Advanced Engineering Mathematics

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Lecture – 04
Applications to the Problems of Potential Flow - I

Hello friends, welcome to my lecture on applications to the problems of potential flow, the Laplace equation del square u = 0 is one of the most important partial differential equations and engineering mathematics because it occurs in connection with gravitational fields, electrostatic fields, a steady state heat conduction in incompressible fluid flow etc.

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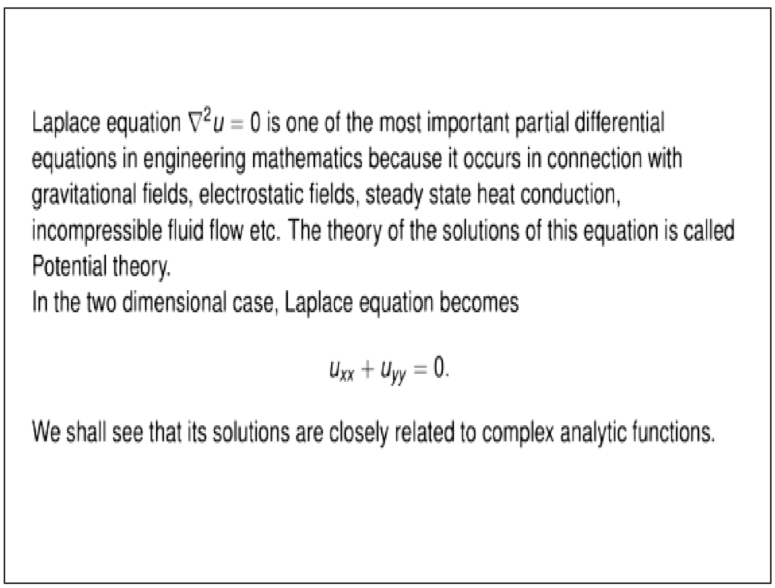

The theory of solutions of the Laplace equation is called the potential theory. Now, in the 2 dimensional case, the Laplace equation del square u = 0 becomes uxx + uyy = 0, we shall see that its solutions are closely related to complex analytic functions, when we will look at the solution of 2 dimensional Laplace equation, we are going to see that they are closely related to complex analytic functions.

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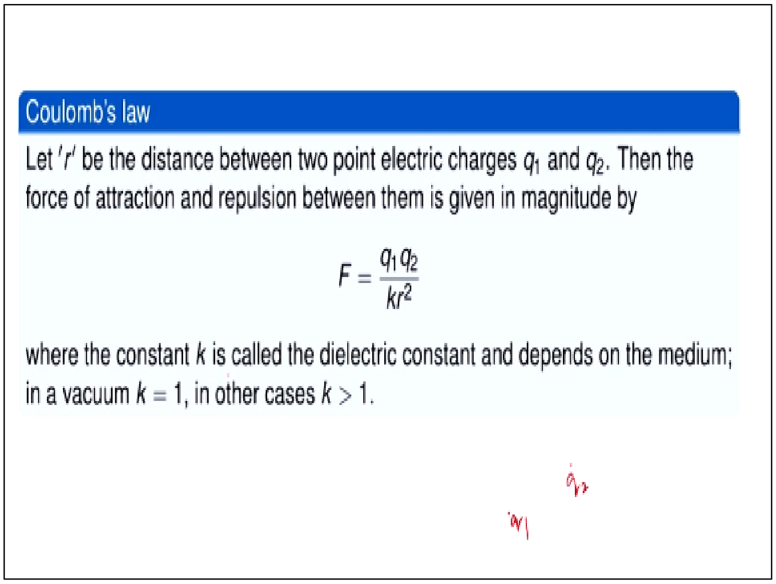

Let us first discuss Coulombs law; let r be the distance between 2 point electric charges q1 and q2, let us say 2 point electric charges q1 and q2, okay then the force of attraction and repulsion between them is given in magnitude by F = q1 q2 over kr square. Here, k is the dielectric constant and it depends on the medium in a vacuum, the value of k is 1 and in other cases, the value of k is > 1.

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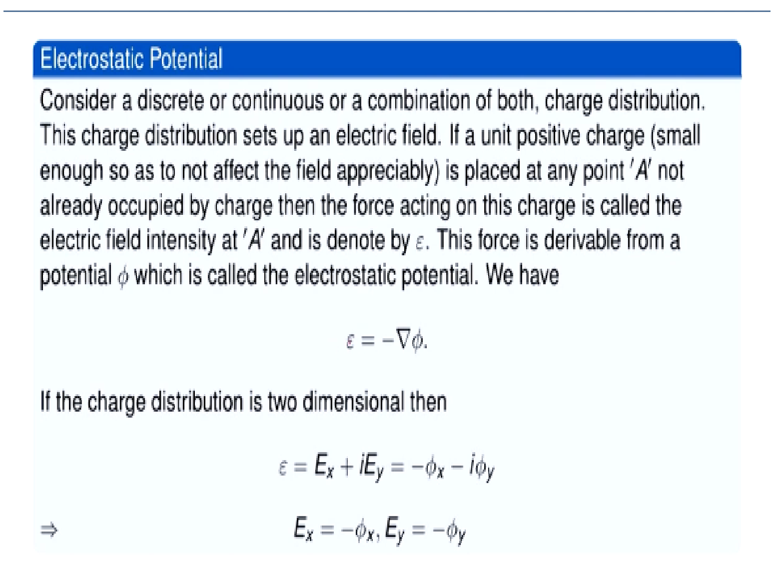

Now, let us consider a discrete or continuous or a combination of both charge distribution, this charge distribution sets up an electric field, if a unit positive charge is small enough so as to not affect the field appreciably is placed at any point A not already occupied by charge then the force acting on this charge is called the electric field intensity at A and is denoted by this symbol, okay.

This force is derivable from a potential phi which is called the electrostatic potential, okay and we have this; we can call it as big epsilon = - del phi, okay. If the charge distribution is 2 dimensional than this big epsilon is Ex + i Ey, okay and this big epsilon is – del phi, so this is = - phi x – i phi y, when we write the; this is the notation in the complex plane, okay as such del phi is i phi x + j phi y in 2 dimensions.

But when we write it corresponding representation in the complex, okay it will be written as -5x – i phi y, so we have – phi x – i phi y and here also you can see this force, epsilon is written as Ex + i Ey, this is the complex representation of the field intensity which is Ex i + Ey j, okay, this iota is the complex number, this iota is not the unit vector, okay, this iota is; here this iota is square root -1, so Ex = then – phi x and Ey = - phi y.

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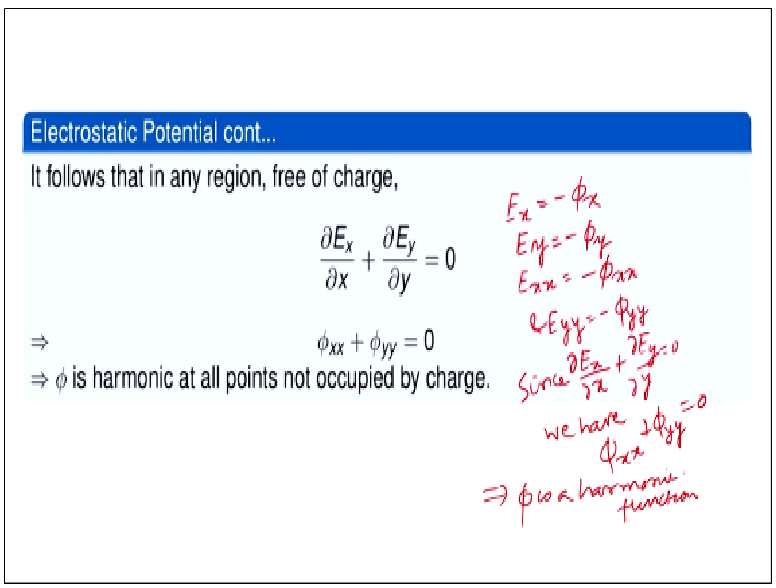

Now, it follow that in any region free of charge, the partial derivatives of Ex + partial derivative of Ey that is = 0. Now, Ex =; we have seen Ex = - phi x and Ey = - phi y, so Exx = - phi xx and Eyy = - phi yy, which implies that we have the partial derivative of Ex with respect to x + partial derivative of Ey with respect to y, okay, this is what we have, okay. So, since we have; since partial derivative of Ex with respect to x + partial derivative of Ey with respect to y is 0.

We have phi xx + phi yy = 0, okay which means that phi is a harmonic function, at all points which are not occupied by a charge.

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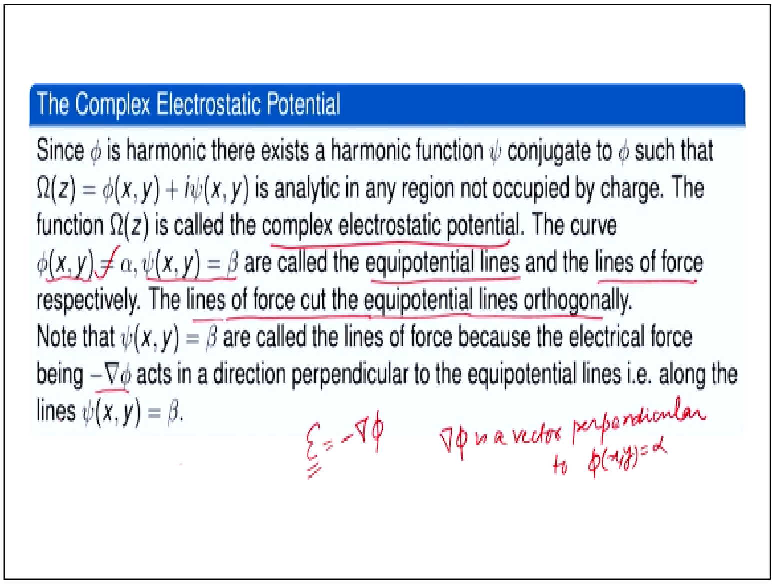

Now, we have harmonic function phi with us, okay, so we can find its corresponding conjugate harmonic functions psi, okay such that the complex function, omega z = phi xy + i psi xy, phi xy is the real part of omega z and psi xy is the imaginary part of omega z. Now, this function is analytic in any region not occupied by the charge, the function omega z is called the complex electrostatic potential, okay, this function is called the complex electrostatic potential.

The curve phi xy = alpha, okay and psi xy = beta, find 2 level curves, family of level curves, okay since phi is the potential, okay, we are writing this big epsilon as – del phi, so that is the phi is the potential there, so phi xy = alpha means, the curves phi xy = alpha, the level curve phi xy = alpha are called as equipotential lines or you can call them as equipotential curves because on every curve, the potential is the same.

And then psi xy = beta are known as the lines of force, okay respectively, lines; we write lines but they can be called as curves, so the lines of force cut the; now we have seen that the family of level curves phi xy = alpha and psi xy = beta, they cut each other orthogonally, okay so, the lines of force cut, the equipotential line orthogonally. Note that psi xy; why we call psi xy = beta as lines of force.

Because pis xy = beta, the direction of the electrical force, okay we had the electrical force – del phi, okay and this del phi is vector which is perpendicular to phi xy, we know this that phi del y; del phi is a vector, a perpendicular to phi xy = constant, okay. So, del phi is a vector perpendicular to phi xy and a psi xy = beta is also perpendicular to this phi xy, therefore the direction of psi xy = beta and the vector; electrical force, this electrical force are same.

And therefore, we call psi xy = beta s, lines of force, so let us note that psi xy = beta are called the lines of force because the electrical force which is –del phi, it reacts in a direction perpendicular to the equipotential lines phi xy = alpha, okay, that is along the lines, psi xy = beta.

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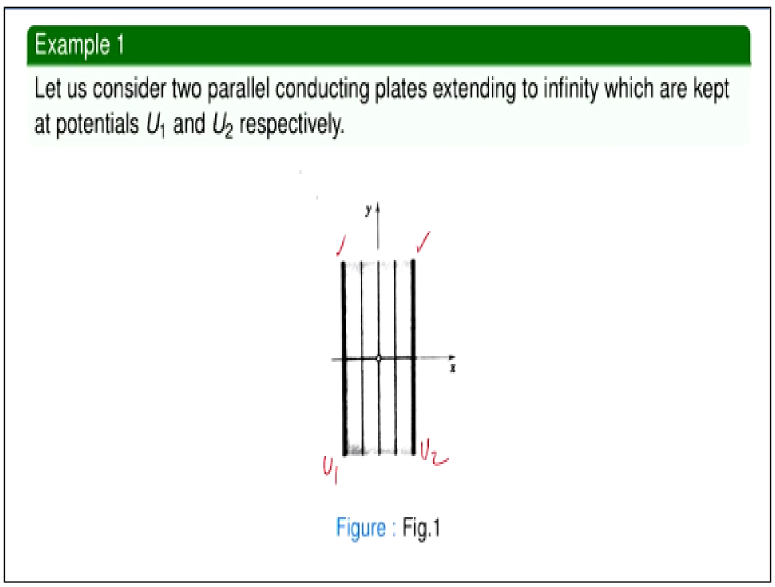

Now, let us consider 2 parallel conducting plates; one conducting plate is this one, the other conducting plate is this one, they are extending to infinity and they are capped at potentials; u1 and u2, so this is having the potential u1 throughout and these having potential u2 throughout and they are infinitely long.

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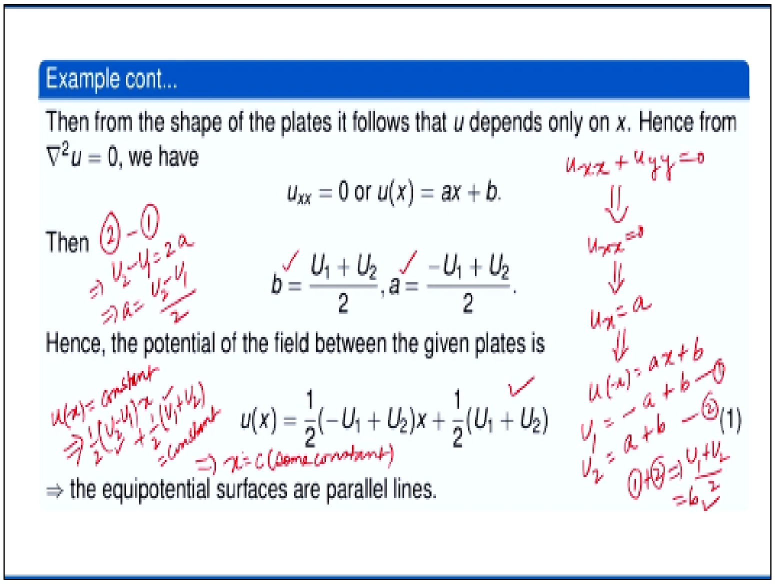

Then from the shape of the throughout; shape of the plates, it follow that u depends only on x, you can see that U is constant on this plate, okay, it does not depend on y, it is E = U1, okay throughout this plate, and this is let us say this plate is x =; this plate is parallel to y axis, so the equation of this plate is x = some constant, okay and so for all values of y, you can see U =; if you denote the potential then U = U1.

And similarly, here this is; this plate is given by another equation h = some constant, okay, so along this plate also for all values of y, okay we have U = U2, so u depends only on x, it does not depend on y, okay. So, since U depends only on x, the Laplace equation del square U =0 gives us Uxx = 0, del square u is Uxx + Uyy = 0, okay. This implies that Uxx = 0 because U is independent of y, okay.

Now, Uxx = 0 is; when we integrate it twice, if you integrate it once, what you get; Ux = some constants say, a okay and when you integrate it again with respect to x, you get Ux = ax + b, so U is the function of x, it is linear function, ax + b. Now, let us use the condition that when x =; we have the condition that and this plate U = U1, on this plate U = U2, so when we use that we get the values of a and b, okay.

So, let us put them as x = 1 and -1, so this is x = -1, okay and this is x = 1, okay, so when you take x = - 1, x = 1, what you get; Ux = U1, so U1 = -a + b, okay, U1 is –a + b and u2 = a + b, okay, so when you add the 2 equations, U1 + we get, adding these, let us call it as 1 and this as 2, so then 1 + 2 gives you U1 + U2/ 2 = b, okay we get the value of U1 =; value of b as U1 + U2/2 and when we subtract the coefficient 1 from 2, so 2 – 1 gives you what?

U2 – U1 = 2a so that gives a = U2 – U1/ 2 okay, so we get the values of and b by taking the 2 plates at left plate at x = -1 and the other plate at x = 1, okay we get the values of a and b and then putting the values of a and b then Ux = 1/2 – U1 + U2* x, 1/2 U1 + U2 =, okay. Now, Ux is potential, okay Ux is potential, so if you take ux potential to be constant then you see that Ux = constant gives -1/2 U2 – U1 1/2 U2 – U1 * x + 1/2 U1 + U2 = constant, okay.

Since U1 + U2/ 2 is the constant, U2 – U1/ 2 is the constant, this implies that x = some constant, okay, some constant c, okay. So, when we take U xy; Ux = some constant we get x = some constant, so means for different values of the constant will get different lines which are parallel to y axis, so equipotential lines are surface lines; equipotential surfaces are parallel lines, they are parallel to y axis.

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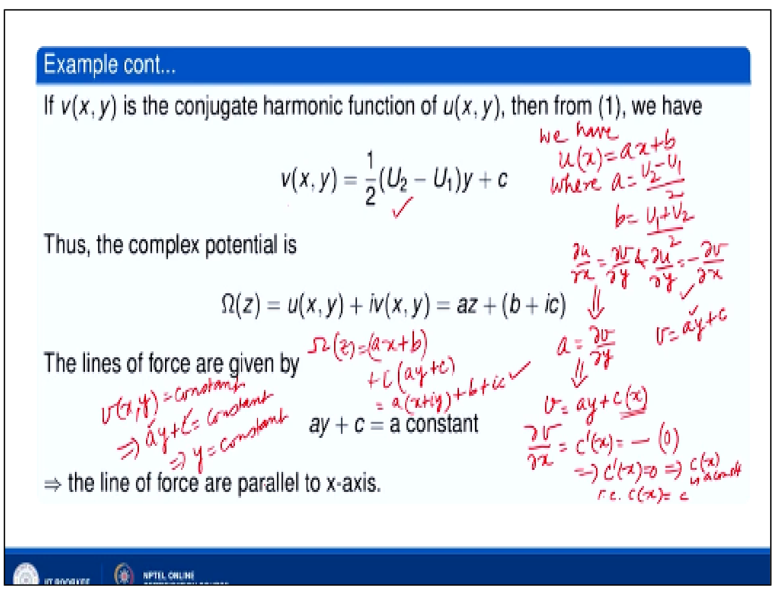

If v xy is the conjugate harmonic function of u xy then we can find the value of v xy, why using Cauchy- Riemann equations, we have u xy =; w have ux; u is not depending on y, so ux = ax + b we have, we have a = u2 – u1/ 2, b = u1 + u2/ 2, okay. Now, let us use the Cauchy- Riemann equations, so ux = by and uy = - vx, so let us first use this equation, so u xb = a, okay, so a = vy, so this implies v = ay + some constant let us say, some constant of integration which will be a function of x, so let us write cx, okay.

Now, we will make use of this determine this unknown function of x, okay, so this gives you; let us differentiate this with respect to x, so derivative of b with respect to x gives you y is in independent effect, so this will give you c dash x, okay but from here this vx = -uy, so this = - times uy is how much; uy is 0, okay, uy is not depending on y, so this implies c dash x = 0 and which implies cx is a constant, okay, it is a constant.

I have denoted this cx/ c, okay, so that is cx is some constant c, okay, so what do we get; by v = ay + c, where a is this, u2 – u1/ 2, okay, so v xy = this one, 1/2 u2 – u1 * y + c and now let us find the complex potential because we have u and y with us, so omega z = u xy + iv xy, u xy = ax + b, so omega z = u xy is ax + b + I times v xy is; this is a, okay, so ay + c, so what do we get; a times x + iy + b + i * c.

Now, this is az + b + ic, okay, so this az + b + ic; a is u2 – u1/ 2 and b is u1 + u2/ 2, c is a real constant, the lines of force are now, what are the lines of force? Lines of force are given by u xy = constant gave us the equipotential lines, the v xy = constant gives us the lines of force, so v xy = constant means v xy = constant, implies ay + c = constant, okay, now c is the constant, a is the constant, so this implies y = constant okay.

So, the lines of force are parallel to x axis, okay, so lines of force are parallel to x axis the equipotential lines are parallel to y axis, so they cut each other at right angles, they define 2 orthogonal families.

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Example 2

Let us consider two conducting coaxial cylinders which extend to infinity on both sides and are kept at potentials U_1 and U_2 respectively.

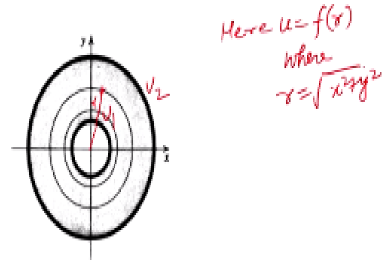

Figure : Fig.2

Now, let us consider two conducting coaxial cylinders, so we have 2 coaxial cylinder, they have the same axis, okay they are coaxial which extend to infinity on both sides okay, so on both sides of the axis they are extending to infinity and are kept at potentials U1 and U2 respectively, so let us say this is U1 okay, this is U1, this is U2, okay, they are kept at potentials U1 and U2 respectively.

So, here what is happening is that the potential U okay at any point is given by at any let us say at this point, okay, it is given by the distance of this point form the origin, let us say this distance of this point from origin is r, then here u is the function of r, okay, we have the 2 conducting coaxial cylinders which are kept at potentials U1 at; so that mean that on the coaxial cylinders, the constant; the potential is remains constant.

So, potential varies when r varies, okay, u is the function only of r, so u is the function of r, and r where r = under root x square + y square.

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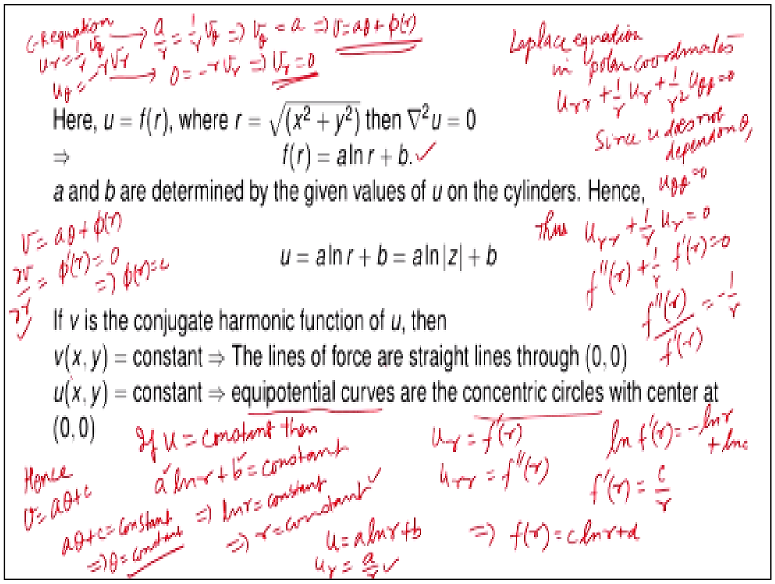

Okay, now let us see what is fr, what is the value of fr, okay, so u is the harmonic function of r, we can see that since u does not depend on theta, okay, we have u = uxx + uyy = 0 so, let us we call the Laplace equation in polar coordinates, urr + 1/r ur + 1/ r square u theta theta = 0. Now, since u is independent of theta, okay, since u does not dependent on theta, u theta theta = 0, okay, thus urr + 1/r ur = 0, okay.

Now, let us find urr, so urr =; ur = f dash r, okay and urr = f double dash r, so what do we get; f double dah r + 1/r f dash r = 0 and this gives us f double dash r divided by f dash r = -1/r, okay, so ln f dash, so let us integrate with respect to r, so ln f dash r = -ln r + some constant let us say, ln c, okay, so then f dash r = c/r, okay, now this integrate; when we integrate it again with respect to r, we get fr = c ln r + some constant d.

Or you can say if you replace the constant cd y, a and b, you get fr = a lnr + b, a and b can be determined by the given values of u and v; 2 cylinders, okay, so u = a ln r + b and r can also be written as mod of z, okay. So, in complex we can write u as a ln mod of z + v, let us say that v is the conjugate harmonic function of u, okay, then what will happen; u = constant means ln r is constant means, r is constant, r is constant means see, if u is constant, then a ln r + b = constant, okay.

A and b are constants, okay, a ln r + b = constant means ln r is constant and which implies that r is constant, okay, r is constant means what; are the equipotential curves, u is the potential, okay, so equipotential curves are concentric circle with centre at 00, r = constant means it defines a circle with centre at 00 and v xy = constant; v xy is what; v xy, let us find v xy, we can find by using the Cauchy- Riemann equations in polar form are ur = 1/r v theta.

So, using Cauchy-Riemann equations, ur = 1/rv theta, what do we get; ur =; we have found fr, fr = a ln r + b, so u = a ln r + b gives partial derivative of u with respect to r; ur = a/r, okay, C-R equations in polar form are ur = 1/r b theta u theta = -r vr okay. Now, u = a ln r + b, so ur = a/r, so a/r = 1/r v theta, first equation gives us this which implies that v theta = a and when we integrate with respect to theta what we get?

V = a theta + some function of r, I can write that as phi r, now here u theta = 0 because u does not depend on theta, so this second equation gives us 0 = -r vr which implies that vr = 0, okay and this mean that when you differentiate this v with respect to r, what do we get; v = a theta + phi r, when we differentiate with respect to r, so we have this, theta is independent of r, so we get phi dash r.

But this is = 0, let us use this, so this is = 0 implies phi r = arbitrary constant say, c okay, so what do we get then, okay hence, v =; let us we call this; v = a theta + some arbitrary constant c, so if v xy = constant, okay, v xy = constant means a theta + c = constant and this will implies that theta = constant, okay, so the lines of force are a straight lines which pass through 00, okay because theta is constant.

So, you can see the lines of force are passing through origin, okay and equipotential curves are concentric circles with centre at origin, so at the point of intersection they are; they cut each other at right angles, with this I would like to end my lecture, thank you very much for your attention.