Advanced Engineering Mathematics
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Lecture - 41
Discrete Fourier Transforms - II

Hello friends. Welcome to my second lecture on Discrete Fourier Transform. We will first show that the vectors Uk given by e to the power 2pi kn/n where n varies from 0 up to n-1, it takes value 0,1,2,3 and so on n-1 they are orthogonal that is they form an orthogonal basis over the set of n dimensional complex vectors. So in order to prove that this vector Uk form an orthogonal basis we have to show that

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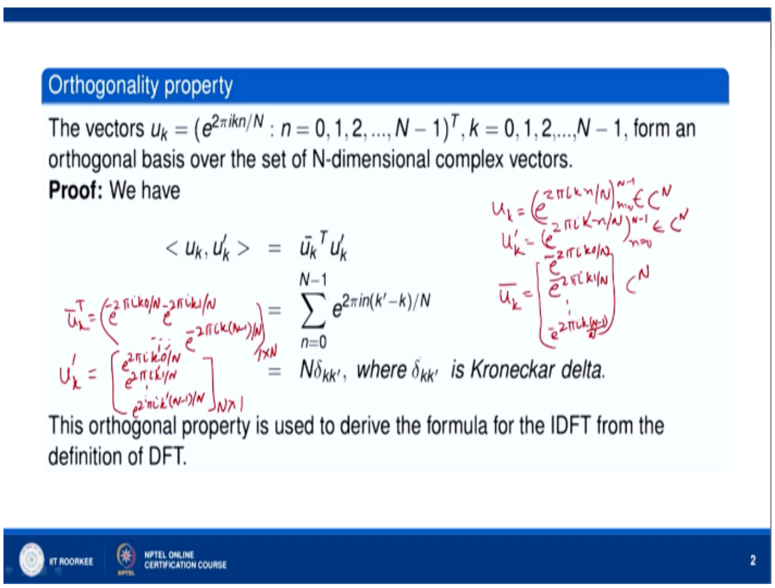

The inner product of Uk with; another vector say Uk dash from the same set is equal 0. Now let us take two vectors Uk and Uk dash from this set e to the power 2pi ikn/n where n takes value 0,1,2,3 and so on up to n-1. So then Uk is e raise to the power 2pi ikn/n Uk dash = e raise to the power 2pi i k dash n/n. This N denotes the transpose. That means we are writing this vectors N vectors in the form of a column, okay.

So that is why we have written that as e to the power 2pi kn/n, okay. They are N vectors, we writing them in the form of a column. So now inner product of Uk with Uk dash, Uk and Uk dash belong to the N dimensional complex vector space that is Cn, okay. So, they belong to Cn, okay. Uk and Uk dash belong to C,n okay. So in Cn the inner product of Uk with Uk dash is defined as Uk conjugate transpose * Uk dash.

Uk conjugate will be equal to e raise to the power iota will be replaced by –iota so -2 iota kn/n, okay. So this will be Uk conjugate, Uk conjugate transpose. Uk conjugate transpose means, we take the transpose of the vector. So, when we take the transpose of the vector, okay then Uk conjugate transpose will be row vector and that will be given by e raise to the power 2pi kn/n, okay where n takes values from 0,1,2,3 and so on up to n-1. So we can write like this.

Let us take n=0 first, so we can write 2pi k0/n then e raise to the power 2pi-, - we have to take because of this –, okay so -2pi k1/n second component and the last component is e to the power -2pi i kn-1/n, okay. So this row vector, Uk conjugate transpose. Uk conjugate is the column vector actually, this column vector where we have n components. So let me write that as follows.

We write it as in the form of a column vector. So we have its component as e raise to the power 2pi i – 2pi i k0/n, e raise to the power -2pi i k1/n okay, e raise to the power -2pi ikn-1/n, n-1/n divided by n. So this is the column vector, Uk conjugate. When you take transpose of this conjugate vector actually these Uk's are all belonging to Cn, okay they are belonging to Cn. K varies from; sorry n varies from 0 to n-1.

They all r vector in Cn, okay. So Uk conjugate transpose is this. When you multiply by Uk dash Uk dash is what Uk dash is column vector, okay. So e raise to the power 2pi i k dash 0/n, e raise to the power 2pi i k dash 1/n and so on to the power 2pi ik dash n-1/n, okay. So when you multiply Uk conjugate transpose with Uk dash okay this being 1/n matrix and this being n/1 matrix when you multiply you get 1/n matrix that is you get this summation.

Sigma n=0 to n-1 e to the power 2pi in, okay this 0, n varies from 0 to n-1, so that we write as n. So e to the power 2pi in and that k dash – k/n, okay. So 1/n*n/1 matrix gives you 1/1 matrix. And here when you sum this series, this is a geometric series, here you can see that, if k=k dash okay, if k=k dash e to the power 0, okay will be equal to 1. And this n is varying from 0 to n-1 so this total sum will be then n, okay. So if k=k dash this is equal to; let us do it here.

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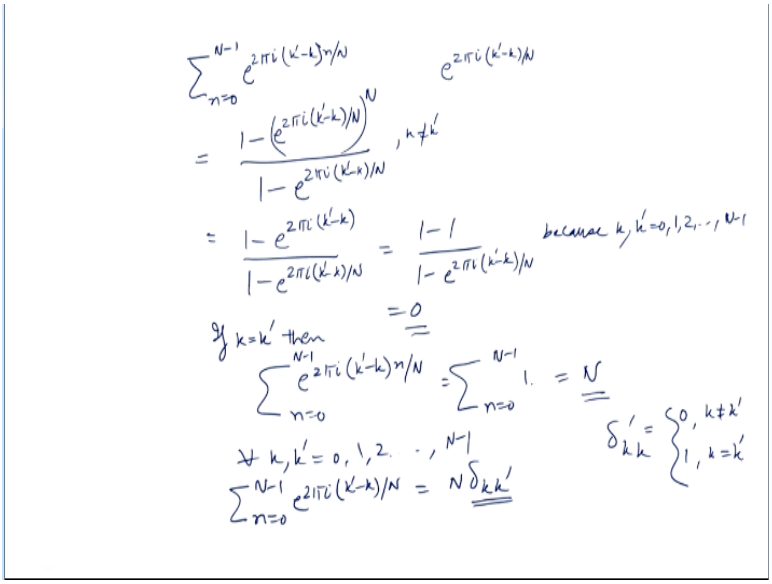

We have sigma n=0 to n-1 e to the power 2pi i k dash-k * n/n, okay. So this is actually a geometric series, if k is not equal to k dash. If k; let us first find the sum when k is not equal to k dash. So this is equal to 1-e to the power 2pi i k dash – k raise to the; or divided by n raise to the power n, okay. The geometric ratio here is e to the power 2pi i k dash – k/n, okay.

And so this divided by e to the power n divided by 1-e to the power 2pi k dash – k/n if k is not equal to k dash, okay. And this will be equal to 1-e to the power 2pi k dash – k, all right divided by 1-e to the power 2pi i/ k dash – k/n. Now k and k dash are integers, so e to the power 2pi ik dash – k will be equal to 1, so we have 1-1/1-e to the power 2pi ik dash – k/n, okay. Because, k and k dash vary from 0 up to n-1 okay.

Now; so this is equal to 0. So when k is not equal to k dash sum of this series is 1 and if k=k dash then we have sigma n=0 to 2pi k dash – k/*n/n=summation n=0 to n-1, 1, okay. So 1 will be summed n times and therefore this is equal to n, okay. So when k is not equal to k dash the value is 0, when k=k dash the value is n, so we can combine both the cases, okay.

So for all k, k dash varying from 0 up to n-1 we can say that, sigma n=0 to n-1 e to the power 2pi ik dash – k/n = n times delta kk dash, okay. Where delta kk dash is a Kronecker delta. Okay. And we know that delta kk dash = 1 when k is not equal to k dash and 0 when k=; sorry it is 0; it is 0 when k is not equal to k dash and it is 1 when k= k dash. Okay. So this is how we can find the value of this series, so we have this. N times delta kk dash, where delta kk dash is a Kronecker delta.

This orthogonal property; so now we can see that when k and k dash, okay Uk and Uk dash are any two vectors from this set of n vectors okay these n vectors are there, so from the set of n vectors then their inner product is 0 whenever k is not equal to k dash, this means that these vectors form an orthogonal basis. So this orthogonal property of the e to the power 2pi i iota kn/n will be used to determine the inverse discrete Fourier transform using the definition of discrete Fourier transform. Let us see how we do this.

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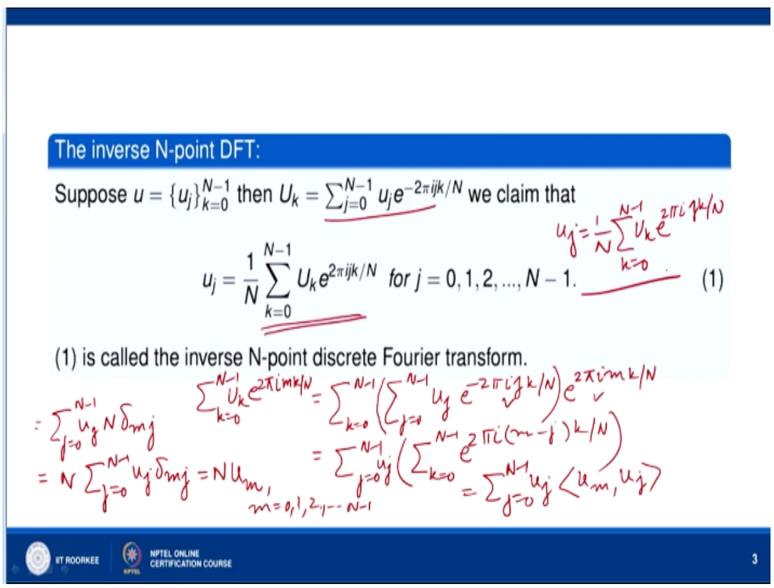

So let us arrive the formula for the inverse N-point DTT, okay. Suppose u = Uk or uj where j varies from 0 to n-1 then Uk is the discrete finite discrete transform; discrete finite transform of the vector uj, so it is given by sigma j=0 to n-1 uj e to the power -2pi iota jk/n. We claim that, the inverse discrete Fourier transform IDFT is given by uj=sigma 1/n sigma k=0 to n-1 Uk e to the power 2pi jkn/n where j varies from 1 to and so on up to n-1. So let us see how we prove this.

So we have Uk, Uk is given to us, so uj is the definition of DFT, Uk sigma k=0 to n-1, okay sigma k=0 to n-1. Uk will be = sigma k=0 to n-1 the value of Uk is sigma j=0 to n-1, okay uj e to the power -2pi ijk/n, okay e to the power -2pi ijk/n that is Uk right sigma = j=0 to n-1; sigma j=0 to n-1; uj e to the power -2pi iota this. Now this can be written as; these are finite sums we can interchange them. So we can write as sigma j=0 to n-1 uj, uj can be written outside, okay, sigma k=0 to n-1 okay, e to the power -2pi ijk/n, okay.

Now, let us see, we have to show; now you okay; okay let us go to the previous slide. In the previous slide, we have shown that these vectors u e to the power 2pi i; this e to the power 2pi ikn/n okay satisfies this orthogonal property that is Uk Uk dash the other product of Uk with Uk dash is equal to n delta kk dash. So here what will happen, here let us see; we will have sigma okay, we have to multiply Uk by 2pi ijk/n that we have not done.

So let me write this as; we can write like this so sigma k=0 to n-1, okay. Uk e to the power 2pi ijk/n, okay. So this is sigma k=0 to n-1, I have put the value of Uk, this is the value of Uk. I have to multiply by e to the power 2pi ijk, j is already there. So what I should do, I should take some other index not j, let me take here m. Let us take here e to the power 2pi i, because j already I am using here so I should not write j here, so 2pi ink/n let us write this. Okay. So e to the power 2pi ink/n.Okay.

So, what I do, I consider this right hand side, okay. I consider sigma k=0 to n-1, Uk e to the power 2pi i, instead of j I write m, okay. So 2pi imkn/n. Then this is equal to sigma k=0 to n-1, I put the value of Uk from here, okay. Sigma j=0 to n-1 uj e to the power -2pi ijk/n and then multiplied by e to the power 2pi ink/n. Now let us see, I interchange the sums here because sums are finite, so sigma j=0 to n-1; uj I write outside then sigma k=0 to n-1 to the power -2pi i.

So I remove this here and write this as; I combine this in this and write e to the power 2pi i m-j * k/n. So I write like this. So I write e to the power 2pi i m-j * k/n, okay. So we have seen that, let us look at this one. We have seen that sigma n=0 to n-1 e to the power 2pi ink dash – k/n which is the inner product of Uk dash, okay. So here we can say that we are having the enough product of Um will Uj okay. So this is nothing but sigma j=0 to n-1 uj inner product of Um with Uj. Okay. Okay, so this we have seen from the orthogonal property.

Now this is sigma j=0 to n-1 I write here = sigma j=0 to n-1 uj and then delta; n times delta mj, okay. Because inner product of uj is n times delta mj okay so this is equal to n times sigma j=0 to n-1 uj delta mj. And delta mk is the kronecker delta. So it will be equal to 1 then m=j otherwise it is 0. So when this j will be equal to m this sum will be equal to Um okay otherwise delta mj will be 0 for every other j which is not equal to m, so then other terms will be 0.

So this is n times Um, okay. So we can see that sigma k=0 to n-1, Uk e to the power 2pi ik/n is =n times Um, where n is taking any value from 0,1,2 up to n-1. Now we can replace m by j okay and then we have uj from here uj=1/n okay. This n will go here, a sigma k=0 to n-1, okay Uk e to the power 2pi i, m I am replacing now with j, so jk/n, okay. So this is how we find the inverse N-point discrete Fourier transform. Okay.

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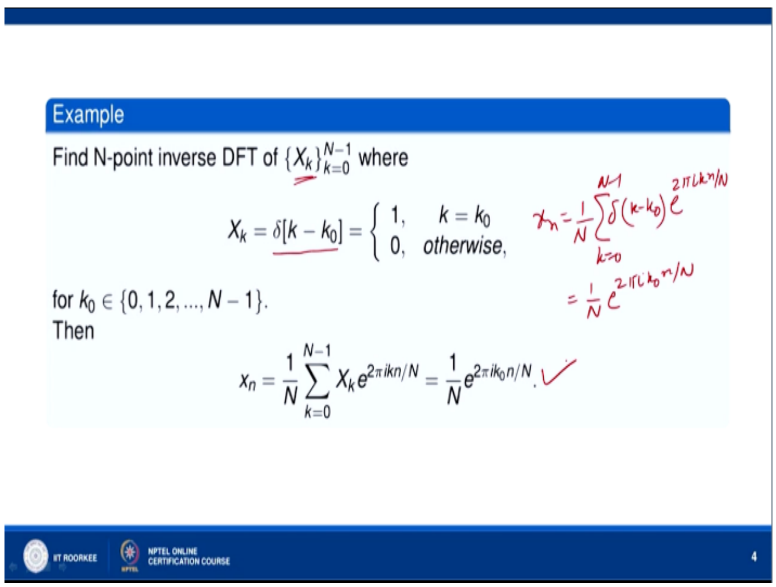

Now let us take example, find N-point inverse discrete Fourier transform of this frequency magnitude Xk, k runs from 0 to n-1. Xk is given by delta k-k0. Delta k-k0 is equal to 1 when k=k0 otherwise it is 0. And k0 is any number from the set 0,1,2,3 and so on up to n-1, okay. Now let us use the N-point inverse DFT formula, okay. So N-point inverse DFT formula gives instead of uj I am writing now Xn okay because our sequence is Xn.

For the sequence Xn we are using the discrete Fourier transform by Xk. So Xn = 1/n sigma k=0 to n-1; instead of j we are using n. So 1/n sigma k=0 to n-1 Uk e to the power 2pi ikn/n, okay. So here, when you put the value of Xk. So Xn, Xn will be equal to 1/n sigma k=0 to n-1, delta k-k0, e to the power 2pi ikn/n, okay. Now by definition of delta, delta k-k0 is 1 when k=k0. So only the term where k becomes equal to k0 that term will survive other terms will all vanish.

So this will be 1/n. And when k=k0 we have delta k-k0 = 1, so we get e to the power 2pi iota k0 * n/n, okay. So we get this.

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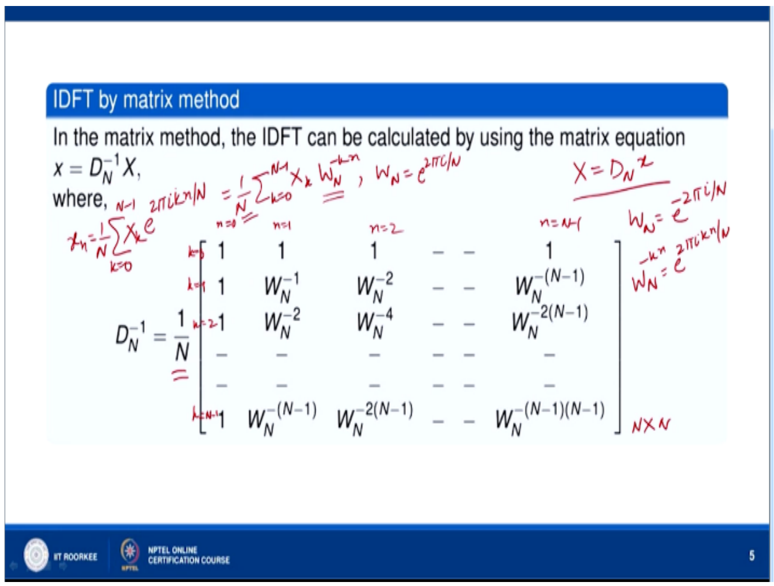

Okay, now we have discussed in the lecture, first lecture on DFT, how we can calculate DFT of a given sequence by using matrix method. Here also, the inverse discrete Fourier transform can be determined by using the matrix equation X=Dn inverse X. Okay, we had used the matrix equation X=D X there. Okay. Here we have Dn inverse given by 1/n * this matrix, this matrix is n/n matrix okay.

So in the first row we have 1 1 1 each. Let us see how we get this matrix, we this sequence Xn= let us use this one, Xn=1/n sigma k=0 to n-1 we have this sequence Xn= sigma 1/n sigma k=0 to n-1 and Xk e to the power 2pi kn/n, okay, e to the power 2pi kn/n, okay. So you can see here, when this is k runs along the rows, okay. K runs along the rows, this is k=0; this is k=1; this is k=2 and so on this k=n-1, okay, k=n-1. And your, we would be writing it as kn; okay so n runs along the columns.

This is n=0, n=1, n=2 and it is so on = n-1. And we can write it as 1/n sigma k=0 to n-1 XkWn k – kn, okay. So we had; if you recall in the matrix method for DFT we had taken Wn to be equal to e to the power -2pi i/n, okay, e to the power -2pi i/n, so here Wn to the power –k and Wn to the power –kn will be Wn-kn will be equal to e to the power 2pi ikn/n, okay. So we use same Wn here, Wn= e to the power 2pi i/n, okay.

So this is 1k=0, n=0, Wn to the power 0 will be 1, so when k=0 Wn to the power –kn it will be 1, so we get this 1 here. And then when k=1 okay, but n=; yeah when k is taking any value from 0 to n-1 but n=0 again Wn= e to the power 0 will be equal to 1, so we get column first column also 1. This column will be k=1, n=1, so Wn to the power -1*1 that is Wn to the power -1. Then k=1, n=2, so Wn to the power -2 and so on.

Then Wn to the power –n-1 when k=1, n=n-1. And similarly, we get the last row. So 1/n, this 1/n comes here, okay. Xk is the column vector, okay. X0, X1, X2 and so on Xn-1 and this matrix Wn to the power –kn, this matrix is here, okay. So Dn inverses this. Dn inverse will be multiplied by X to get the sequence X.

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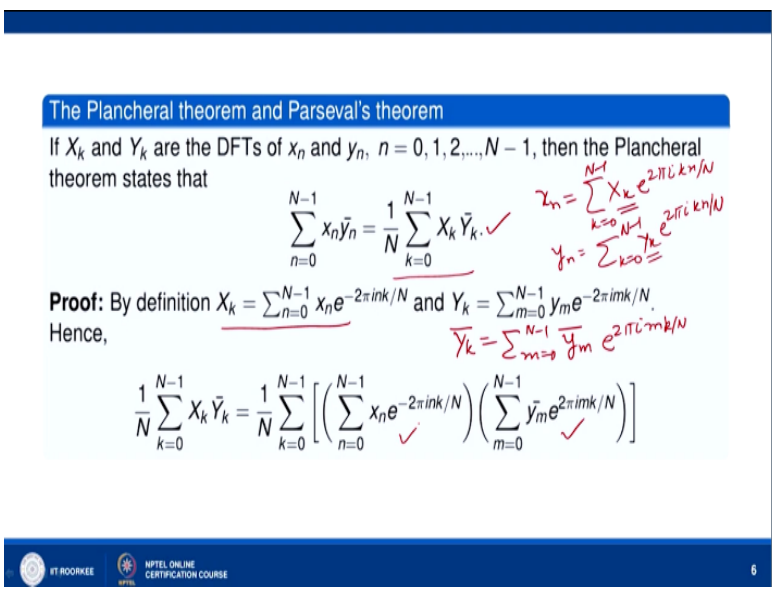

Now let us discuss the Parseval's theorem. Plancheral theorem and Parseval's theorem. Suppose Xk and Yk are discrete Fourier transforms f Xn and Yn, okay. So Xn=sigma Xk, k=0 to n-1 e to the power 2pi ikn/n. And by n=similarly, sigma k=0 to n-1/k e to the power 2pi kn/n. Okay. So those Xk Yk are discrete Fourier transforms of the sequence Xn Yn. Then the Plancheral theorem states that sigma k=1 to n-1 XnYn bar is n, sigma k=1 to; 0 to n-1 XkYk bar. Okay.

So Xn Yn bar, Xn you multiply by Yn bar, okay. What we will get? See, Xk from the definition of; we are taking the right hand side to prove this equation, to prove this theorem. So by the definition of discrete Fourier transform Xk is given by sigma n=0 to n-1 Xn e to the power -2pi iota nk/n and Yk will be given by sigma m=0 to n-1/n e to the power -2pi i iota mk/n, okay. So let us take the right hand side of this equation.

So 1/n sigma k=0 to n-1 XkYk bar, okay will be 1/n sigma k=0 to n-1 then you put the value of Xk you put the value of Yk conjugate, Yk conjugate will be conjugate here, okay Yk conjugate = sigma m=0 to n-1 Ym conjugate e to the power conjugate of this quantity, so we get 2pi iota mk/n. Okay. So this is what we get and this, okay. Let us now see what happens to this.

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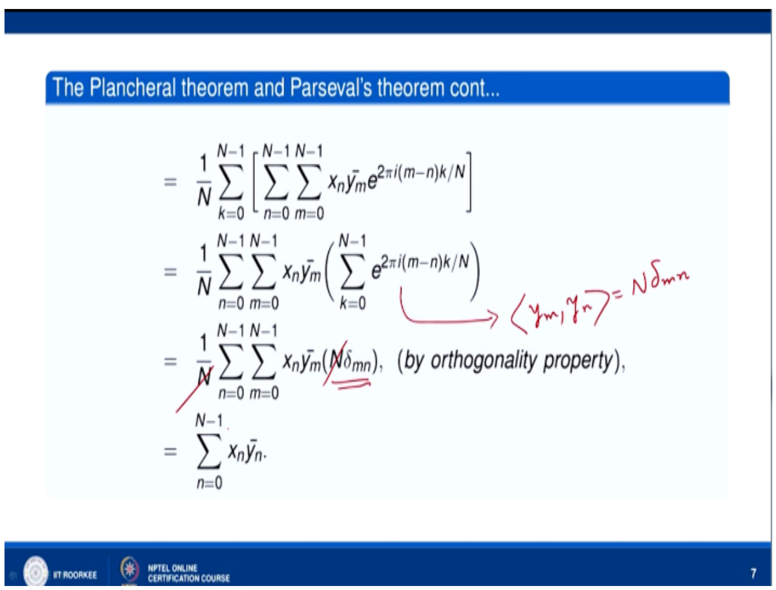

So 1/n sigma k=0 to n-1, we can write it as 1/n sigma k=0 to n-1, we put the two sigma together, sigma n=0 to n-1; sigma m=0 to n-1 then XnYm bar okay XmYm bar e to the power 2pi iota m-n*k/n, okay. So e to the power 2pi i m-n*k/n, okay. Now, so now these are all finite sums, okay. We can interchange and write 1/n sigma n=0 to n-1, m=0 to n-1 the summation over k can be brought inside. So XnYm bar sigma k=0 to n-1 e to the power 2pi I m-nk/n.

Now this is again by using orthogonal property. This is nothing but in a product of sequence Ym and Yn, so this is equal to n times delta mn, okay. So we get 1/n sigma n=0 to n-1 m=0 to n-1 XnYm bar and delta mn, okay. So delta mn is Kronecker delta it will be equal to 1 when m=n otherwise it will be 0. So all those values of m, okay which are equal to n they give the value of delta mn as 1, okay otherwise it is 0.

So this n, this n will cancel with this n and XnYmbar delta mn with this double summation to reduce to single sum, okay. So n=0 to n-1 XnYm bar, okay when m=n. So this is how we get the left hand side. Okay. This left hand side. So this prove the Plancheral theorem. Let us see how we get the Parseval's theorem from here.

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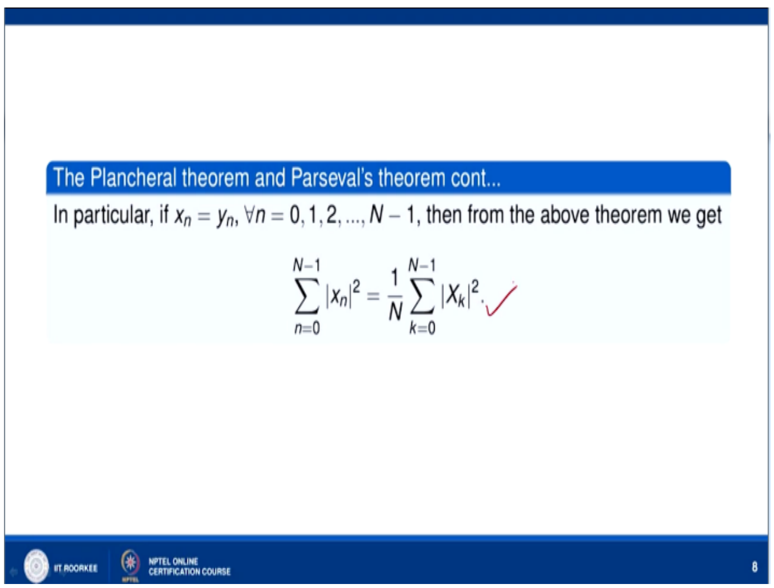

So we go to particular case here, let us take the sequences Xn and Yn to be same okay. Xn = Yn for all n. So when Xn=Yn for n=0,1,2,3 and so on up to n-1 then from the Plancheral theorem what do we notice, XkYk bar will be Xk bar now because at Yn and Xn are sequences are same, so XnXk bar, XnXk bar will be mod of Xk square. And here Xn bar will be mod of Xn square, so we have this result. Sigma n=0 to n-1 mod of Xn square is equal to sigma k=0 to n-1 1/n times this, okay = to this yeah.

So we have sigma n=0 to n-1 mod of Xn square = 1/n sigma k=0 to n-1 mod of Xk square, okay. So we get this result. So this is Parseval's theorem.

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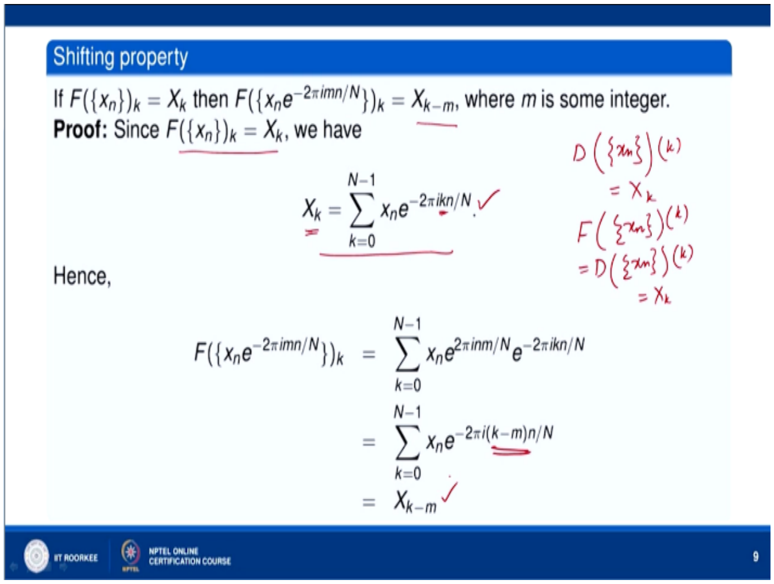

So if f(Xn) k = Xk that means discrete Fourier transform of the sequence Xn=Xk. Earlier, when we started we had written the notation for the discrete Fourier transform as D, D of Xn k=Xk we had used earlier, but here in this Shifting property D is replaced by F, so there is just change of notation but it is the same thing So f(Xn)k is same as D of Xn k it is equal to Xk. So then this Shifting property says that if you multiply Xn/e to the power 2pi imn/n then the discrete Fourier transforms shift to the right okay it becomes Xk-m where m is sum integer.

So let us use the definition of discrete Fourier transform we can see that this Xk is not given by sigma k=0 to n-1 Xn e to the power -2pi i kn/n. Now multiply Xn by e to the power 2pi imn/n, okay. So we will replace here Xn by Xn e to the power 2pi mn/n to get the discrete Fourier transform of this sequence. So we have sigma k=0 to n-1 Xn e to the power 2pi imn/n * e to the power -2pi ikn/n and this I can write as sigma k=0 to n-1 Xn e to the power -2pi ik-mn*n/n, okay.

And by definition of discrete Fourier transform, this is discrete Fourier transform of Xn sequence. Okay. So here this k, now corresponding to this k we have Xk here, okay. So here this k is replaced by k-m, okay so we have the discrete Fourier transform Xk-m, okay. So when you multiply Xn/ e to the power 2pi mn/n the discrete Fourier transform shifts to the right, it becomes Xk-m.

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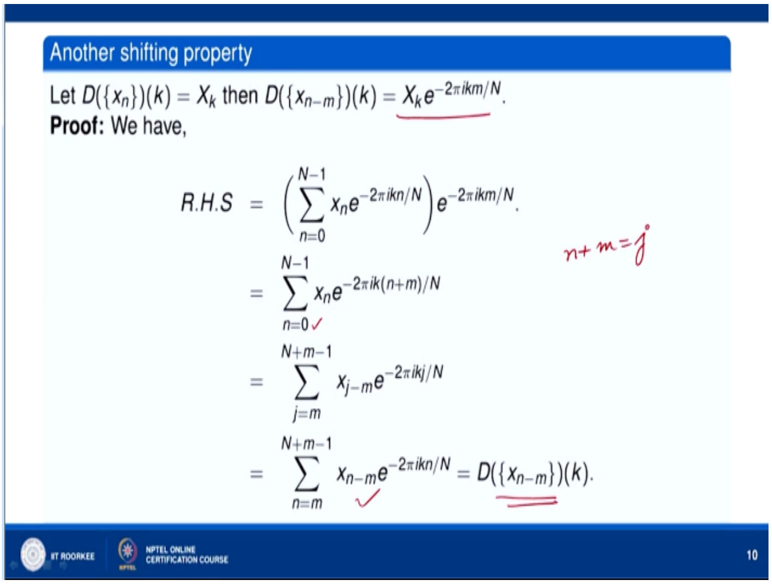

Now another shifting property. Suppose the Xn (k) = Xk then the Xn-(k) the Xn-nk is equal to Xk e to the power – 2pi iota km/N, okay. Let us see how we get this. So we use this right hand side, right hand side by definition of the discrete Fourier transform it is sigma n=0 to n-1 e to the power -2pi iota km/n and we are multiplying it by e to the power -2pi ikm/n. So the summation is over n here so I can bring it inside, sigma n=0 to n-1 Xn e to the power -2pi ik, e to the power -2pi ik and we get n+m/n, okay. And so let us now repeat n+m/j.

So when we write n+m=j then earlier was equal to 0, so now j will be equal to m and n was equal to capital N-1 so j will be n+m-1, so n becomes j-m so Xj-m e to the power -2pi ikj/n. Okay. And therefore, this is let us replace j/n so n=m to n+m-n Xn-m e to the power -2pi ikm/n. So this is nothing but the discrete Fourier transform of the sequence Xn-m.

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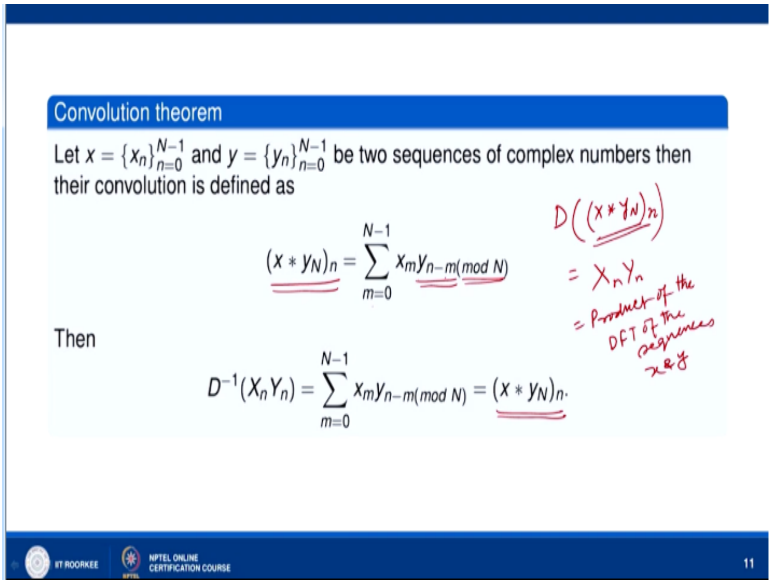

Now Convolution theorem. Suppose X=Xn n=0 to n=1 one sequence Yn n=0 to n-1 is the other sequence of complex numbers then their convolution is defined as x * yN n okay. Sigma m=0 to –n-1 XmYn – m(modulo N), okay. This theorem says that, the discrete Fourier transform of this convolution of the sequence Xn and Yn, okay. X and Y, this is the discrete Fourier transform of the convolution of the sequence X and Y which we denote by this notation is equal to sigma m=0 to n-1 XmYn-m(modulo N) okay.

So which is nothing but; yeah when you take the inverse discrete transform it is Xm*Yn. Xm is the discrete Fourier transom of X and Yn is the discrete Fourier transform of the sequence Y. So discrete Fourier transform of, so we can otherwise, I mean in other words we can say that discrete Fourier transform of X*Yn okay, this equal to Xn*Yn that is product of the DFT of the sequence, sequences X and Y. Okay.

So discrete Fourier transform of the convolution of the two sequences X and Y is equal to product of their discrete transforms, so this is; and I repeat the convolution of the sequences X and Y is defined as sigma m=0 to n-1 XmYn-m and them modulo n, modulo n is used so that the values of Yn do not go beyond this N-1 they remain in between 0 and N-1, okay.

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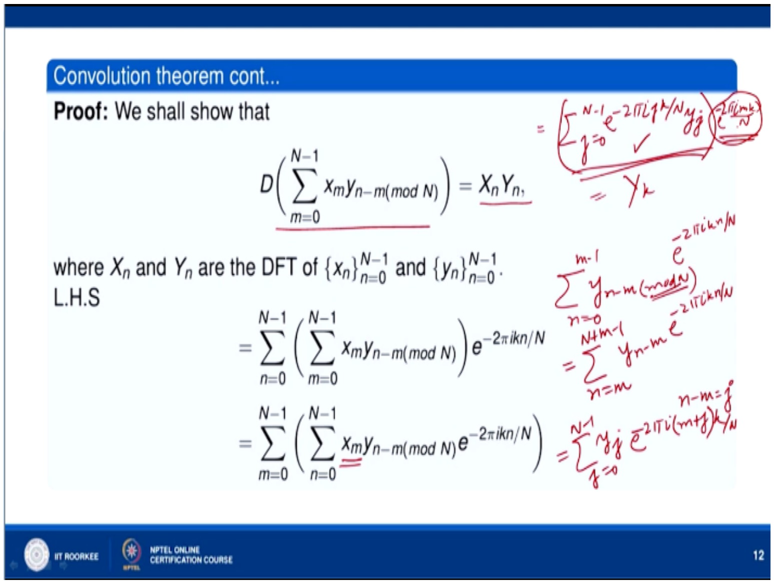

Now let us prove this. So discrete Fourier transform of, we are going to show that discrete Fourier transform of the convolution of the sequence X and Y= product of their discrete transforms. Okay. So let us take this left hand side, okay. Left hand side will be equal to sigma m=0 to; okay sigma m=0 to n-1 this convolution okay we are writing here m=0 to n-1 XmYn-m modulo N and then multiplying by e to the power -2pi ikn/n, okay.

So this is simply discrete Fourier transform of the convolution of X and Y. And then what we do; these are finites sums so we can interchange them, so sigma m=0 to n-1 this is written at outside and inside we write sigma m=0 to n-1 XmYn-m modulo N e to the power -2pi ikn/n. Okay. Now then, this Xm this Xm is independent of N so I can write it outside the summation over N and what we will get; then we will get, yeah so sigma m=0 to n-1 Xm and this sigma m=; we are interchanged, we have already interchanged okay.

So Xm will go outside the summation over n-1. Now we have sigma n=0 to m-1 by n-m okay modulo n into e to the power -2pi ikn/n. So because of this modulo this is equal to sigma n=m to n+m1, okay sigma n=m to n+m-1 by n-m okay, e to the power -2pi ikn/n, okay because of this modulo okay I can write this expression as sigma n=m to n+m-1 by n-m. Now let me write n-m=j, okay.

So I will be getting j=0 to n-1 by j and here n will be n+j so e to the power -2pi im+j * n/n, okay which we can write in as follows. Sigma j=0 to n-1 e to the power -2pi iota mn/n okay. So thus we write iota jn/n * Yj. Okay, this Yj. This is one thing multiplied by e to the power -2pi iota mn/n. Okay. So this quantity e to the power -2pi iota mn/n will be multiplied to this Xm which is outside the summation over n, okay, so this here, okay.

And we will get sigma m=0 to n-1 Xm e to the power -2pi imn/n, okay, e to the power -2pi imn; yeah you can write K okay what we have put here, we have put j right. So when we put j here, okay so I get here this sum as sigma j=0 to n-1 Yj e to the power -2pi ikj. Where we getting kj? Okay, this was; this is outside, okay. Inside, okay we are getting here 2pi iota kn/n. And what was (()) (40:49)?

Okay, so, I put here okay wait I put here of this not n, n will be put as j okay. So this should be k, okay. So this should be k here not n, okay. So what we have here, actually this is now coming to this, this is nothing but Yk, okay this is Yk and this quantity okay multiplied to Xm sigma m=0 to n-1 gives you Xk, okay so we get this. This and this, okay their product is Xk Yk. Thus the convolution of two sequences is the inverse transform of the product of the individual transforms. So this is how we prove the convolution theorem.

With this we have finished the discussed on discrete Fourier transforms. Thank you very much for your attention.