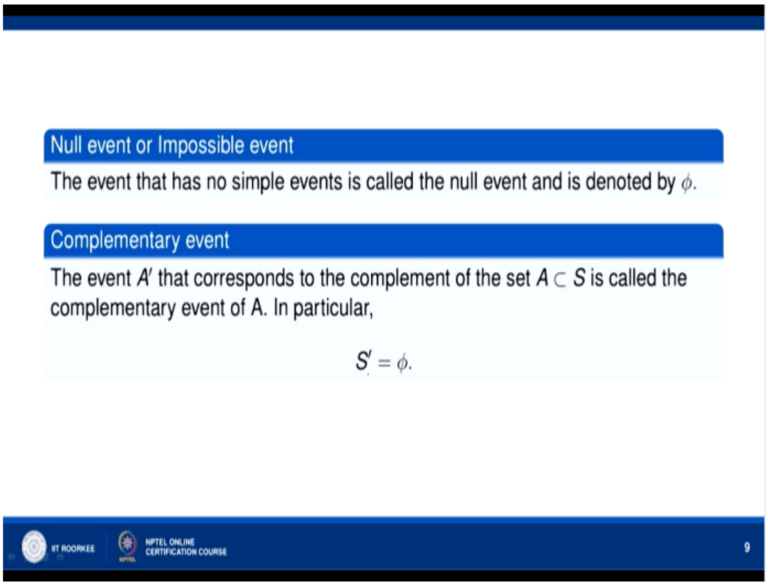

Now the event that has no simple events is called the null event okay and is denoted by phi. It is also called impossible event. The event A that corresponds to the complement of the set A subset of S is called the complementary event of A and so in particular if S denotes the sample space, then S complement, S complement is=phi.

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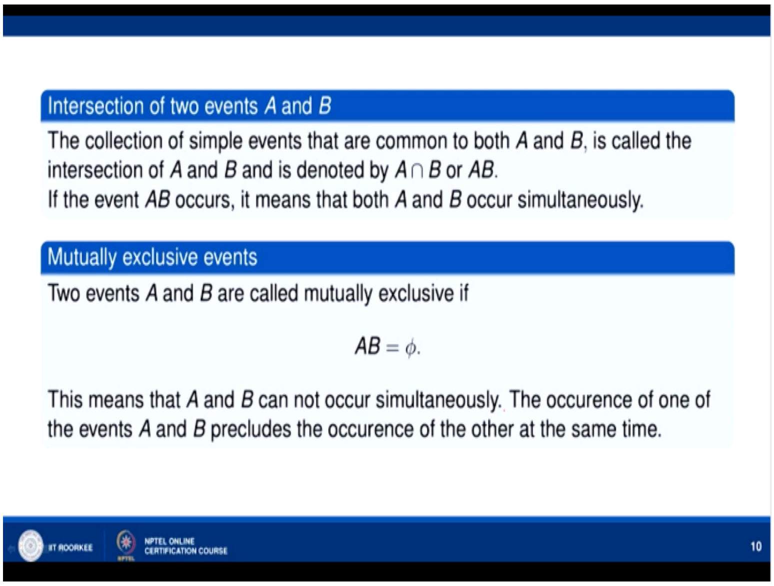

Now the collection of simple events that are common to both A and B is called the intersection of A and B and is denoted by A intersection B or we also denote it by AB. If the event AB occurs, it means that both A and B occur simultaneously. Two events A and B are called mutually exclusive if A intersection B=phi. This means that A and B cannot occur simultaneously okay.

Suppose you toss a coin okay, then if A is the event that the head appears and B is the event that the tail appears, then the event A and B are mutually exclusive. When you toss the coin either head will come or tail will come. If head comes tail cannot come, if tail comes head cannot come. So the two events A and B are mutually exclusive. So A and B cannot occur simultaneously.

The occurrence of one of the events A and B precludes the occurrence of the other at the same time.

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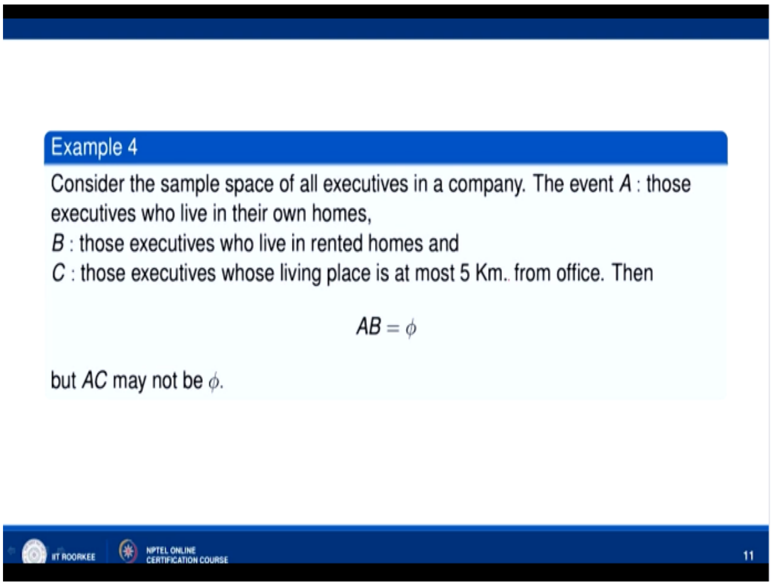

Consider the sample space of all executives in a company. The event A is the event that those executives who live in their own homes okay. A is the event that those executives who live in their own homes, B is the event of those executives who live in rented homes and C denotes the event that those executives whose living space is at most 5 kilometers from office okay. Then, A intersection B okay.

A is the event those executives who live in their own homes and B is the event those executives who live in rented homes. So A intersection B is=phi but A intersection C may not be phi because A is the event that those executives who live in their own homes and C is the event those executives whose living space is at most 5 kilometers from office. So there may be some executives whose living space is at most 5 kilometers from office and they live in their own homes okay. So AC may not be=phi.

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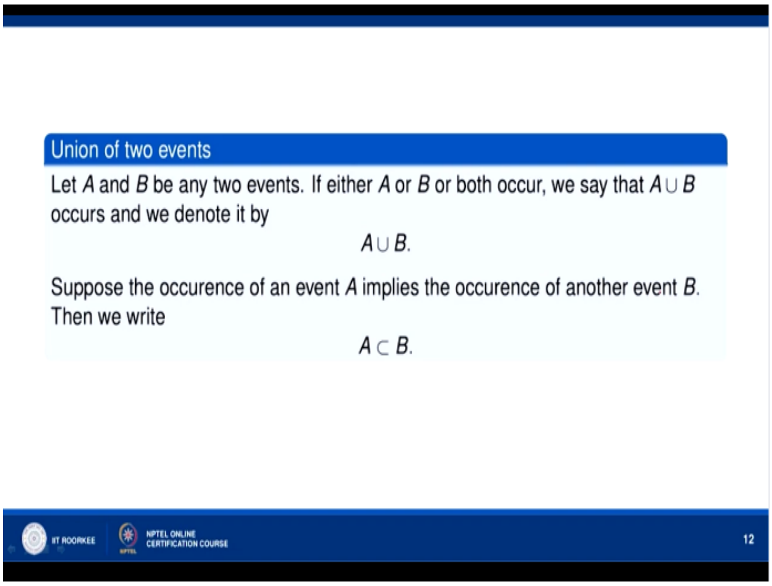

Now let us consider the union of two events. Let A and B be any two events. If either A or B or both occur, we say that A union B occurs and we denote it by A union B. Suppose the occurrence of an event A implies the occurrence of another event B. Then, we say that A is subset of B.

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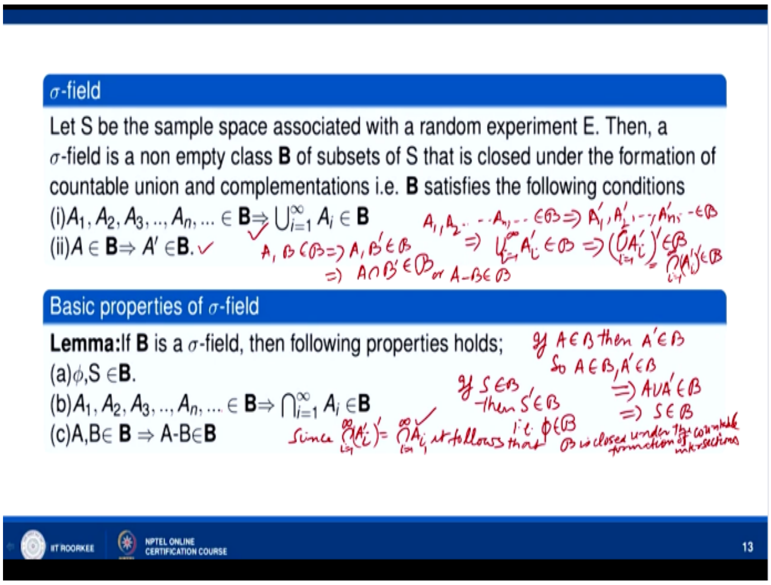

Now let us define the sigma field. Let S be the sample space associated with a random experiment E. Then, a sigma field is a nonempty class B of subsets of S that is closed under the formation of countable unions and complementations. That is the set B, class B satisfies the following conditions. If A1, A2, A3, An and so on are the sets belonging to the class B, then they are countable union.

Union i=1 to infinity Ai also belongs to B and if A belongs to B then it is complement, A complement also belongs to B. Now from these definitions of sigma field, we can drive the basic properties of sigma field. If B is a sigma field, then following properties holds okay, phi and S belong to B. So let us first show the part a of the lemma. We have to show that phi, S belongs to B.

So from the condition 2 from this condition okay if A belongs to B then A complement belongs to B okay. So since B is a nonempty set okay, nonempty class okay, so there must be some set at least there in B, so let us say the set is A. So if A belongs to B then its complement also belongs to B. So A belongs to B, A complement belongs to B implies that A union A dash belongs to B okay.

A union A dash is S okay, so this implies that S belongs to B okay. Now if S belongs to B, then S complement also belongs to B okay. If S belongs to B, then S complement also belongs to B. S complement means that is phi belongs to B because S is the whole space okay. Now A1, A2, An belongs to B implies that intersection i=1 to infinity i belongs to B that means B is closed under the formation of countable intersections okay.

So this we can prove as A1, A2 and so on An belongs to B implies that their complements belong to B okay. Now B is closed under the formation of countable unions, so union i=1 to infinity Ai complement belongs to B okay by this property. Now this implies that by second property that if A belongs to B, A complement belongs to B. So union Ai i=1 to infinity complement okay complement of this belongs to B.

By De Morgan's law, union Ai dash dash is=intersection Ai dash dash i=1 to infinity okay. Now A complement complement is A okay, so this is intersection i=1 to infinity Ai okay belongs to B okay. So that is to say since intersection Ai complement complement is=intersection Ai i=1 to infinity okay. It follows that B is closed under the formation of countable intersections okay.

Now if AB belongs to B then A-B belongs to B okay. So A, B belongs to B implies that A, B dash belong to B okay and A, B dash belongs to B means by this property that the class B is closed under the formation of countable intersections. From here, we can say that A intersection B dash belong to B okay. A intersection B dash is A-B or now here you may say that this is countable intersection, how we can say about finite intersection?

We can take say A1, A2, A3 and so on An belong to B. So you can take say finite number of them and rest you can take as S okay.

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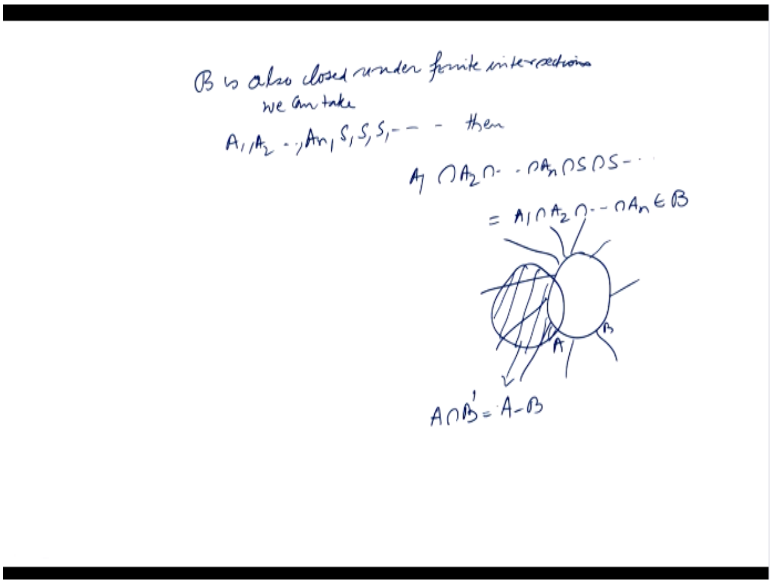

If B is also closed under finite intersections, we can take A1, A2 and so on An finite number of them S An+1 as we can as S An+2 we can take as and so on. Then, intersection A1 intersection A2 intersection An intersection S intersection Sn so on. This will be=A1 intersection A2 intersection An. So B is also closed under the formation of finite intersections okay. So using this property, we can say that when A and B belong to B then by second condition A and B dash belong to B and so A intersection B dash belong to B.

But A intersection B dash is=A-B. We can see from here. Suppose this is our A, this is B okay, then so A intersection B dash okay. B dash is this complement of B okay. This is B dash and clearly B dash intersection A, A intersection B dash is this which is nothing but A-B okay. So A intersection B dash is=A-B okay. So this is how we prove these properties of the sigma field.

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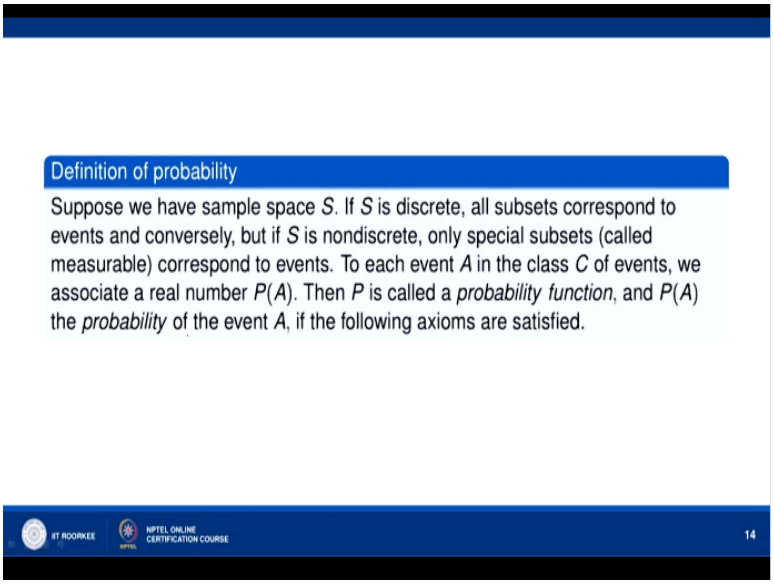

Now let us go to definition of probability. Suppose we have a sample space S. If S is discrete, all subsets correspond to events and conversely and if S is nondiscrete, only special subsets called measurable correspond to events. To each event A in the class C of events, so let us consider class C of events. C is actually sigma field here, the sigma field which we have defined just now that we are taking as C here.

So C is the sigma field, so to each event A in the class C of events, we associate a real number PA okay. So you take any event belonging to the class C associate a real number PA with that then P will be called probability function and PA the probability of the event A if the following axioms are satisfied.

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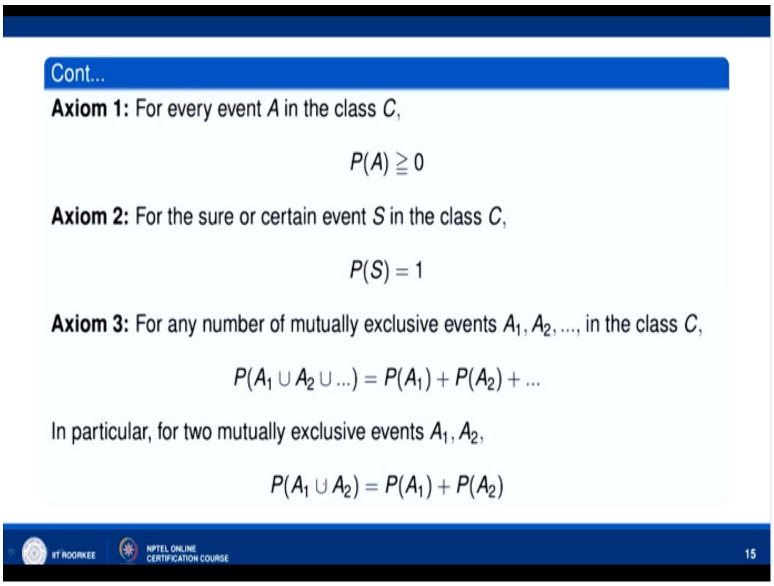

First axiom is that for every event A in the class C, we must have PA>=0 okay. So the second one is for the sure or certain event S okay, S is the whole space so it is sure to happen, so for the sure or certain event S in the class C, PS must be=1. For any number of mutually exclusive events A1, A2 and so on in the class C, P A1 union A2 and so on is=PA1+PA2 and so on.

In particular, for two mutually exclusive events, A1, A2 P A1 union A2=PA1+PA2. So we will say that P is the probability function provided these conditions are satisfied.

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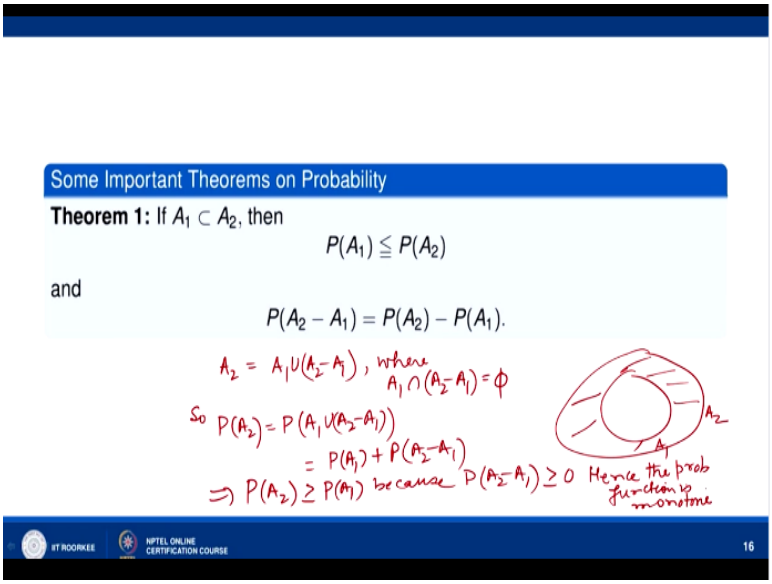

Now let us look at some important theorems on probability. If A1 and A2 are any two events belonging to the class C of events and A1 is subset of A2, then PA1 is<=PA2. So let us see, by Venn diagram we can say this is say A1 and this is say A2 okay. So A1 is contained in A2 okay A1 is contained in A2, so we can say that A2 is=A1+A2-A1 or you can put a union here okay A2=A1 union A2-A1 okay.

Moreover, that A1 and A2-A1 are mutually exclusive because their intersection is phi okay where A1 intersection A2-A1=phi. So probability of A2 will be=probability of A1 union A2-A1 okay. Since A1 intersection A2-A1 is phi, we have PA1+P A2-A1 okay. Let us go to this condition axiom 3, P A1 union A2=PA1+PA2 if A1 intersection A2 is phi okay. So here and you see that A1, A2 are events, so A2-A1 is also an event okay belonging to the class C.

And therefore P A2-A1 will be>=0 okay, so PA2 is>=PA1 because P A2-A1 is>=0. So if A1 is subset of A2 then PA1 is<=PA2. So we can say that the probability function is monotone okay. It satisfies the monotone property.

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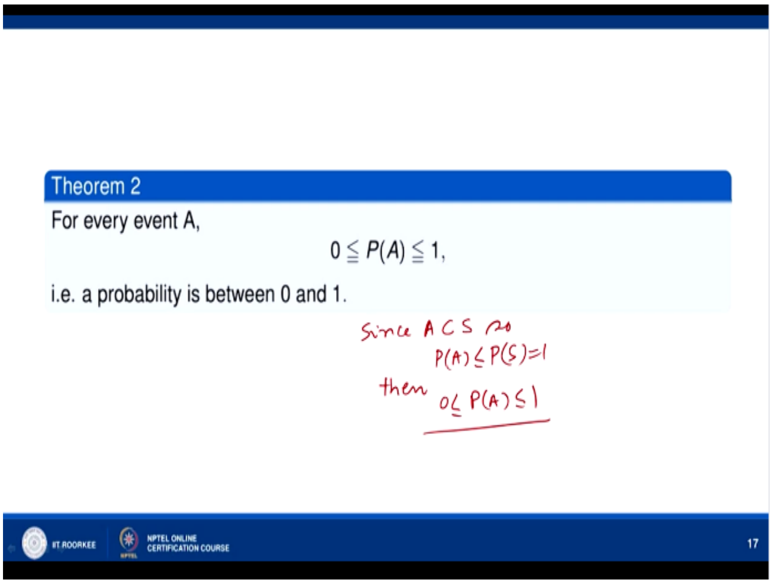

Now for every event A 0 is<=PA<=PS okay since A is subset of S. You take any event A belonging to the class C of events, then A is subset of S. So PA is<=PS okay and PS is=1 okay. This we have already assumed PS=1 in the axiom 2, so you take any event A subset of S then PA is<=PS and therefore 0 is<=PA<=1 okay because you take any event A belonging to the class C of events PA is>=0 okay so 0 is<=PA<=1.

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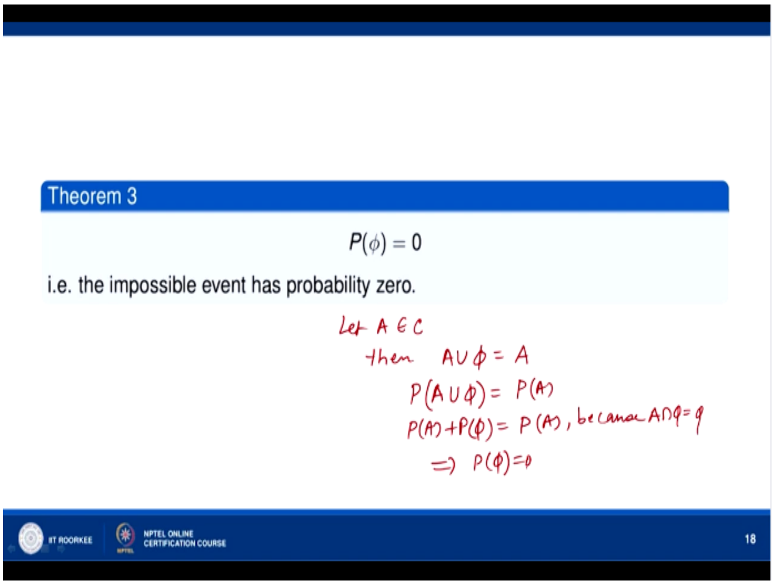

Now probability of phi is=0 okay. So let us prove this. Let A be any event belonging to the class C okay. Then, A union phi=A okay. Since A and phi are mutually exclusive, P A union phi okay is=PA okay. P A union phi is PA+P phi=PA. This is because A intersection phi=phi okay. They are mutually exclusive. So then this implies that P phi is=0 because we can cancel PA both sides, PA is line between 0 and 1, so we can cancel and therefore P phi=0.

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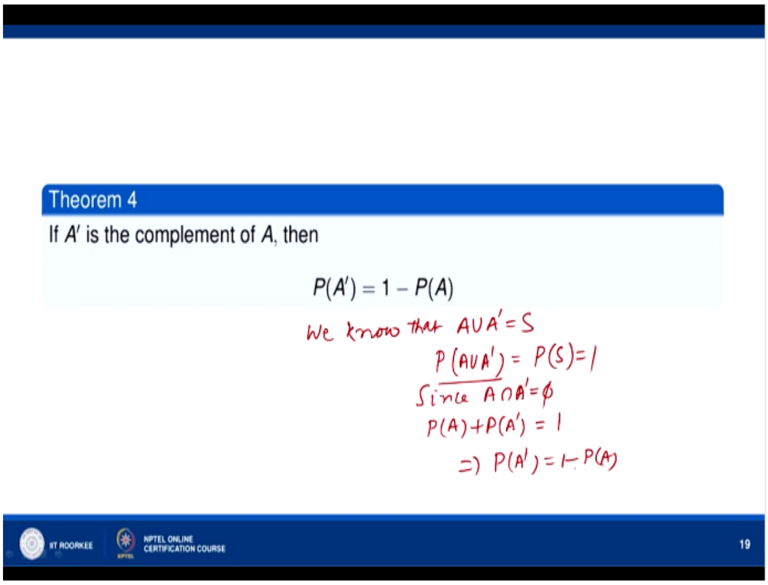

Now let us say prove PA dash=1-PA. So we know that A union A dash=S okay. Then, P A union A dash=PS=1 okay. Since A intersection A dash=phi, A and A dash are mutually exclusive. So PA+PA dash we shall get okay. P A union A dash will give us PA+P A dash this is=1. This implies that PA dash is=1-PA. So probability of the complement of A is=1-PA.

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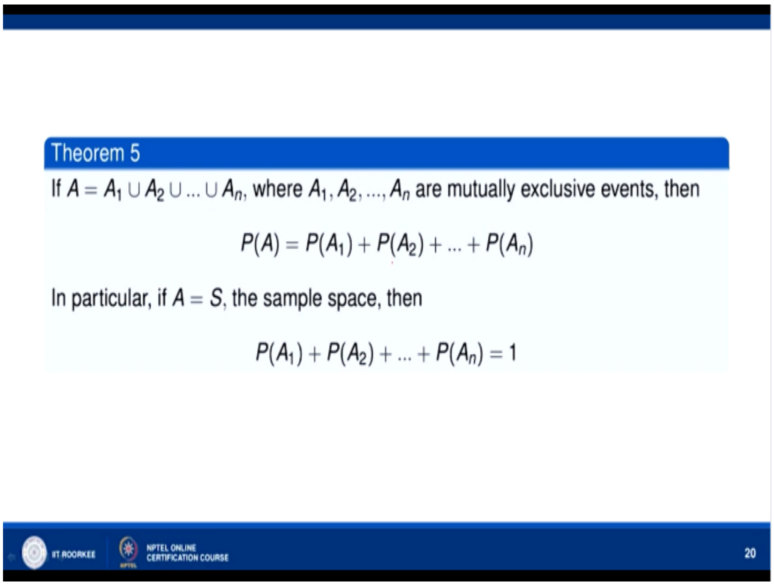

Now let us come to if A=A1 union A2 and so on An where A1, A2, An are mutually exclusive events, then PA=PA1+PA2+PAn. In particular, if A=S the sample space, then PA1+PA2 and so on PAn is=1 okay. So we have this axiom here P A1 union A2=PA1+PA2 okay. So in particular for two mutually in case of two mutually exclusive events, P A1 union A2=PA1+PA2. This we can by mathematical induction, from here we can say that P A1 union A2 and so on union An=PA1+PA2 and so on PAn.

If A=S then PS will be=PA1+PA2 and so on PAn so PS=1 therefore PA1+PA2 and so on PAn=1. Now here how we get in particular for two mutually exclusive events, this gives us PA1+PA2 okay. So this can be shown once you have P phi=0 okay. So we can take after A1, A2, A3, A4, A5, A6 all we can take as phi, then P A1 union A2 union phi and so on will give us P A1 union A2 and P phi=0, so we will get PA1+PA2, so this comes from that.

We can take A3, A4, A5, A6 all as phi and then P A1 union A2 union phi union phi union phi gives P A1 union A2 and we get here PA1+PA2 because of the fact that P phi=0.

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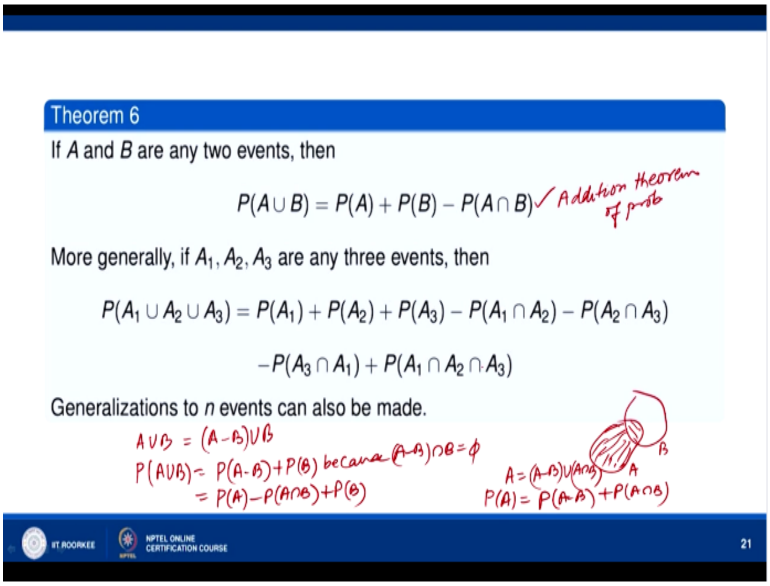

So if A and B are any two events, then P A union B=PA+PB-P A intersection B okay. So let us prove this. We have suppose we have this is our set A, this is another set B okay. Then, A union B can be written as A-B, this is A-B, A-B+B okay, so A-B union B okay. So probability A union B=probability of A-B+probability of B because A-B intersection B is=phi because A-B intersection B is=phi okay.

So now A-B, A-B is A is=A-B union A intersection B. This is A intersection B, so A=A-B union A intersection B and therefore PA=P A-B+P A intersection B because A-B and A intersection B are mutually exclusive. So this gives us P A-B=PA-P A intersection B. So this is=PA-P A intersection B+PB okay. So P A union B=PA+PB-P A intersection B. This is also called the addition theorem of probability okay.

More generally if A1, A2, A3 are any 3 events, then P A1 union A2 union A3 is=PA1+PA2+PA3-P A1 intersection A2-P A2 intersection A3-P A3 intersection A1+P A1 intersection A2 intersection A3. Let us prove this result because we shall need it later on.

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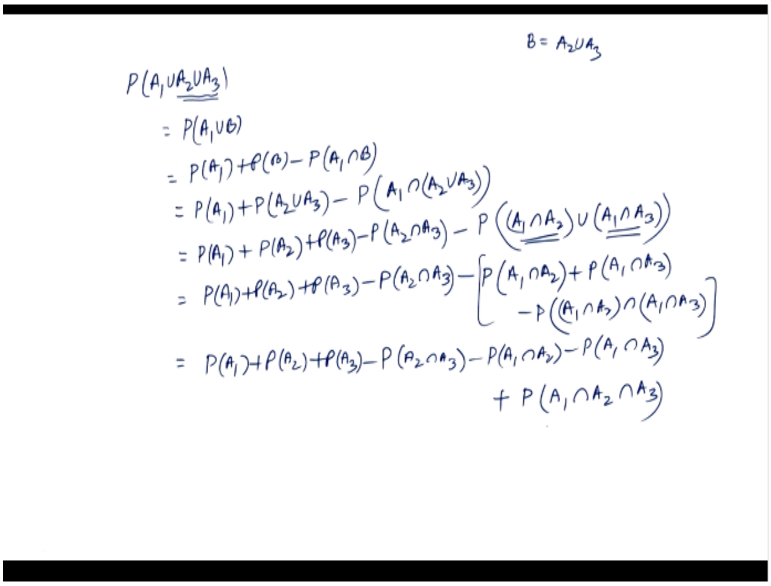

So we have P A1 union A2 union A3 okay. We can use the result for 2 events. So let us consider A2 union A3 as B, let B=A2 union A3. So then we will have P A1 union B. We have already proved that P A union B is PA+PB-P A intersection B. So let us apply that result. So we have PA1+PB-P A1 intersection B, A1 is=sorry B is=A2 union A3, so PA1+P A2 union A3 we have and here we will have P A1 intersection A2 union A3 okay.

Now again the result P A union B=PA+PB-P A intersection B. We then write PA1+this is PA2+PA3-P A2 intersection A3 and here this will be=A1 intersection A2 union A3 will be A1 intersection A2 using the property of set theory union A1 intersection A3 okay. So this will be=PA1+PA2+PA3-P A2 intersection A3. Now we have one event here, one event here okay. So P A1 intersection A2+P A1 intersection A3-P A1 intersection A2 intersection A1 intersection A3 okay.

So this gives us PA1+PA2+PA3-P A2 intersection A3-P A1 intersection A2-P A1 intersection A3. Now A1 intersection A2 intersection A1 intersection A3 is A1 intersection A2 intersection A3 okay. So this is how we prove the result in case of 3 events okay. So generalizations to n events can be then similarly.

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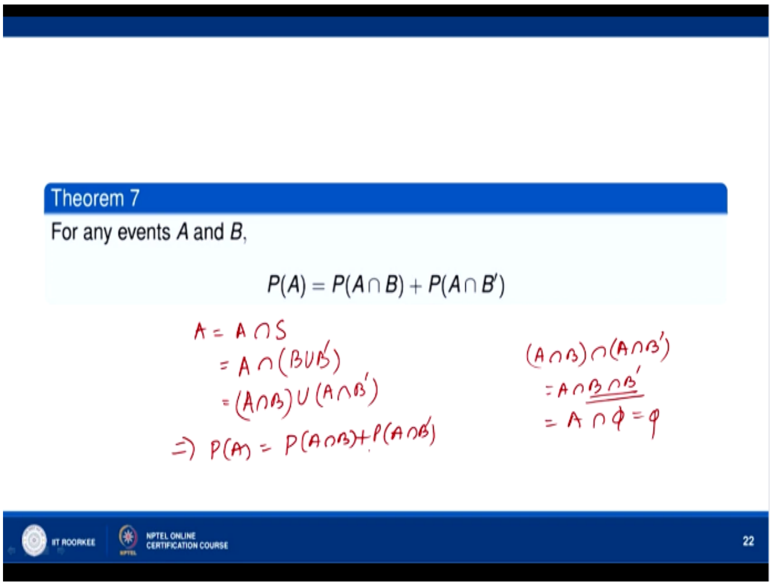

Now for any events A and B, PA=P A intersection B+P A intersection B dash okay. So let us prove this. We can say A is=A intersection S okay, S is the sample space okay and this I can write as A intersection B union B dash okay, B is any other event, so B union B dash is=S. Now this is=A intersection B union A intersection B dash okay. Now A intersection B intersection A intersection B dash=phi okay.

A intersection B intersection A intersection B dash is=A intersection B intersection B dash, B intersection B dash is phi, so we have A intersection phi, so this is=phi and therefore probability of A is=probability of A intersection B+probability of A intersection B dash okay.

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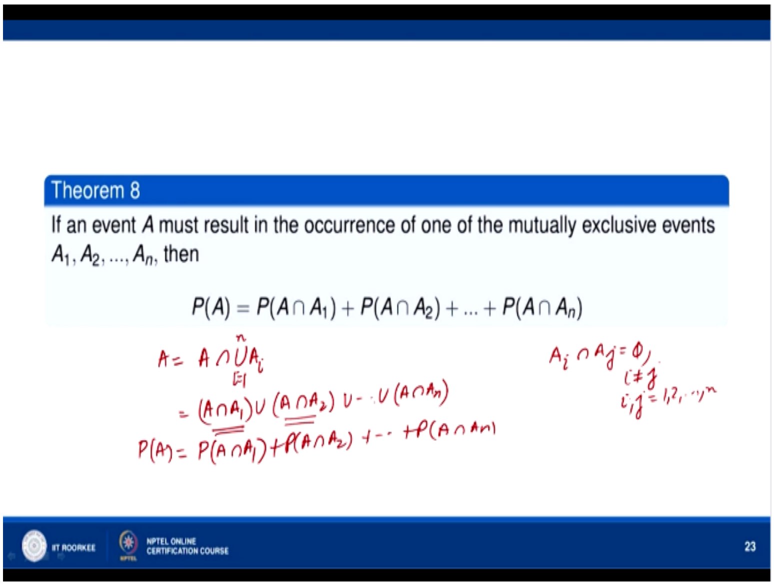

Now if an event A must result in the occurrence of one of the mutually exclusive events A1, A2, A3 and so on An okay. If an event A must result in the occurrence of one of the mutually exclusive events A1, A2, A3 and so on, then we shall have A=A intersection A1, A2, A3, and so on, so we can say A intersection union Ai i=1 to n okay. If we take the intersection of A with i=1 to n union Ai, then it will be=A.

So this will be=A intersection A1 union A intersection A2 union A intersection An. If an event A must result in the occurrence of one of the mutually exclusive events. A1, A2, An are mutually exclusive means Ai intersection Aj=phi whenever i is not equal to j and ij vary from 1 to n okay. So this gives you PA= now A intersection A1, A intersection A2, A intersection An are also mutually exclusive because their intersection is phi.

So probability of A intersection A1+probability of A intersection A2 and so on probability of A intersection An because they are mutually exclusive.

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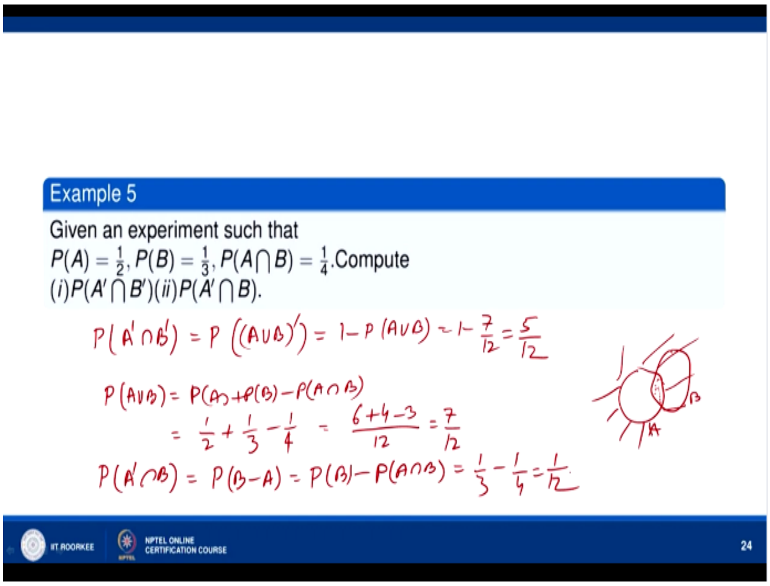

Now let us take some exercises. So we are given a random experiment where A and B are any two events. So probability of A is=1/2, probability of B=1/3 and we are given that probability of A intersection B is 1/4. We have to compute the probability of A complement intersection B complement. Now probability of A complement intersection B complement is=probability of A union B complement.

A union B complement is A complement intersection B complement and probability of A dash we know probability of A dash is 1-probability of A. So this is 1-probability of A union B okay. Now probability of A union B we know, probability of A union B is=probability of A+probability of B-probability of A intersection B by addition theorem of probability. So PA=1/2, PB=1/3 and P A intersection B is 1/4, so this is how much it is?

This will be 12, so this is 6 here and here we have 4 and here we have 3 okay. So 6, 4, 3 and we shall have it 7/12 okay. So 1-7/12 gives you 5/12. So P A dash intersection B dash is 5/12. Now probability of A dash intersection B okay, so let us see we have A here, this is A, this is B okay. So A dash is this one sorry A dash is this is A dash okay, this is A dash, so A dash intersection B will be B-A okay.

So this is probability of B-A okay. Probability of B-A we have to find okay, so from B we have to subtract A intersection B, this is A intersection B okay. So this is=probability of B-probability of A intersection B and probability of B is 1/3 and this is 1/4. So we get 1/12 okay.

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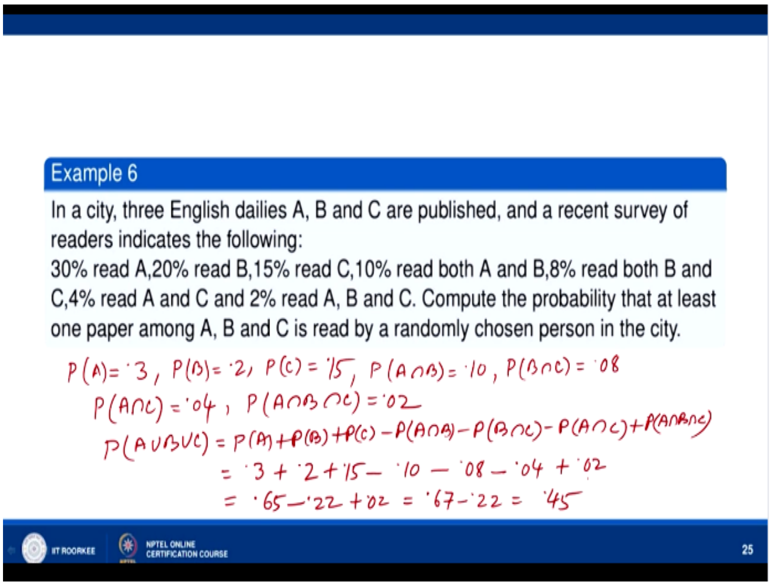

So now let us now go to next question. In a city, three English dailies A, B and C are published and a recent survey of readers indicates the following. A people read 30%, 30% people read A paper okay, so probability of the paper A is=30% that is 3.3. The paper B is read by 20% people, so probability of B is=0.2. C paper is read by 15%, so probability of C is 0.15 okay, 10% people read both A and B.

So probability of A intersection B is=0.10, 8% read both B and C, so B intersection C probability of B intersection C is=8% that is 0.08 okay. Probability of A intersection C okay 4% that is 0.04 okay. Now 2% people read A, B and C, so probability of A intersection B intersection C is=0.02 okay. Now compute the probability that at least one paper among A, B and C is read by a randomly chosen person in the city.

So we need the probability of A union B union C okay and we have done this formula. This is probability of A+probability of B+probability of C-probability of A intersection B-probability B intersection C-probability of A intersection C+probability of A intersection B intersection C okay. So PA=0.3 PB=0.2 PC=0.15, probability of A intersection B is 0.10, probability of B intersection C is 0.08, probability of A intersection C is 0.04+probability of A intersection B intersection C is 0.02 okay.

So let us calculate this. So 0.30+0.20 is 0.50+0.15, this is 0.65 okay and here we have 0.10, 0.08 so 0.18 and then 0.04 is 0.22 and here we have 0.02. So how much we get? 0.65+0.02 is 0.67, so 0.67-0.22 is 0.45. That means 45% people okay read at least one of the 3 papers at A, B or C okay.

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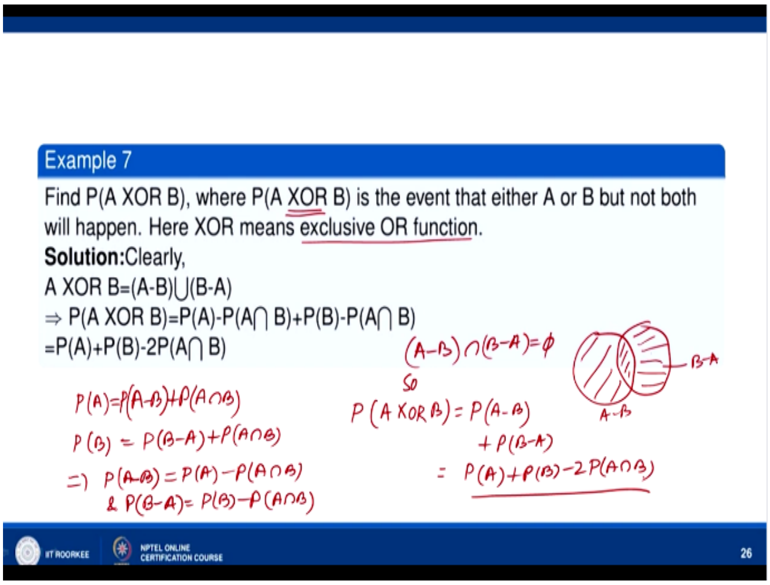

Now find P A XOR B okay, P A XOR B, A XOR B means what? Probability of A XOR B is the event that either A or B but not both will happen okay. So if you take A here and this is B okay. Then, this portion is A-B, A occurs B cannot occur and this portion represents B-A that is B can occur but A cannot occur okay, B will occur but A will not occur okay. So when we say that probability of A XOR B okay, it means that we need the probability that either A or B occurs but not both happen together okay not both will happen.

So XOR means exclusive OR function. This means exclusive OR function okay. Now A XOR B means A-B, this is A-B union B-A. So probability of A XOR B will be=probability of A-B+probability of B-A because A-B intersection B-A is=phi, A-B intersection B-A is=phi, so probability of A XOR B is=P A-B+P B-A okay. Now we know that PA=P A-B+P A intersection B because this is=P A-B+P A intersection B.

Because A-B and A intersection B are mutually exclusive, so probability of A will be probability of A-B+probability of A intersection B. Similarly, B-A and A intersection B are mutually exclusive, so PB=P B-A+P A intersection B okay. So this give you the values of probability of A-B, this is=probability of A-probability of A intersection B and probability of B-A which is probability of B-probability of A intersection B okay.

So let us put the values of P A-B and P B-A what we will get? PA+PB-twice P A intersection B okay. So this is how we get the value of A XOR B.

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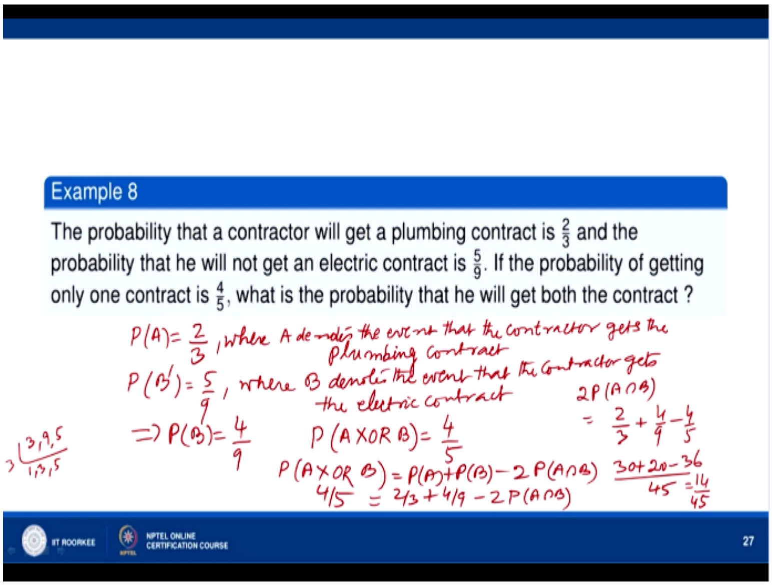

Now let us consider this problem. The probability that a contractor will get a plumbing contract is 2/3. So let us say A denotes the event that the contractor gets the plumbing contract okay. Then, probability of A is=2/3 and the probability that he will not get an electric contract. So let us say that B denotes the event that he gets the electric contract okay. So then we are given that PB complement is=5/9 okay where A denotes the event of that the contractor gets plumbing contract.

Further let B denotes the event where B denotes the event that the contractor gets the electric contract okay. So PB dash is=5/9 which implies that PB=4/9, PB dash=5/9, so PB=4/9. Now if the probability of getting only one contract, probability of getting only one contract is 4/5, what is the probability that he will get both the contracts? So here we are given the probability of getting only one contract that is either A or B but not both okay.

So probability of A XOR B is given to us, probability of A XOR B is 4/5 okay. Now A XOR B is=PA+PB-twice P A intersection B okay. So PA XOR B is=PA+PB-twice P A intersection B okay. So this is 4/5=PA, PA=2/3, PB=4/9-2 P A intersection B okay. So what do we get? 2 P A intersection B is=2/3+4/9-4/5 okay, so from here we can get the value of A intersection B.

So this is 3 9 5 okay 3 9 5 LCM let us find. So this is 45 okay, so we get 15*2=30 then 9*5=45, so 5*4=20 and then 5*9=45, 9*4=36 so we get 50-36 that is=50-36 is 14, 14/45 okay, so that is the answer in this case okay.

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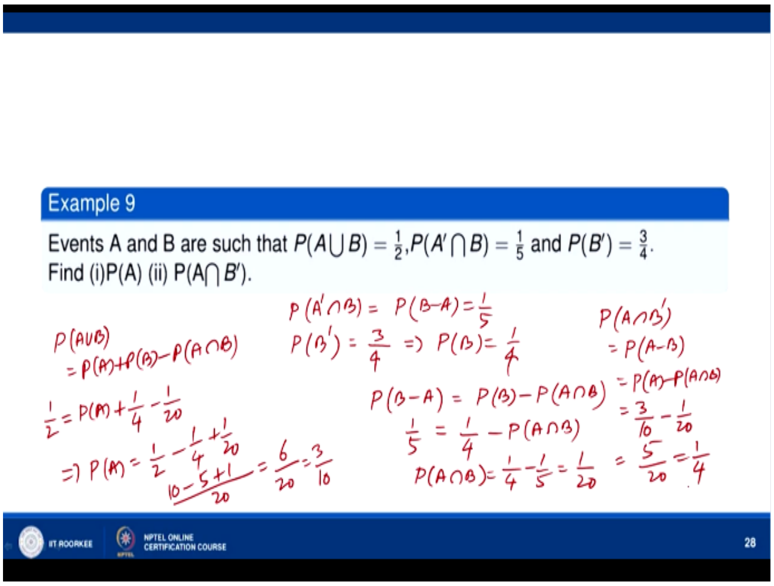

Now let us go to this question. Events A and B are such that P A union B=1/2, P A dash intersection B is 1/5, PB dash is 3/4. We have to find the probability of A okay. So we are given the probability of P A dash intersection B means that this is probability of B-A okay. This is=1/5. Probability of B dash is=3/4 implies that probability of B is 1/4 okay. Now we need the probability of A okay.

So P B-A okay P B-A is=PB-P A intersection B okay. From B when you subtract A intersection B, you get B-A okay. So B-A is 1/5, so this is=PB 1/4-P A intersection B okay. So we get P A intersection B as 1/4-1/5 means that we have 1/20 okay. Now P A union B is=PA+PB-P A intersection B. So P A union B is 1/2, so 1/2=PA we have to determine, PB we know, this is 1/4 and P A intersection B is 1/20.

So we get PA=1/2-1/4+1/20 and we have the LCM 20, so we get 10 here okay, here 5, here 1 okay. So 11-5, so 6/20. This is least 3/10 okay. So 3/10 we get and P A intersection B dash is=P A-B which is=PA-P A intersection B. So this is 3/10, PA is 3/10-P A intersection B is 1/20. So we get this will be 6/20-1/20, so 5/20 and this is 1/4 okay. So that is how we get the value of P A intersection B dash. With this I would like to end my lecture. Thank you very much for your attention.