Advanced Engineering Mathematics
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 40
Discrete Fourier Transforms - I

Hello friends, welcome to my lecture on Discrete Fourier Transforms. There will be two lectures on this topic; we have at the first lecture now. So in the Discrete Fourier transforms the discrete dataset are converted into a discrete frequency distribution. We shall see that. The discrete dataset are converted into a discrete frequency distribution. Now since the resulting frequency distribution is discrete the computers commonly use discrete Fourier calculation, one frequency information is required.

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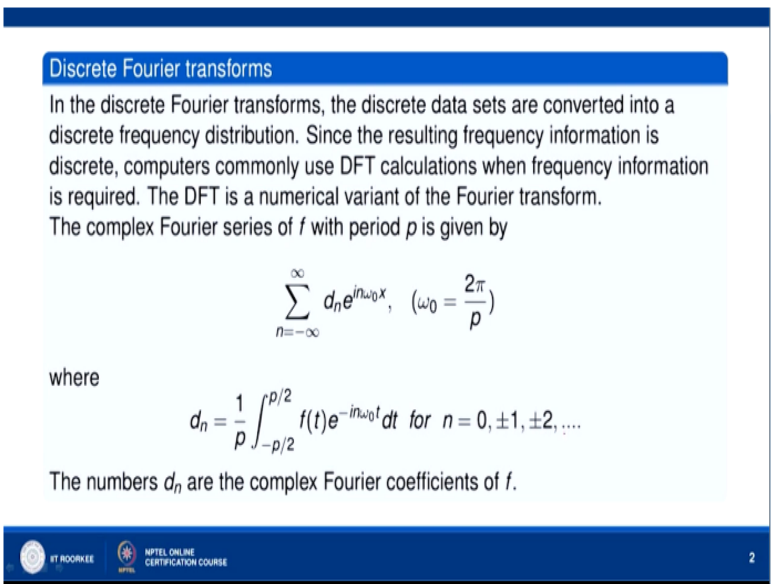

Now the DFT is also numerical variant of Fourier transform. Let us see how the motivation for DFT came about. The complex Fourier series of F with period p is given by sigma n=-1 infinity dn e to the power –in omega0 x where omega0 is suppose to 2pi/p. The values of this Fourier coefficients dn are given by 1/p integral/-p/2 to p/2 f(t) e to the power – in omega0 t dt when n is taking values 0, +-1, +-2 and so on. The numbers dn's are called the complex Fourier coefficients of f, since f is given to be p periodic.

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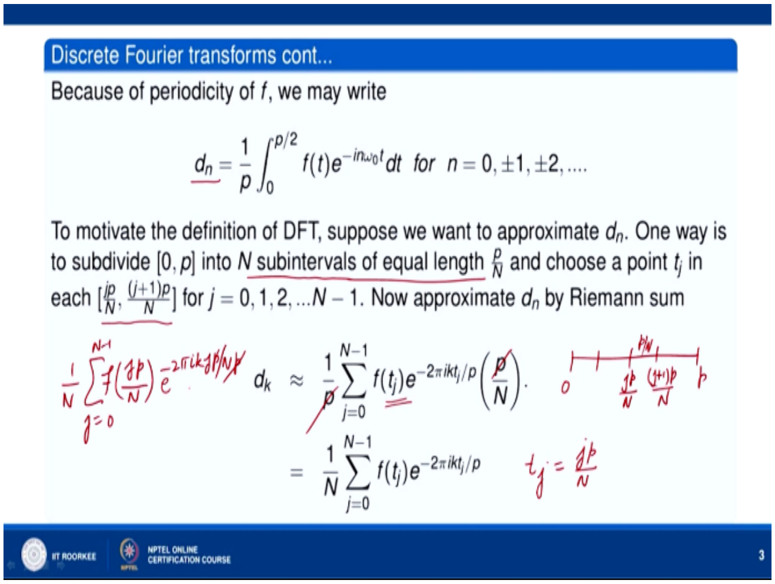

We may replace this integral over –p/2 to p/2 by any other integral where the length of the interval is p, okay, so we can replace integral over –p/2 to p/2 by integral over 0 to p/2, so dn=1/p integral over 0 to p f(t) e to the power –in omega0 t dt where n=0, +-1 ,+-2 and so on. Now to motivate the definition of DFT suppose we want to approximate the value of dn, okay. Suppose we want to; because we have to calculate this integral.

We can consider numerical integration, so if we want to approximate the value of dn, okay then one way is to sub-divide 0p interval, okay so 0p interval be subdivide into n's of intervals of equal length p/n and choose a point tj in each subinterval, jp/n to j+p/n for j=0,1,2 and so on up to n-1. This is your interval 0p divided by divided into n's of intervals of length p/n each. Okay. So suppose your j; this interval is jp/n j+1 and p, okay. The length of each interval is p/n.

So this we do for; I mean this is the js subinterval where j takes values from 0 to n-1. Now approximate this dn by Riemann sum, okay. So 1/p sigma j=0 to n-1 f of tj, okay tj is any point in this interval, so f of tj is to the power –in omega0 is 2pi over p, okay. So we put 2pi over p for that and then we have tj, okay. So we put here ktj because we want the value of d(k), so d(k) is ktj here instead of n we are writing k, okay.

So e to the power -2pi iktj/p and then we multiply by the length of the subinterval, so this length is p/n, okay of each interval, the length of each interval is p/n. Okay, so now this is what, this p will cancel with this p, okay. And take tj to be let us, okay so here we will take it later the tj to be jp/n. Let us first simply this, so 1/n okay sigma j=0 to n-1 f(tj) e to the power -2pi ktj/p. Now choose tj to be equal to jp/n. tj can be any point in this subinterval so we choose tj to be jp/n. If you choose tj to be jp/n then what do we get?

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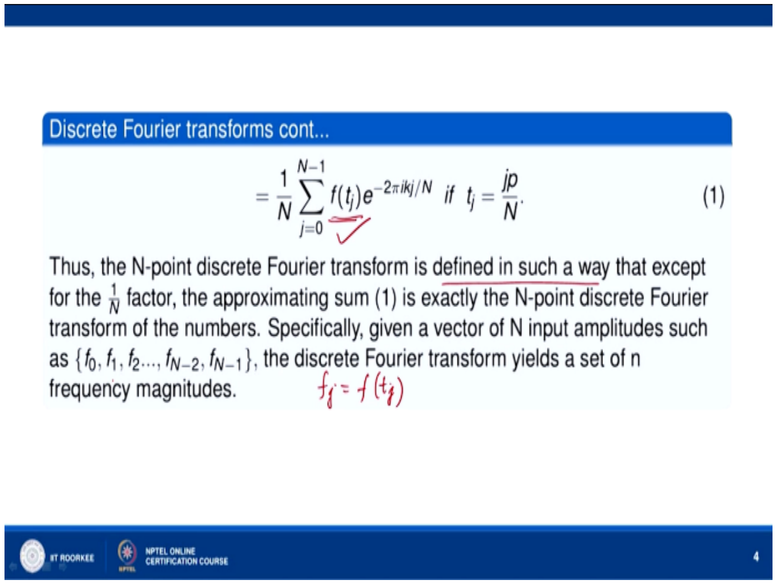

We get 1/n sigma j=0 to n-1 f(tj) e to the power -2pi ikj/N if tj = jp/N, okay. Then, what we get here, put tj to be jp/N so we will get 1/N sigma j=0 to n-1 f(tj) tj is jp/N * e to the power – 2pi ikjp/Np. This we will cancel, okay. So we will get 1/N sigma j=0 to n-1 f(jp/n) e to the power -2pi ikj/N, okay. this is what we get.

Thus, the N-point discrete Fourier transform is defined in such a way that except for the 1/N factor, except this 1/N factor the approximating sum is exactly the N-point discreet Fourier transform of the numbers, f(t0), f(t1), f(t2) and so on f(n-1) okay. Specifically, we can say given a vector of N input amplitudes say f0, f1, f2, fN-2, fN-1. So here I write fj for f(tj), okay. f(tj) I am writing fj, okay.

So given a vector of an N amplitudes, these are N amplitudes such that f0, f1, f2, fN-2, fN-1 the discrete Fourier transform yields a set of n frequency magnitudes. So sigma j=0 to n-1 f(tj) e to the power -2pi ikj/N will give us a set of n frequency magnitudes. And the magnitudes will be given by this dk’s, okay k will run from 0 to n-1. So we will get corresponding to n amplitudes, okay we will get n frequency magnitudes, okay.

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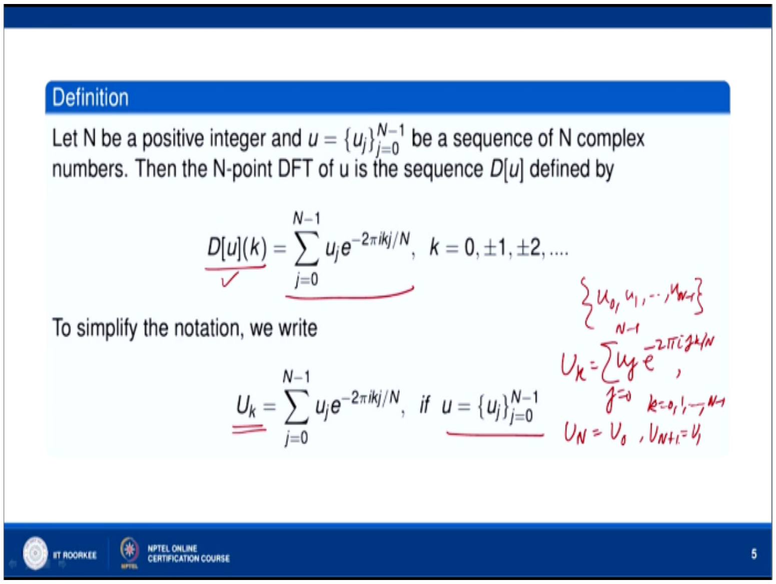

So let us see how we define the discrete Fourier transform. Let N be a positive integer and u be u= {uj}j j=0 to N-1 be a sequence of N complex numbers. Then the N-point DFT of u is the sequence D[u] given by D[u](k) = sigma j=0 to N-1 uj e to the power -2p ikj/N where k takes values 0, +-1, +-2 and so on, okay you can see here. Here this k taking value infinite number values 0, +-1, +-2 and so on.

But you will see that only N out of those infinite number values, only N are distinct remaining are repeated because it turns out that this d[u][k] is a N periodic sequence, okay. So the k value take the value of k from 0 to N-1, those frequency magnitudes are distinct others are just repeating, okay because of periodicity of order; periodicity with period N. So here you can see if you compare this definition, this definition if you compare with this definition, okay. Then exact; except this 1/N factor remaining sum is the same, okay.

We write f(tj) uj here okay and then e to the power exponential function is the same exponential signal e to the power -2pi ikj/N, here also e to the power -2pi ikj/N, j running from 0 to N-1, here also j running from 0 to N-1, okay. Now we can simplify this notation, D[u][k] we write as Uk, so Uk is the discrete Fourier transform of the sequence of n numbers, u0, u1, u2, uN-1. The sequence of numbers is u0, u1, uN-1 and they are complex numbers. Okay.

So this is the sequence. Now, let us; so that the N-point DFT, this N-point DFT is periodic with period n. So we need not write infinite number of values of k, we can simply write k=0, 1, 2, 3 and so on up to n-1, okay. So let us see how it is periodic with period n.

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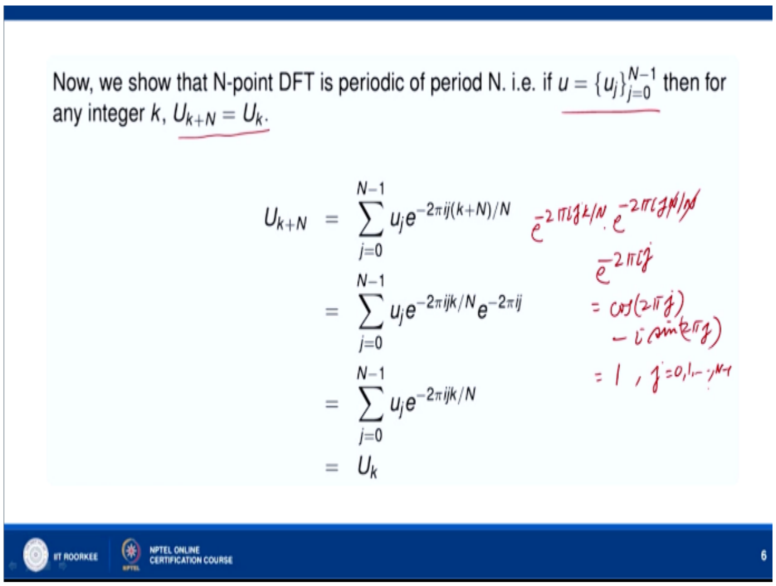

So this is a sequence, okay given sequence of n amplitudes magnitudes and we want to show that Uk+N = Uk for any k, okay. Uk+N by definition you can see, Uk+N is sigma j=0 to N-1 uj e to the power -2pi ij k+n*j/n, okay. U to the power -2pi ij (k+n) n, we can separate it into two parts, e to the power -2pi ijk/n * e to the power – 2pi ij N/N. So this N will cancel with this, okay. Now e to the power -2pi ij, this is equal to a cos 2pi j – i sin 2pi j, okay. j is an integer, okay.

And when j is an integer cos 2pi j=1; sin 2pi j=0, so this is equal to 1, okay when j varies from 0 to n-1, okay. So this becomes this quantity, this quantity becomes e to the power -2pi ijk/n, okay. This is equal to 1, okay. So we get this, and this is equal to Uk. So the Uk, Uk is discrete Fourier transform is N periodic and therefore, Uk is equal to this; we can write Uk = sigma j=0 to N-1 uj e raise to the power -2pi ijk/N where k vary from 0 to 1, 2 and so on up to N-1, okay.

Because Un will be nothing but U0. Un+1 will be equal to U1, okay like this. Okay, so when you take k=n or you take k=n+1 and so on this will be simply U0, U1, U2 and so on, so it will repeat. Similarly, for negative values of K.

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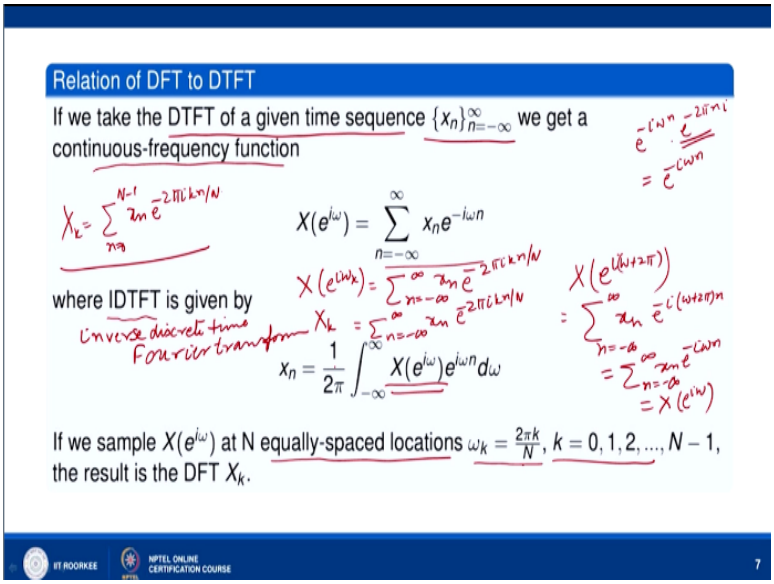

Now let us see how the discrete Fourier transform is related to the discrete time Fourier transform. If we take the discrete time Fourier transform of a given sequence Xn n infinity = to – infinity we get a continuous-frequency distribution, okay. Continuous-frequency distribution we get and it is defined as X e to the power i omega = sigma n= - infinity to infinity Xn e to the power –i omega n. With; inverse IDTFT, okay this is inverse DTFT – Discrete Time Fourier Transform.

This is Discrete Time Fourier Transform; this is Inverse Discrete Time Fourier Transform. So inverse discrete time Fourier transform is given by Xn=1/2pi – infinity to infinity X ei omega e to the power i omega n d omega. Here you can see that, X ei omega i is a continuous frequency distribution, we are considering integral, okay. So it is a continuous frequency distribution and it is 2pi periodic X e to the power i omega you can see, X e to the power i omega +2pi if you consider, okay.

Then this is sigma n= - infinity to infinity okay Xn e to the power –i omega + 2pi * n. So this is e to the power –i omega, -i omega n * e to the power -2n pi i, 2n; no, 2pi ni okay. This value is equal to 1, so we have e to the power –i omega n, so this is equal to sigma, that is x omega ei omega. So it is continuous frequency distribution with period 2pi, okay.

Now, if you, if we sample X ei omega at N equally-spaced locations omega k=2pi I; 2pik/n, okay. k varies from 0 to n-1, the result will be DFT Xk. Now let us see, how we get that. See here you take, X ei omega k, okay. So this is sigma n=-infinity to infinity Xn e raise to the power – omega is 2pik/N, so we have 2pi I, kn/n, okay, 2pik 2pi kn/n.

Now, Xn is defined only for this range, okay 0 to N-1, okay. So this sum will be reducing to; when we replace X e to the power; okay X e to the power i omega k we consider as Xk, okay. Xk=sigma n=-infinity to infinity Xn e raise to the power -2pik n/n, okay so we take omega k=2pi ik/n okay. And consider this at equally-spaced locations, okay equally-spaced locations. And these are N equally-spaced locations, okay.

So then the sequence Xn is taken to be define for N value from 0 to N-1. Okay. So this will be Xk, Xk will be sigma n=0 to n-1, okay Xn e raise to the power -2pi ikn/n okay and which is nothing but DFT, DFT of Xn sequence. So Xn sequence is given for n=0,1,2,3 and so on up to N-1, okay and remaining values we can take as 0s then this Xn then this X ei omega reduces to DFT. Okay.

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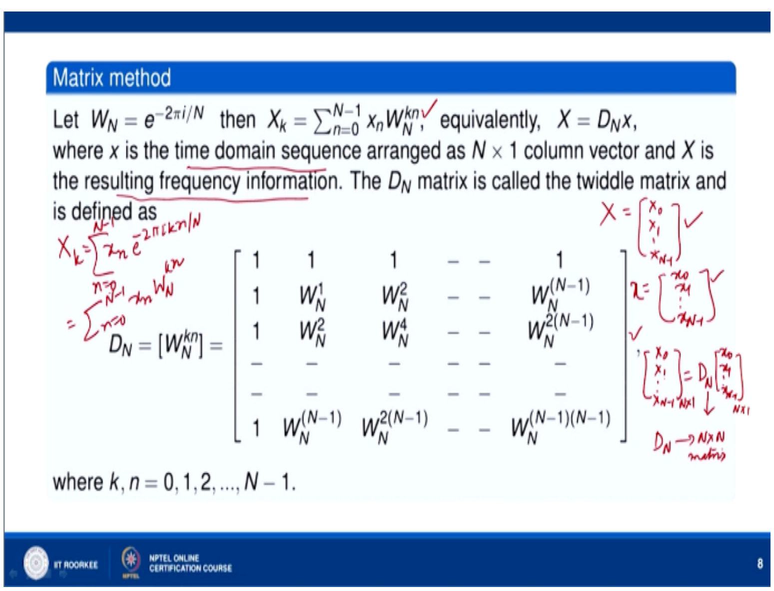

So let us know consider, there is an another method of computing the discrete Fourier transform. This is by Matrix method. Let us denote, Wn e to the power – 2pi i/N by Wn. We have Xk=sigma n=0 to n-1 Xn e raise to the power -2pi i kn/n, okay 2pi i kn/n. So e raise to the power -2pi ikn denoted by by omega n, so this is sigma n=0 to n-1 Xn W n kn. Okay. Xn W n kn.

So this is what we get, okay. Xk=sigma n=0 to n-1 Xn W n kn and equivalently we can write it in the form of a matrix equation, Xn=dnx, okay. So x is this vector, X is column vector, X1, X2 and so on X0, because k takes values from 0 to n-1, so we get here X0 X1, Xn-1. This equal to X, okay. Dn is this matrix, multiplied by X. This small x is the time domain sequence; time domain sequence is that Xn sequence. Okay. So X0, X1, Xn-1.

Okay. So X is this one, X is this one, okay. Capital X is the frequency, resulting frequency formation, column vector, this frequency information and this N/1 column vector is the time domain sequence okay. So X is this one, okay so we have like this. X0, X1, Xn-1 equal to dn sequence matrix * X, okay. So this dn is n/n matrix, okay. This is n/1 matrix, this is also n/1 matrix, dn is n/n matrix, okay. Okay, so this Xk can be written in this form, okay. We let see how we are writing it, okay.

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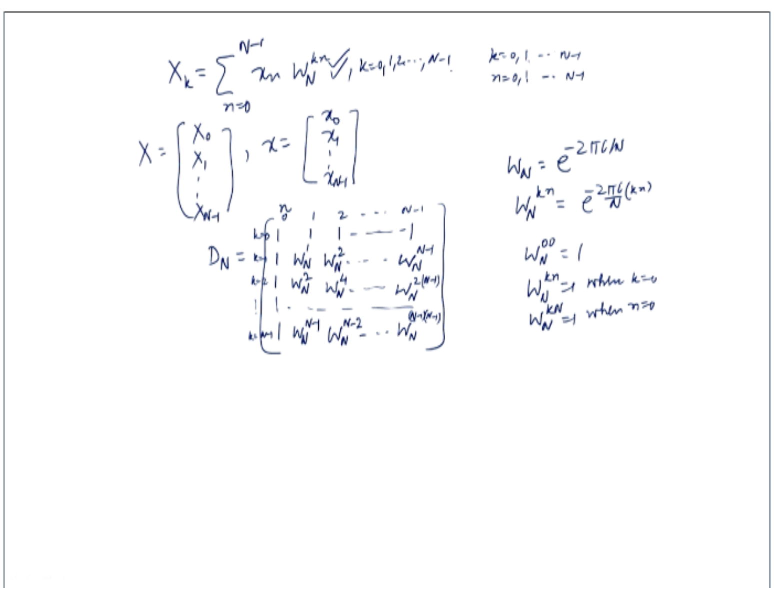

Where Xk = sigma n=0 to n-1 Xn Wn Kn, okay. Now, let us see how we write it. See, X is this one, X0, X1, Xn-1, okay. And small x is small x vector is that time domain sequence, so x0, x1, xn-1, all right. Now, so we let us write Dn sequence, okay. Dn matrix will be, okay. So; see k varies on this Wn Kn, K varies along the rows, and n varies along the columns. So n varies along the columns and k varies along the rows. So first we write when k=0, n=0.

K varies from 0 to n-1 and n also varies from 0 to n-1. Okay. So when, okay, so when this is 0,1,2 and so on n-1 column, okay. Here k=0 then k=1 and then k=2 and then k=n-1 as row, okay. So when k is 0, n is 0. Wn 0 0. Wn=; we have taken Wn to be equal to e to the power -2pi/n, okay. So Wn Kn = e to the power -2pi i/n*Kn, okay. So when K=0 n=0. We will have Wn 0 0. Wn 0 0 = Kn is 0, so e to the power 0 is 1, so we get first element 1.

Now in the first row k is 0, okay. So k is 0 means Wn Kn=1 when k=0. So all elements are 1, okay. Similarly, Wn Kn = 1 when n=0, okay. That is 0; first column okay. So 1 1 and so on 1, okay. Then you have, in this first row, okay k=1 and then 1=1 so we get Wn 1 1, okay Wn 1 1 product I overwriting okay. 1*1 is Wn 1. Then you write Wn K=1 N=2 so you get Wn 2 and so on, Wn K*N. K is 1 N is N-1 so we get N-1. Here K=2 okay, K=2 N=1 so we get Wn 2.

Then K=2 N=2, so Wn 4. Okay. And so on we get K=2, N=N-1 so Wn two times N-1, okay. And then we can similarly write last row okay. K=N-1, N=1, so we get Wn n-1. Then K=N-1 N=2 so Wn N-2 and so on. We get Wn lastly. K is n-1; N is n-1 so N-1 N-1. Okay. This is how we write the matrix Dn. If this matrix Dn when multiplied by X0, X1, X2, Xn-1 equated to this column vector, this give us the all n equations here, okay there are n equations K=0, 1, 2 and so on up to N-1, okay.

So we write these n equations in the form of a matrix equation, like we do in the case of solution of a system of linear equations okay. So here, this is the matrix. This when you write this DFT in the matrix form it is easy to compute as we said by examples, so let us now move onto examples.

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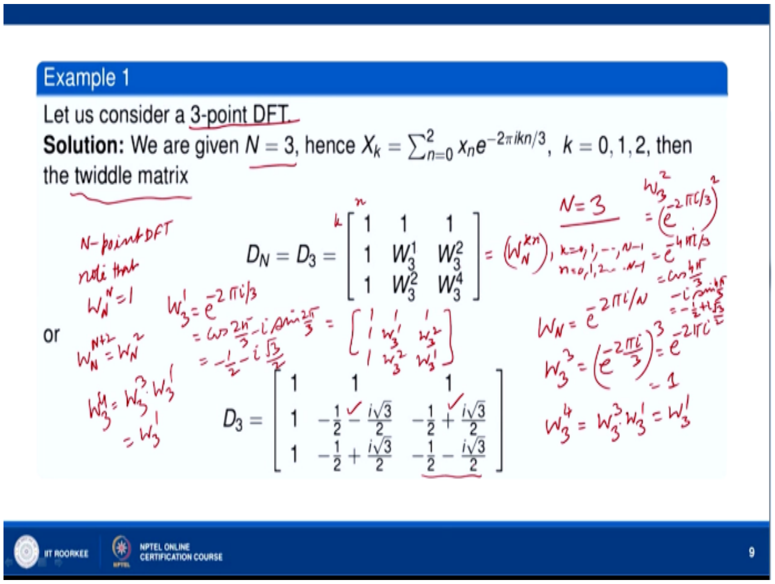

Let us consider a 3-point DFT, okay. 3-point DFT is means N=3. Okay. This DFT is called as N-point DFT, okay. So we are given N=3, okay. Now Xk therefore, sigma n=0 to n-1, n=0 to n-1 means n=0 to 2 Xn e to the power -2pi ikn/3, 3 is n, okay. K=0,1,2. So then the twiddle matrix. This is called as the twiddle matrix, this matrix D, is called the twiddle matrix. Okay.

So twiddle matrix Dn = D3 will be (()) (25:18) okay. So 1 1 1; 1 Wn1 Wn2 1 Wn2 Wn4, okay so this is what you get. N=3. So 1 1 1; 1 W31 W32; 1 W32 W34. As I said here, K varies along the rows and N varies along the columns, okay. And is the matrix Wn Kn, okay. K varies along the columns from 0 to n-1 and n varies along the; K varies along the rows; n varies along the columns. Okay, now let us note one thing. Wn= e to the power -2pi i/n okay.

So, okay N=3, so W33 will be e to the power -2pi i/3 raise to the power 3, okay. W33. Wn Kn means Wn raise to the power Kn, okay. So W3 to the power 3 will be e to the power -2pi i, so we get equal to 1, okay. W3 raise to the power 4, okay. W3 raise to the power 4 when we have, we can write it as W3 raise to the power 3*W3 raise to the power 1. So W3 raise to the power 3 is 1, so we can write it as W31, okay.

So when we have N-point DFT in; for N-point DFT we have to note that Wn n = 1, okay. Wn n = 1. So any power of Wn more than n okay can be reduced to a power less than n, okay. Wn say Wn raise to the power n+2, okay if you have. You can write it as Wn to the power 2, okay. So here, we have W31, W32, W34, W34 will be equal to W33*W31 okay. And W33 is 1 so we get W31. Okay. So this will be, this matrix will be 1 1 1; 1 W31 W32; W32 and then W31, okay.

If we have a N-point DFT the power of Wn; if it is n it will be 1, if it is more than n it will be, can be reduced to power less than 1, less than n okay. Now, let us see what is W31. W31= W3 is e to the power -2pi3, so it is cos 2pi i/3 – i sin 2pi/2 and its value is cos 2pi/3 is -1 sin 2pi/3 is root 2/3, so –i root 3/2. Okay. So this is W31. And W32, W32 is e to the power -2pi i/3 raise to the power 2. So e to the power -4pi i/3.

So this is cos for pi/3 – i sin 4pi/3. Cos pi/3 is cos 4pi/3 is cos pi + pi/3 so that is –cos pi/3. And –cos pi/3 is -1/2, okay. Sin 4pi/3 is sin pi+pi/3 which will be –sin pi/3 is –root 3/2 so this is +I root 3/2. Okay. So you get this value. So W31 is this, W32 is this, W32 is same, this is -1/2 + i root 3/2, we have calculated already and this W34 we have seen, W34 is W31. W31 we have already found, this is -1/2-i root 3/2. So this is how we calculate D3.

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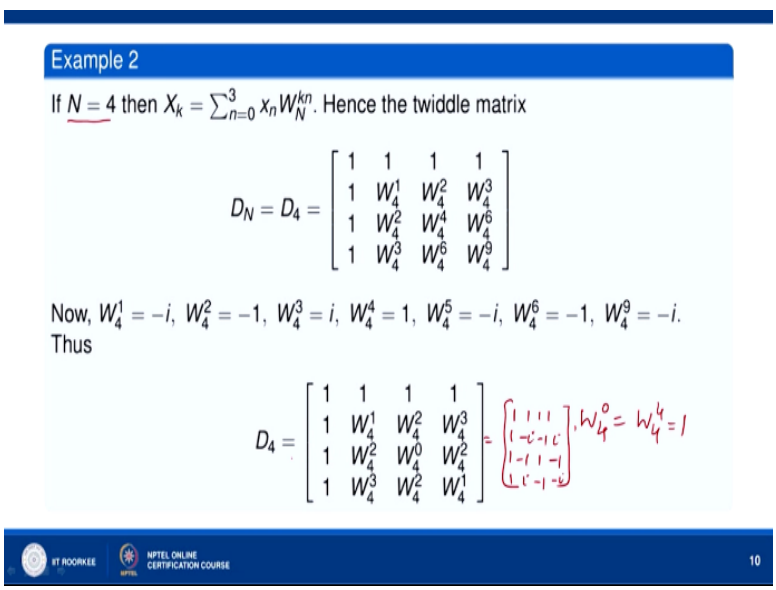

Now, let us consider 4-Point DFT. When you consider 4-Point DFT that is N=4, Xk will be sigma n=0 to 3, Xn Wn Kn, okay. Twiddle matrix will be what, again let us see, let me write twiddle matrix for N=4, okay.

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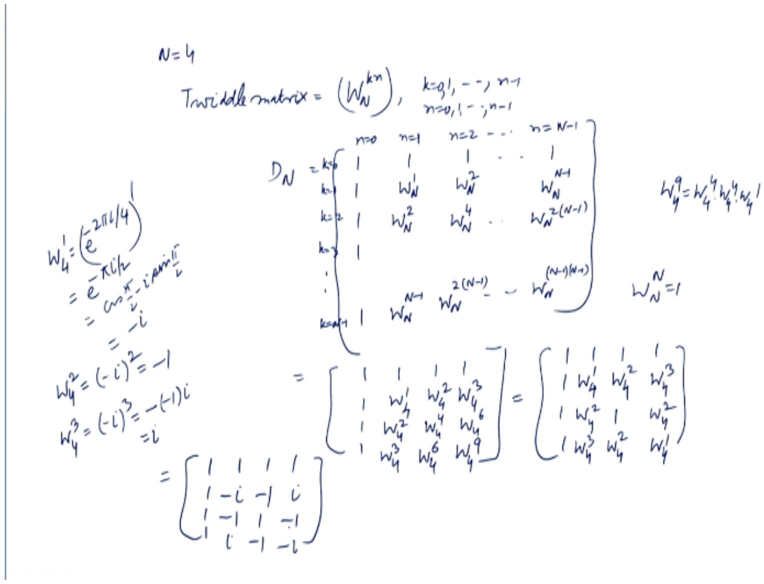

So twiddle matrix, okay. So this equal to Wn Kn. Okay. Where K runs from 0 to n-1 and n runs from 0 to n-1. As I said, twiddle matrix Dn be denoted by Dn and Dn is given by, okay. So k varies along the rows, okay this is k=0, k=1, k=2, k=3 and so on k=n-1 of row. And here $n=0^{th}$ column n=1 column, 2, n=n-1. So n=0 is the first column, n=1 is the second column okay. Wn k*n, k*n when k is 0 or n is 0. Okay. Wn to the power 0 means 1, okay.

So this is 1, this is 1, this is 1, 1, okay. Either k is 0 or n is 0. And then when n is 0 in the first column we also have 1. Then, first k=1, n=1 so we get Wn to the power 1. When k is 1, n is 2 we get Wn to the power 2, okay. When k is 1, Wn n=n-1 we get Wn to the power n-1, okay because k is 1. Then, k is 2 so we get Wn 2 * n is 1, okay. Then, Wn 2*2 that is 4. And then Wn 2*n-1 and so on. Lastly, we write n-1. So Wn, okay k is n-1, n is 1 so we get n-1. K is n-1, this n is 2, so two times n-1. And we get Wn-1, sorry Wn n-1 * n-1. Then we put n=4 in this; this general matrix.

So 1 1 1 1; 1 1 1 okay then we get W41 okay W42 and we get N=4, W43, okay N=4. Then we get Wn1 Wn4 then we get N=4, so this is Wn6, okay. Then 1 Wn to the power 3, Wn to the power 4-1 so that is 4*2; no 4-1 is 3, 3*2 is 6. And then 4-1 is 3, 3*3 9 so Wn9, okay. Now, we can reduce it using the fact that Wn to the power n=1, okay. So this can be written as further 1 1 1 1; 1 W4 W4; 1 W2 W3 then 1 W4 this N should be 4 here, okay everywhere.

Okay, W42 W42. W44 is 1, okay W44 is 1. W46 is W42 then 1 W43 then W46 is W42 okay and this W49, W49 is W44 * W44 * W41, so W41, okay. W49 = W44 * W44 * W41. Okay. Now W41 is how much? W41 is e to the power -2pi i/4. Okay raise to the power 1 because Wn is e to the power -2pi/n, okay. So this is e to the power –pi i/2 so this is cos pi/2 – i sin pi/2 so this is –pi, okay. Then W42, okay. W42 you want. Okay, W42 is W41 square, okay.

So –i square. So this is i square is -1, so we get -1. W4 cube, W4 cube is –i whole cube, okay. So –i whole cube means -1 whole cube is -, okay. i cube is i square * i. i square is -1 * i so +i. Okay. So we get the value of W41, W42, W43 okay. So let us put these values, so then what do we get? The matrix as 1 1 1 1; 1 1 1; W41, W41 is –i W42, W42 is -1 W43 is i okay. And then W42, W42 is -1, W41, W44 is 1 okay, W42 W42 is -1 okay.

And then W43, W43 is i, okay W42, W42 is -1 and W41, W41 is –i. So let us see whether we have this same matrix here. Okay, so we have, we have this matrix, okay. D4 is this matrix. W40, W40 is nothing but W44. Okay. So this is equal to 1. All right. So let us see whether we have this same values or not.

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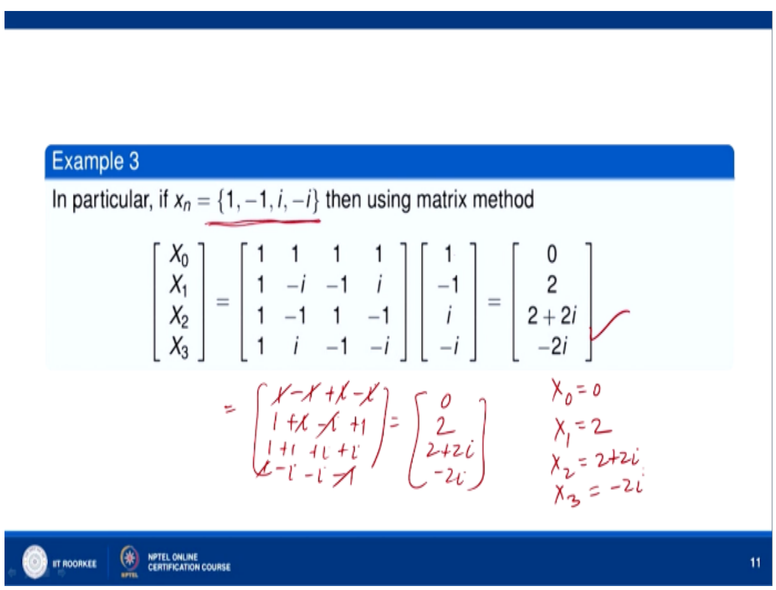

We have X0, X1, X2, X3 this is the left hand side frequency they may have some column, this is the amplitude column, this one. We are taking an example here. Here you can put the values, okay. Let me see; I have not put the values here. This is equal to, if you put these values which I have calculated, this is equal to 1 1 1 1 and then you get 1 and W41=-i and W42=-1; W43=i, okay. And then we have 1 W42=-1, W40=1; W42, W42=-1, okay.

And then we have 1, W43 is i, W42 is -1 and W41 is –i. So let us see, first row; first column they always 1. In the second row we have 1 –i -1 i, okay. 1 –I -1 i. 1 –i +1 i, so okay. Then, we have here 1 -1 1 -1; so we have 1 -1 1 -1 and then we have 1 i -1 –i; 1 i -1 –i, okay. So this is how we calculate this D4. Now let us take an amplitude matrix and let us determine the corresponding frequency matrix, okay. So this time domain, okay time is but 1 -1 i –i okay.

So 1 -1 i – i and these are the values of; this D4 matrix. Let us check. So we multiply by 1 -1 i – i to this matrix and what you get in the numerator, what you get here; this matrix as 1-1 + I - i okay when you multiply this column to first row. When you multiply this column to second row what you get, 1 and –i * -1, so we get +i and -1 * i is –i 1 –i * is –i square so +1.

Okay. And then when, you can multiply to this here 1 -1 – 1 is +1 and then here i we have and here we get +i, okay. So then we get similarly last column, last row we multiply by this and we get 1 then we get –i, we get –i, we get +i square, so -1 okay. So this cancels with this, this cancels with this and this cancels with this. And here we have 1 cancels with 1.

So this is what you have. 0 and then you have 2 you have 2+2i and we have -2i, okay. So this is what we get. So X0, X0 = 0, okay X1=2; X2=2+2i and then X3=-2i, so we can get the frequency of magnitudes when you know the time amplitude, okay. So let us consider the constant sequence. Suppose we are given the constant sequence.

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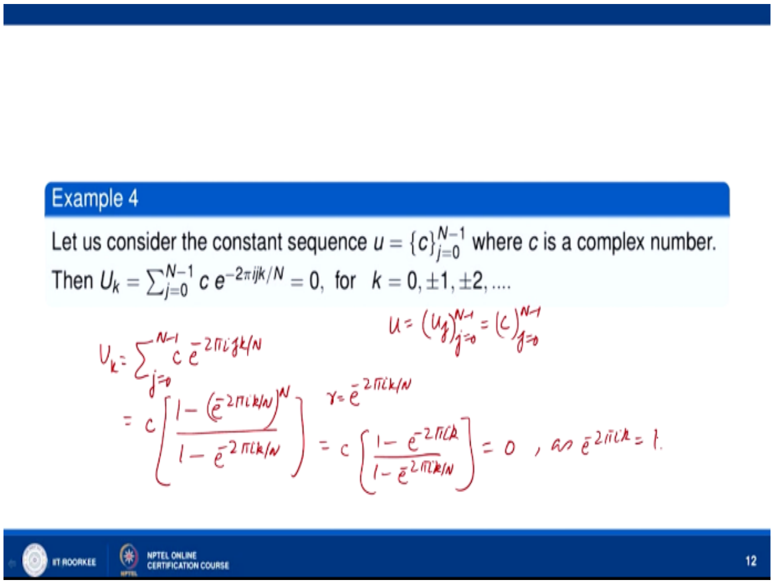

Uj, here u=c; j=0 to 10-1. We have taken u=(uj) okay j=0 to n-1 as the sequence. So here we are given uj as c for all j, okay. So if you want here Uk, Uk=sigma j=0 to n-1, uj is c, e to the power -2pi i, jk/n, okay. C can be taken outside because it is constant and then this is a geometric series, okay. This is a geometric series, you can see.

And the ratio is e to the power -2pi ik/n okay, geometric ratio, r is e to the power -2pi ik/n. So we can sum it up, sum will be 1- e raise to the power -2/i k/n raise to the power because there are n terms, so raise to the power n/1- e raise to the power -2pi ik/n, okay. Now here what will happen, this is c times 1- e raise to the power -2pi ik/n so we get e to the power -2pi ik/1- e raise to the power -2pi ik/n. And k is an integer, okay.

So e to the power -2pi ik will be equal to 1, okay. So this is 0, okay. As e raise to the power -2pi ik=1, okay. So if we take constant sequence then it is discrete for a transform is 0 for all values of k. Okay. K is taking; we are taking writing here +1 -1 + -2 and so on as we have already seen Uk is a n periodic sequence, so we can reduce this k to 0,1,2,3 and so on up to n-1. Okay. Now let us we take another example.

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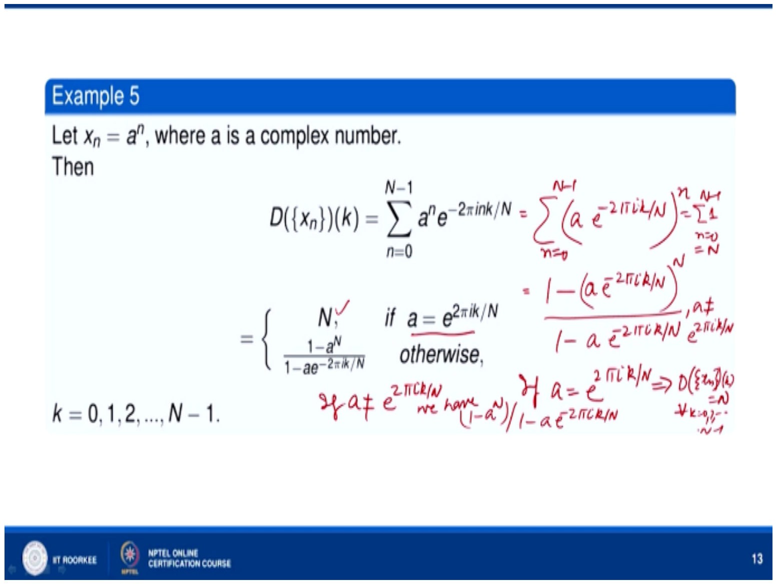

Suppose Xn is a to the power n. Here a is a complex number. Then, the discrete Fourier transform of the sequence Xn, k is given by sigma kn=0 to n-1 a to the power n e to the power -2pi i and k/n. And this can be written as sigma n=0 to n-1 a e raise to the power -2pi ik/n whole to the power n, okay. So this is a geometric series again, okay. So sum will be 1 -1 e a raise to the power -2pi ik/n whole to the power n, okay divided by 1-a e raise to the power -2pi k/n, okay.

Now this will be equal to; if a=2pi ik/n okay if a=2pi ik/n yeah okay here, you can see. Here you can see that, if a=e raise to the power 2pi ik/n okay. We can divide by this only when a is not equal to e to the power 2pi ik/n. If a=e to the power 2pi ik/n then this will be equal to 1, okay and we will sigma n=0 to n-1 okay 1 which is equal to n. So if a=e to the power 2pi ik/n then we get the Xn k=n for all k values of k okay.

And if a is not equal to e to the power of 2pi ik/n then we get this. So here, when you raise to power n we get 1-a to the power n because e to the power -2pi ik will be equal to 1, so we get the if a is not equal to e raise to the power 2pi ik/n, this is here we are taking a not equal to e raise to the power 2pi ik/n okay, so then we get 1-a to the power n/1-a e raise to the power -2pi ik/n. And if, as I said if a=e to the power 2pi ik/n we get the value 1. Dxn k=n for all k. Okay. So this is how we do this problem.

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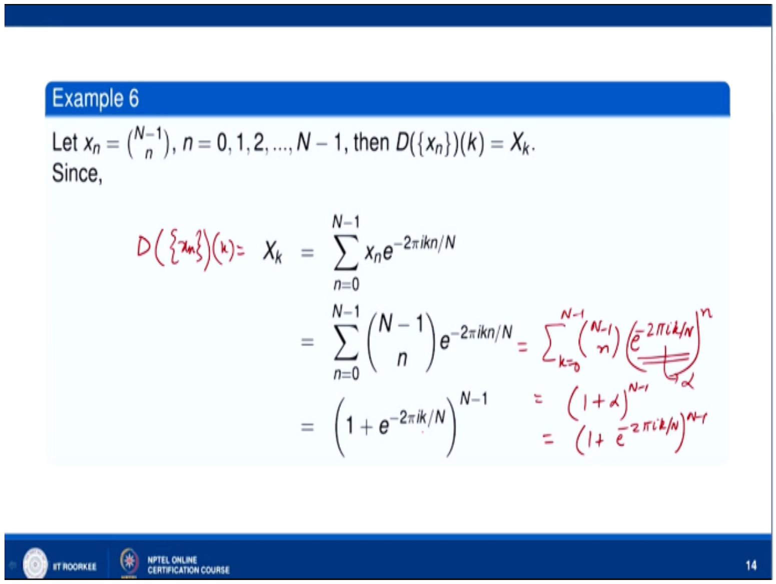

If Xn is the binomial coefficient n-1 cn, okay then Xk, Xk is again D of Xn at k okay. So that we are writing as Xk, sigma n=0-1 Xn e to the power 2pi ik/n, so this is sigma n=0 to n-1, n-1 cn, e to the power -2pi ikn/n. And this can be regards as sigma k=0 to n-1, n-1 cn e to the power -2pi ik/n raise to the power n, okay so then this by binomial theorem this is nothing but 1+; if it is suppose alpha then 1+alpha to the power n-1, okay.

Alpha is e to the power -2pi ik/n, okay. So this is 1+e raise to the power -2pi I; this I am writing as alpha. Okay. Okay, so this how we can do this problem. Again, for k=0,1,2,3 and so on up to n-1.

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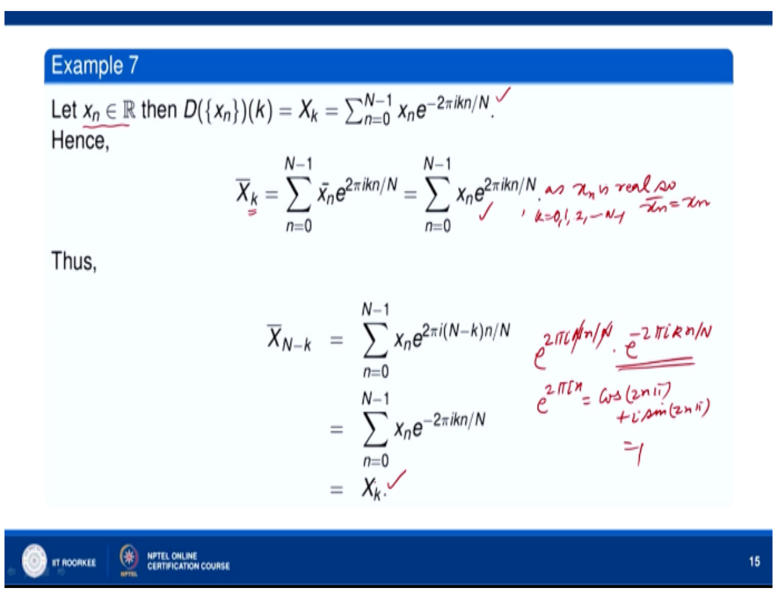

Now if; this is a very important question. If Xn is real sequence, we are going to show that, if Xn is a real sequence then discrete sequence of Xn is also real, okay. So let Xn belong to r then then Xn k = Xk = sigma n=0 to n1 xn e raise to the power -2pi ik/n by definition we have this, okay. Hence, let us take the conjugate of Xk. We want to show that Xk is real for every k, so let us take conjugate of this and we shall then show that Xk bar = Xk for every k.

So Xk bar = sigma n=0 to n-1, Xn bar and the conjugate of e to the power -2pi ik/n is e to the power 2pi ik/n, okay. Now, Xn is real, Xn bar = Xn. Okay. So this is because as Xn is real. So Xn bar = Xn. So this question, this is equal to this, okay. Now, let us replace, let us consider Xn, this is valid for any k, okay. So let us consider X bar nk, then k can be; here k varies from 0 to n-1, okay. k varies from 0 to n-1. Okay. So let us replace k/n-k then, okay.

So X bar n-k will be from here sigma n=0 to n-1 Xn e to the power 2pi i, we are replacing kn-k, so n-k * n/n. And we can write it into two parts. e to the power 2pi n*n/n and the other part is e to the power -2pi k * n/n, okay. So this n cancels with this. e to the power 2pi in=1, e to the power 2pi in=cos 2n pi + i sin 2n pi, so this is equal to 1 because n is an integer. Okay. So we get, this is equal to 1, we get only this, okay. So we get Xn e to the power -2pi i kn/n. So X bar n – k = Xk.

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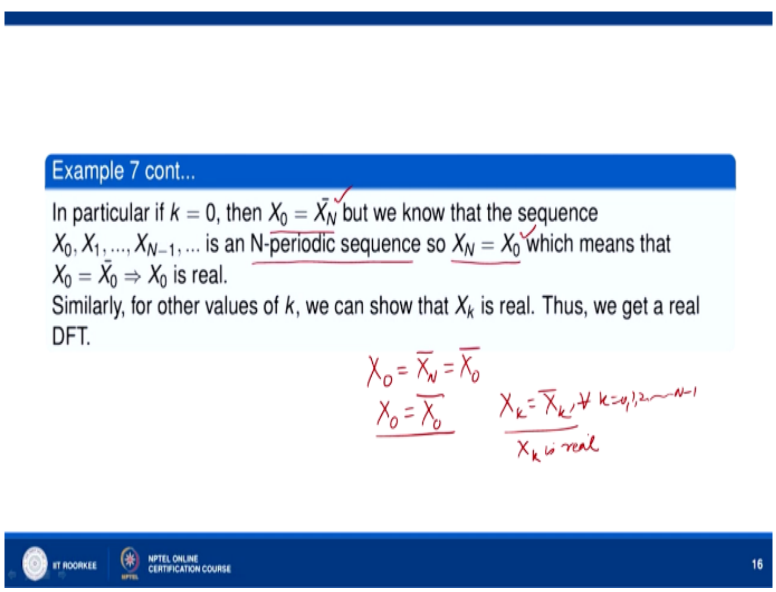

Now let us put k=0 in this, okay let us put k=0, so what do we get Xn bar = Xn 0. Xn bar = X0. Now we know that X0, X1, X2, Xn-1 is an N periodic sequence, okay. So Xn=X0, okay. Xn=X0, so what do we get, X0 = Xn bar. X0 = Xn bar so and Xn= X0 so Xn bar = X0. Okay. We know that, we have found that X0 = Xn bar but because of n periodicity Xn=X0 var. So combining this and this okay, we get X0 = Xn bar = X0 bar, So S X0 = X0 var.

So this means that X0 is real, okay. Similarly, we can take k=1,2,3 and so on up to n-1 and show that Xk=Xk bar for all var. So we can show for other values of K that Xk is real. Okay, so Xk is real. If, so if the input sequence is real the output is also real, so we get real DFT.

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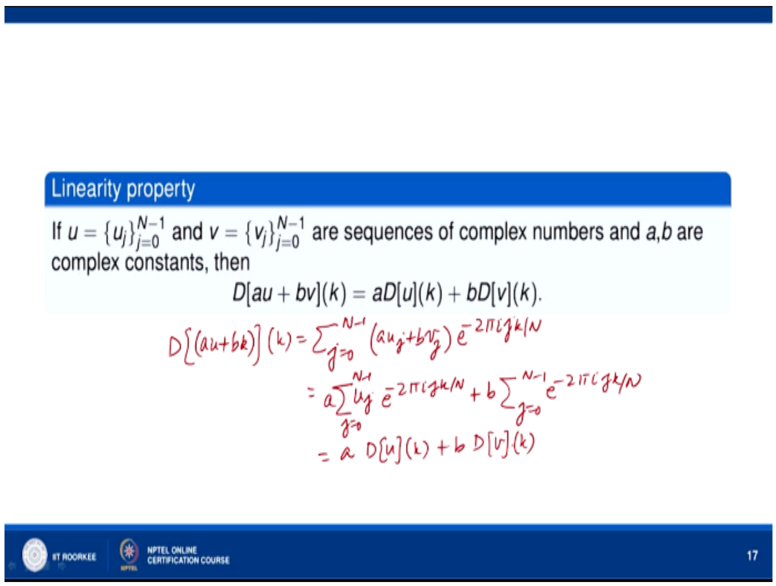

Okay. Now Linearity Property. If you u=uj j=0 to N-1; v = vj j=0 to N-1 are sequences of complex numbers and a, b are complex constants, then D of au + bv k = aD u k + bD v k. Okay. So this we can easily show D of au + bk by definition = sigma j=0 to N-1; au j + b bj, e raise to the power -2pi jk/n okay. Now, since this is a finite summation I can write it as a uj+ b bj, b breaks in two parts then take a and b common a times, a will be out so a times uj here + b times okay so this is a times D u k + b d v k, okay.

So this discrete Fourier transform satisfies linearity condition. It satisfy linearity property.

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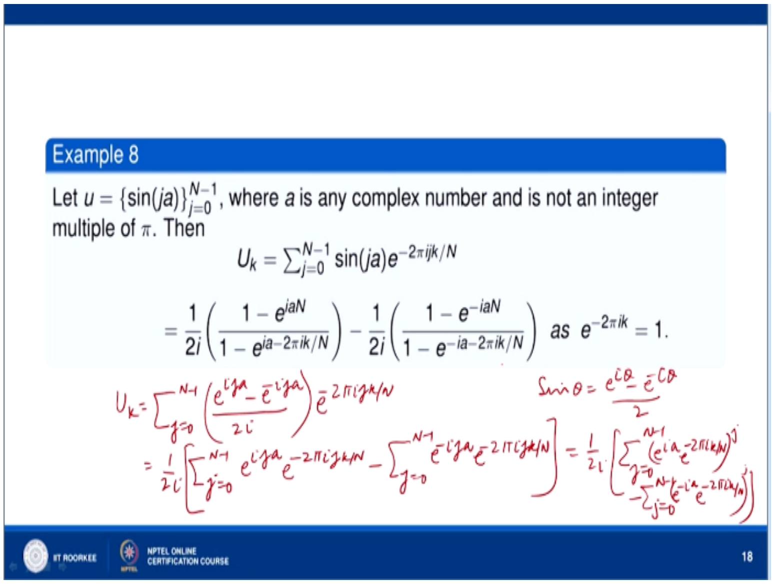

Okay, now let consider this problem. u = sin ja j=0 to N-1 where a is any complex number and it is not an integer multiple of pi. Okay, so then if you find the discrete Fourier transform of this sequence then Uk by definition will be equal to sigma j=0 to N-1 sin ja e to the power -2p ijk/N, okay. And let us put then the value of sin ja. Sin theta we know, sin theta = e to the power i theta – e to the power –i theta/2, okay. So sin ja we can put some values.

So Uk = sigma j=0 to N-1 e raise to the power ija – raise to the power –ija/2i multiplied to e to the power -2pi ijk/N, okay. So now let us use linearity property. So by linearity property we have 1/2pi times sigma j=0 to N-1 e to the power ija e to the power -2pi ijk/n and then sigma j=0 to N-1 e raise to the power –ija e to the power -2pi ijk/n okay. Now we can write; let us find sum of this. So this is actually can be written as 1/2i sigma j=0 to N-1 e to the power ia sin; e to the power ia; e to the power -2pi ik/n raise to the power j, okay.

And then similarly the other one, sigma j=0 to N-1 e to the power –ia e to the power -2pi k/n raise to the power j, okay. They are both geometric series, okay. So we can write it sum.

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$$= \frac{1}{2i}\left[\sum_{j=0}^{N-1}\left(e^{ia-\frac{2\pi ik}{N}}\right)^j - \sum_{j=0}^{N-1}\left(e^{-ia-\frac{2\pi ik}{N}}\right)^j\right]$$

$$= \frac{1}{2i}\left\{\frac{1-\left(e^{ia-\frac{2\pi ik}{N}}\right)^N}{1-e^{ia-\frac{2\pi ik}{N}}} - \frac{1-\left(e^{-ia-\frac{2\pi ik}{N}}\right)^N}{1-e^{-ia-\frac{2\pi ik}{N}}}\right]$$

$$= \frac{1}{2i}\left\{\frac{1-e^{iaN}}{1-e^{ia-\frac{2\pi ik}{N}}} - \frac{1-e^{-iaN}}{1-e^{-ia-\frac{2\pi ik}{N}}}\right] \qquad \left(e^{-\frac{2\pi ik}{N}}\right)^N = 1$$

So we have, we have 1/2i sigma j=0 to N-1 e to the power ia – 2pi ik/n, okay 2pi k/n raise to the power j, okay. So then we have this and then we have sigma - we have in the middle sigma j=0 to N-1 e to the power –ia -2pi ik/n raise to the power j, okay. And we can write it sum 1/2i, so this is geometric series. So 1-e to the power ia-2pi ik/n raise to the power n, okay divided by 1- e to the power ia – 2pi k divided by n, okay.

Here, we will have 1-e raise to the power – ia – 2pi ik/n whole to the power n, divided by 1-e to the power –ia -2pi ik/n, okay. Now this is nothing but 1/2i, 1-e to the power ian 1/1-e to the power ia-2pi ik/n, okay. And then -1-e raise to the power –ian/1-e raise to the power –ia-2pi ik/n. This is because the e to the power -2pi k/n raise to the power n = 1, okay. So this is what we get, okay as the answer to this problem, okay. Now; so with this I would like to end my lecture. Thank you very much for your attention.