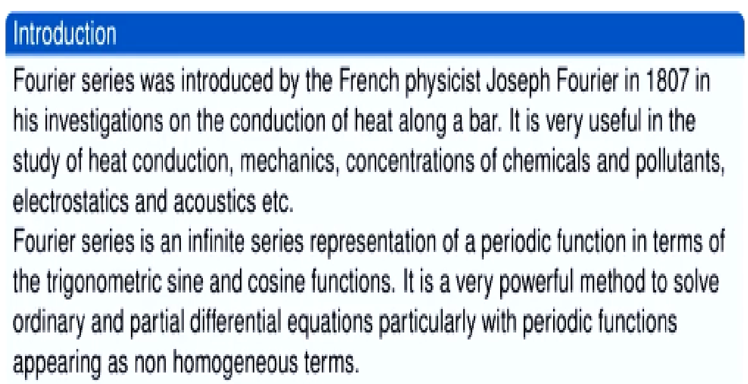
Advanced Engineering Mathematics
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology- Roorkee

Lecture – 39
Fourier Series

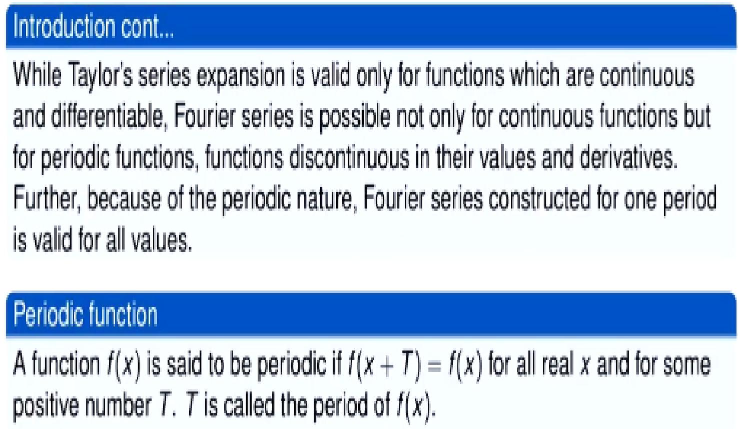
Hello friends, welcome to my lecture on Fourier series was introduced by the French physicist Joseph Fourier in 1807, while he was investigating the conduction of heat along a war and it was very useful in the study of heat conduction mechanics concentration of chemicals and pollutants electro statics and acoustics etc. Fourier series is an infinite series representation of a periodic function in the terms of the trigonometric sine and cosine functions. It is very powerful method to solve ordinary and partial differential equations.

(Refer Slide Time: 00:59)



Particularly with periodic functions appearing as non-homogeneous terms

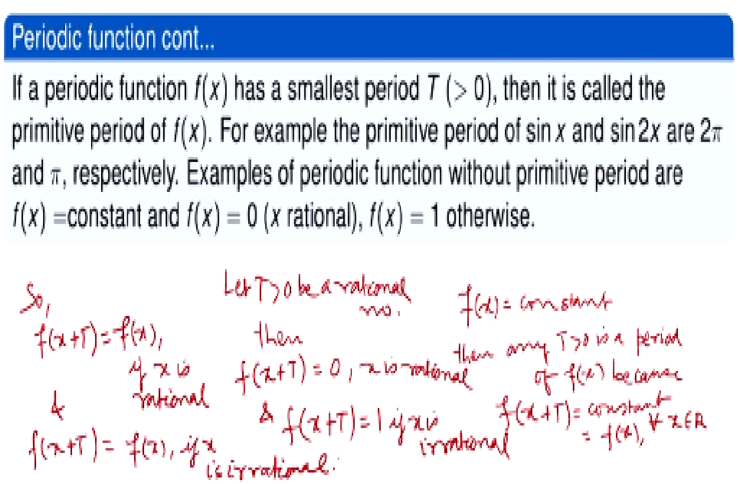
(Refer Slide Time: 01:08)



By Taylor's series expansion we know is valid only for functions which are continuous and differentiable Fourier series is possible not only for continuous functions but for periodic functions which are discontinuous in their values and derivatives further because of the periodic nature Fourier series constructed for 1 period is valid for all values of x let us see how we define a periodic function a function fx is said to be periodic if fx+T = fx.

For all real x and for some positive real number T is then called the period of fx now if a function fx has a smallest period T>0.

(Refer Slide Time: 01:55)



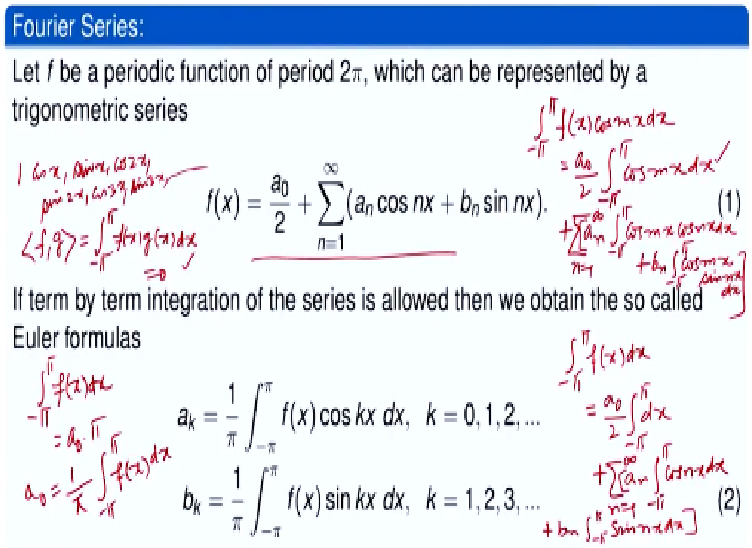
Then it is called the positive period of fx for example the primitive period of sinx and sin2x are 2Pi and Pi respectively examples of periodic function without primitive period are fx = constant if you take fx = constant then any T>0 is a period of f then you take any T>0 is a period of fx because fx+T=constant when you take any T >0 x +T is also real number x=T by definition is a constant which is constant and which is = fx.

And this valid for all x belonging to R so if the function fxis the constant function it can be recorded as the periodic function with any period T >0 and so if you take T to be any number >0 they we do not have any smallest value of T>0 so if fx is constant then it does not have a primitive period now if you take fx = 0 when x is rational an fx =1 otherwise in this case you can see you can take any positive real rational number T.

Let T >0 be a rational number then if x is rational number because it is rational x+T is rational and so fx +T will be =0 and when x is irrational and we assume T to be rational then x+T will be rational and fx+T = 1ifx is irrational here we are assuming x to be rational so fx+T =fx if x is rational and fx +T=fx if x is irrational now we know that set of positive rational number does not have any smallest positive values.

So, that does not exist smallest positive rational number so we can say that the example of fx = 0 when x is rational and x=1 otherwise does m 0 have a primitive period every positive rational number is a period of this function and set of positive rational numbers does not have a smallest positive rational number so fx does not have a primitive period now let us consider the Fourier series.

(Refer Slide Time: 05:29)



Let f be a periodic function of period 2Pi which can be represented by a trigonometric series thjis is the trigonometric series fx = a0/2+ sigma n=1 to infinity an cos nx + bnsin nx you can see that it is made up of trigonometric functions cosnx + sin nx which are by periodic such a series is called a trigonometric series now if term by term integration of series is allowed then we obtain the so called euler formula.

You can integrate this on both sides of this equation with respect to x / the interval -Pi to Pi then integral /-Pi to Pi fxdx will be = a0/2intergal/=Pi to Pi dx+ sigma n =1 to infinity an integral / -Pi to Pi cosnx dx+bn integral/-Pi to Pisinnxdx now integral/-Pi to Pi dx will be = 2Pi while integral/-Pi to Picosnxdx is 0 and integral/-Pi to Pi sinx is also 0 integral /-Pi to Pi sinnxdx is 0 because sinnx is an odd function of x.

And integral /-Pi to Pi fxdx is 0 fx is an odd function so integral /-Pi to Pi sin nxdx is 0 integral /-Pi to Pi cos nxdx is 0 because integral of cosnx is sin nx/ and it is value at –Pi n- Pi is 0 so this is 0 therefore what do we have – integral/- Pi to Pi fxdx= a0/2 * 2Pi so that means a0Pi or we can say a0=1/Pi integral/-Pi to Pifxdx now if we want to derive this formula for ak then you multiply by those sides by say cos mx.

Integrate with respect to x / interval – Pi to Pi so what we will have integral/-Pi to Pi we are multiplying by cos mx so fx cos mxdx = a0/2* integral/-Pi to Pi cosmxdx+ sigma and n= 1 to infinity an integral /- Pi to Pi cosmx cosnx dx+bn integral/-Pi to Pi cosmx sinnx dx now integral/-Pi to Pi cosmx dx is 0 because integral of cosmx is sinmx/m its value at Pi n-Pi 0 so this integral is 0 integral /-Pi to Pi cosmx cosnxdx is also 0.

Actually we know that 1 cosx sinx cos2x sin2x cos3x sin3x and so on they are to satisfy orthogonal property that they are orthogonal functions so that means that if you take any function f here n and function g there then the inner product of f with g = integral/-Pi to Pifx*gxdx this is = 0 so these functions 1 cosx sinx cos2x sin2x cos3x sin3x and so on cos nx sinnx sinnx they satisfy orthogonal property that means if you take any 2 functions f and g.

From this set where f is not = g f and g are distinct functions then integral /-Pi to Pi fxdx is 0 so because of this property we have integral/ -Pi to Pi cosmx cosnxdx is 0 when m is not = m an integral/-Pi to Pi cosmx*sin nxdx is 0 now what will happen when m= n so when m=n what will happen is that we will have integral/-Pi to Pi

(Refer Slide Time: 10:28)

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m \int_{-\pi}^{\pi} \cos^2 mx\,dx$$
$$= a_m \int_{-\pi}^{\pi} \left(\frac{1+\cos 2mx}{2}\right) dx$$
$$= a_m \left[\frac{x + \frac{\sin 2mx}{2m}}{2}\right]_{-\pi}^{\pi}$$
$$= a_m \cdot \pi$$
$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

Here m is any arbitrary positive integer,

So we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx,\quad k=1,2,3,\ldots$$

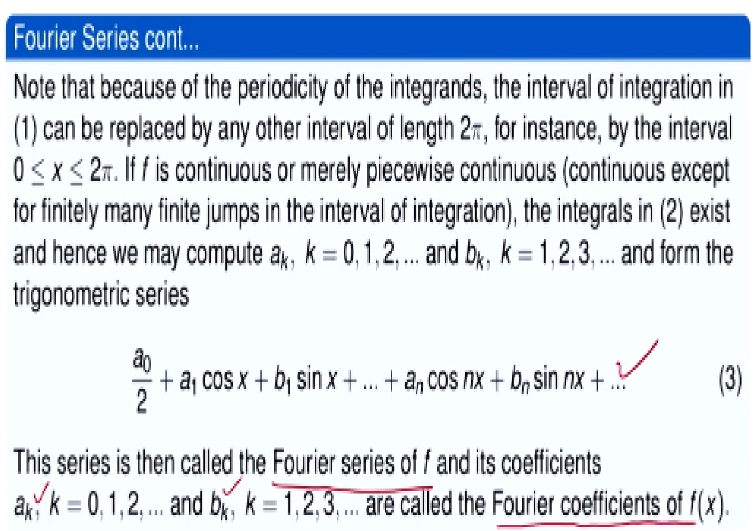
So, we multiply by cosmx and therefore integral/-Pi to Pi fx cos mxdx = am because when n is not = m integral of cosmx *cosnx dx is 0 / interval of -Pi to Pi so when n= m we have -Pi to Pi cos square mxdx and this can be written as am integral/-Pi to Pi 1+ cos2mx/2dx we can integrate this expression and we get x+sin 2mx/2m/2 this is = am*Pi because sin 2mx is 0 when x is Pi R-Pi, so this implies that am=1/Pi integral/-Pi to Pi fx cos mxdx.

Now here m is any positive arbitrary positive integer so we get the value of ak all k so we get ak=1/Pi integral/-Pi to Pi fx cos kxdx where k takes value 1 2 3 and so on now in order to find this bk we multiple this equation by sinmx and then integrate/-Pi to Pi so thus like we have found the value of ak is we multiple fx/sinmx integrate /-Pi to Pi then what will happen integral/-Pi to Pi sin mx dx will be 0 here.

And here we will have integral /-Pi to Pi sin mx * cos nxdx that will also be 0 integral /-Pi to Pi sin mx*sinnxdx will be 0 then n will be = m and when m=n will have integral /-Pi to Pi sin square mxdx integral/-Pi to Pi sin square mxdx also Pi so we will get bm *Pi = integral /-Pi to Pi fx * sin nxdx and therefore we will get the value of bm as 1/Pi integral/-Pi to Pi fx sin mx dx but m is arbitrary positive integer so we will get the value of bk for all k.

Bk = 1/Pi integral/-Pi to Pi sin x sin kx dx k takes value from 1 2 3 and so on now the value of a0 and the value of ak the value of a0k emerged with the value of ak because the value of a0 can be retrieved from the expression for ak if you put k=0 in the expression for ak you get a0x 1/Pi integral/-Pi to Pi fxdx we get the value of a0 from the expression for ak therefore a0 is emerged with the value of ak and k varies in ak k varies from 0, 1, 2 and so on.

(Refer Slide Time: 14:17)

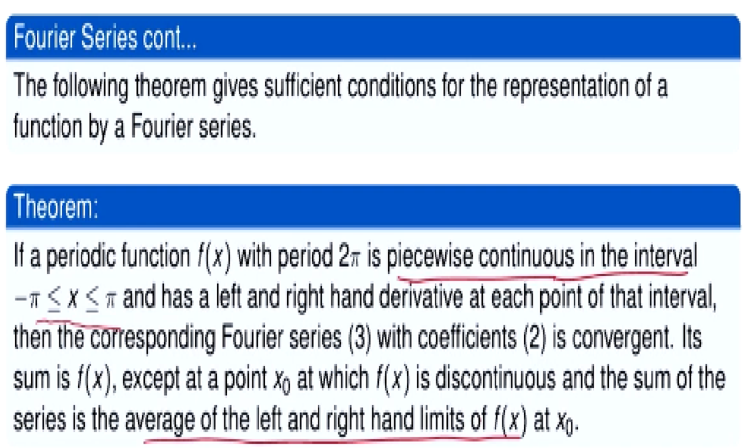


Now let us see that because of the periodicity of the integrands here you can see fx is 2Pi periodic cos kx is 2Pi periodic and sinkx is 2Pi periodic so fx* cos kx*2Pi periodic here fx *sin kx is 2Pi periodic therefore integral/-Pi to Pi can be replaced by any integral where the length of interval is 2Pi so integral/-Pi to Pi can be replaced by integral/ a to a+ 2Pi where a is any real number in particular integral/-Pi to Pi we can replace by integral/0 to 2Pi.

So, because of the periodicity of the integrand the integral of integration in 1 can be replaced by any other interval of length 2Pi for instance by the interval 0<=x<=2Pi now if f is continuous or merely piece wise continuous piece wise continuity means it is continuous except for finitely many finite jumps in the interval of integration the integrals in 2 which we call as euler formulas this integrals exist.

Even when the function of fx is having finitely jump finitely many jumps in the interval of integration so finitely many finite jumps in the interval of integration then also these integrals exist and so we may compute the value of ak per k =0 1 2 and so on and bk for k=1 2 3 and so on n form the trigonometric series nr/2+a1cosx+b1sinx and so on an cosnx+ bnsinnx and so on now this series is then called the Fourier series of f. And its coefficient ak and bk are called the Fourier coefficients of fx.

(Refer Slide Time: 16:16)

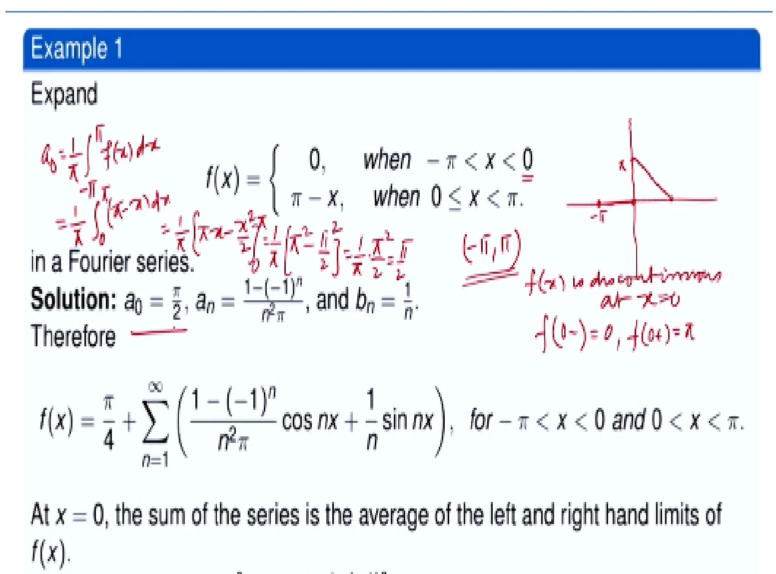


Now let us talk about the convergent of this Fourier series when will it represent the function fx, so they are sufficient conditions for that this sufficient conditions are the following this theorem gives us the sufficient condition for the representation of a function by a Fourier series if a periodic function fx with period 2Pi is piecewise continuous in the interval or –Pi to Pi and has a left and right hand derivative at each point of that interval then the corresponding.

Fourier series 3 with coefficients 2 this Fourier series with coefficients given by these equations 2 is convergent it sum is fx except at a point x0 where the function is except at a point x0 at which this function is discontinuous and the sum of the series then is the average of left hand and right hand limits so the sum of the series will be = fx at each point of continuity of and the sum of the series will be the average of left hand and right hand limits at each point of discontinuity of f.

If the function f is piecewise continuous in the interval –Pi to Pi and has left hand and right hand derivative at each point of the interval-Pi to 2Pi

(Refer Slide Time: 17:38)



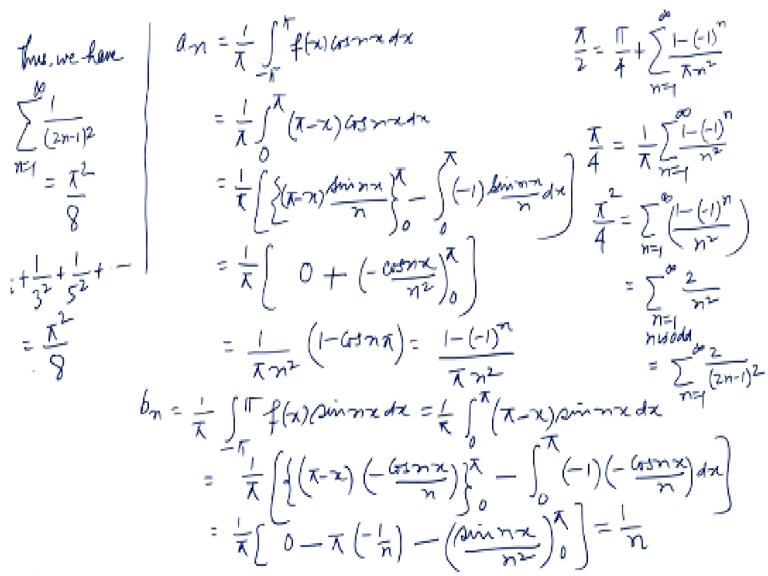
Now let us consider this example fx = 0 when –Pi<x<0 and fx = Pi-x when 0<=x<Pi let us draw the graph of this function /interval –Pi to 0 this function is 0 so we have this graph and when x =0 the value of the function is Pi so here we put the circle this is let us say Pi so here at x =0 the

value of the function is Pi and when x is going to Pi fx goes to 0 so the graph of the function is like this now this is the function defined / the interval –Pi to Pi.

You can see that fx is defined/ the interval –Pi to Pi / the interval-Pi to Pi the point x = 0 is the point of continuity of fx is continuous at x =0 you can see this from the graph because /integral –Pi to 0 fx is 0 and the/ interval 0 to Pi it is given by Pi-x so left hand limit at 0 is 0 while the right hand limit 0 is Pi so left hand and right hand limits are not = and therefore the function is not continuous at x =0 everywhere else on the interval – Pi to Pi fx is continuous.

Now we can find the value of a0 now you can see that the valuer of a0 here is given by 1/Pi integral/-Pi to Pi fxdx so this integral will reduce to the integral/0 to Pi fxdx because / the interval –Pi to 0 fx is 0 so 1/Pi integral/0 to Pi fx is Pi-x so we put Pi –x here for fx and this will give you 1/Pi Pix- xsquare/2 so let us put the values the limits then we have 1/Pi and this will be Pi square –Pi square /2 so we get 1/Pi*Pi square/2 that means Pi/2 so this value a0=Pi/2.

(Refer Slide Time: 20:34)



We can similarly find the value of an given by 1/Pi integral/-Pi to Pi fx cosnx dx again let us use this set that fx is 0/interval 0 to –Pi to 0 so we have 1/Pi integral /0 to Pi and /0 to Pi fx is given by Pi-x so Pi –x cos nx dx lets integrate this by parts then we have Pi-x* sin nx/n then 0 to Pi derivative of Pi –x is -1*sin nx/n, so this is 1/Pi when x takes the value Pi – x is 0 even sin n Pi 0 and when x = 0 this is Pi –x is Pi but sin n0 is 0.

So, we get 0 here- - + and then we have integral of sin nx /n so sin nx integral of sin nx is – cos nx/n so that means – cos nx/n square we have so this will be =1/Pi * n square and we put the limit we get – cos n/+1 so 1-cosnPi which will be = 1—1 to the power n /Pi*n square we can also find bn = 1/Pi-Pi to Pi fx sin nx dx so this integral be also reduce to 1/Pi integral /0 to Pi fx is Pi –xsin xdx this will be = again.

We integrate by parts 1/Pi so Pi- x * - cos nx/n 0 to Pi then we have 0 to Pi derivative of Pi – x is -1* - cosnx/ndx so this is 1/Pi when we put x = Pi – 0 we get the value 0 but when we put the lower limit Pi –x becomes Pi and here what we get – cos n Pi/n – cos n Pi/n and to get with the negative sign so we get cos and Pi /no we have when we put the cos upper limit gives 0 then - we put the lower limit so Pi – 0 means Pi.

And then – x = 0 so Pi – 0 is Pi and this is -1/n, and this is - - - integral of cos nx is sinnx/n so we get sin nx/n square 0 to Pi so this is 0 so sinnx is 0and x = 0 as well as x =Pi so we get 1/Pi*Pi/n that is we get the value n so an is 1/2 bn is 1- -1 power n/Pi nsquare so we get the value of an and we get the value of bn we put these values in the Fourier series of the function f and Fourier series will have the sum fx/ the points.

At the points where the function is continuous in the open end interval –Pi to 0 and / the interval 0 to Pi so fx = a0/2 means Pi/4 and then sigma n = n to infinity an cos nx+bnsinnx this is the Fourier series of the function f in the open interval –Pi to 0 and 0 to Pi now at the point x = 0 we right hand side of the Fourier series will becomes the Fourier series at x = 0in Fourier series becomes put x = 0 in this.

So, Pi/4+sigma n=1 to infinity 1—1to the power n /n* nsquare/Pi cos0 is 1 and sin 0 is 0 so we have Pi/4 sigma n = 1 to infinity 1 - -1 to the power n /n is n square Pi = average of left hand right hand limits f0-+ f 0+/2 now if f0 -0 f0+Pi so this is Pi/2 s what we can say now we can say that we have Pi/2 = Pi/4+sigma n =1 to infinity 1- -1 to the power n/Pi *n square cos nx sp Pi/2- Pi/4 is Pi/4.

And so we get Pi/4= 1/Pi sigma n=1to infinity 1- -1 to the power n /n is square cosnx is 1 at x = 0 so we have to put that also so this is actually 1 cos nx is we can replace this by 1 so we have actually this because at x = 0 cosnx is 1 so we have Pi square /4= sigma and = 1 to infinity 1—1 to the power n/n square now let us notice when n is even integer then 1 – 1 will be 0 because -1 to the power n will be 1 so this is 0 and this expression is 0.

When n is even it is 2Pi n square when n is odd so this = sigma n =1 to infinity and n is odd we are considering only odd integral value of n / the set of numbers 1 2 3 and so on up to infinity so this we have 2/n square and this we can also write as replacing n by 2n-1 we have n = 1 to infinity 2/2n-1 whole square so what we get we get that sigma n = 1 to infinity 1/2n-1 whole square is Pi square/8 thus we have sigma n = 1 to infinity1/2n -1whole square = Pi square/8.

So, we get the sum of the series 1+1/3 square 1/ 1 square + 1/3square so 1/1 square +1/3 square +1/5 square and so on this = Pi square/8 so by Fourier series we can also determine the sum of many infinite series which are otherwise not easy to find so let us now go to next slide so this is how we determine the Fourier series for the given function, and we have also seen that we can determine the sum of the infinite series 1+1/3 square+1/5 square +1/7 square and so on. As Pi square /8 now let us consider now we will consider.

(Refer Slide Time: 29:46)

Fourier series of functions having arbitrary period:

In applications, periodic functions rarely have period 2π. Therefore in order to find the Fourier series of a periodic function $f(t)$ with period, say T, we use the change of scale. We introduce a new variable x such that $f(t)$ as a function of x, has period 2π. Let us define $t = \frac{T}{2\pi}x$ then $x = \pm\pi$ corresponds to $t = \pm\frac{T}{2}$ and $f(t) = f(\frac{T}{2\pi}x) = \phi(x)$, say.

Then $\phi(x + 2\pi) = \phi(x)$, so we can find the Fourier coefficients for the function ϕ by the Euler's formulas.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos kx\, dx, \quad k = 0, 1, 2, \ldots$$

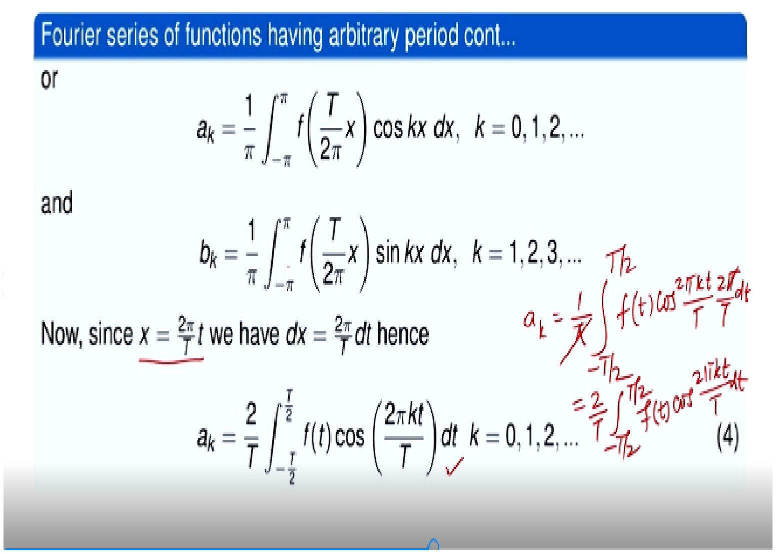
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\, dx, \quad k = 1, 2, 3, \ldots$$

Fourier series of functions having arbitrary period in applications periodic functions rarely have 2 period 2Pi therefore in order to find the Fourier series of a periodic function ft with period let us say T ft is not having period 2Pi. Okay it is having an arbitrary period T then we use the change of scale okay. So, we will use the change of scale we introduce a new variable x such that ft as a function of x, has period 2 pi.

Let us define t=T/2 pix then x== will be pi t will be -t/2 then x will be – pi t will be -T/2 and then x will be +pi t will be pi/2. So, x=+-t will correspond to t=+- T/2 and ft the function ft will change to f of T/2 pi x which we can write as some function phi x. Okay now this function phi x is a 2 pi periodic function you can see because phi of x+2 pi = by definition phi x +2 pi =f T/2 pi x+2 pi = f of Tx/2 pi +T.

Okay we are assuming that f is periodic with period T so f of Tx/2 pi +T will be F (Tx/2 pi F of Tx/2 pi is phi x okay phi is now 2 pi periodic and therefore we can write the Fourier series for the function phi okay we can determine Fourier Co efficient. Fourier Coefficients are given by ak=1/ pi -pi to pi integral over – pi to pi phi x cos kx dx where k takes value 0 1 2 3 and so on and bk is given by 1/ pi integral/- pi to pi phi x sin kx dx k to x value 1 2 3 and so on. We can get the expression of ak and bk in terms of the function phi.

(Refer Slide Time: 32:07)

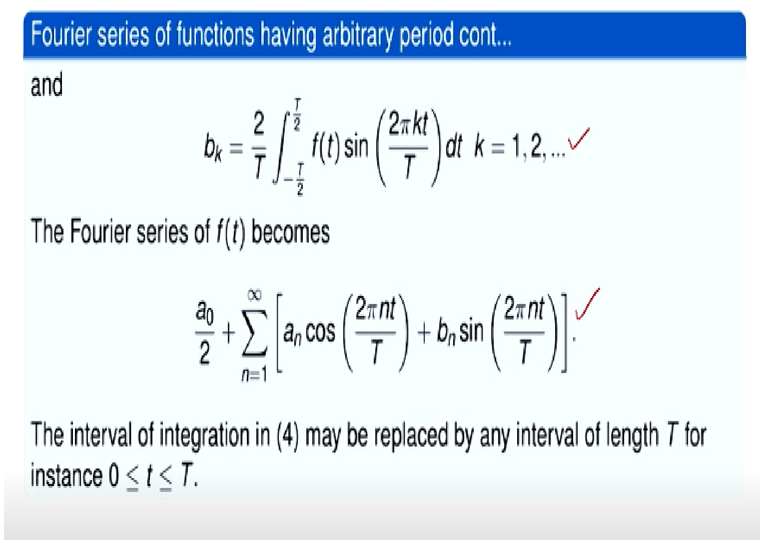


Now let us put the value of phi x okay so phi x is f of tx/2 pi so let us put that value so ak =1/pi integral/ pi to – pi f of Tx/2 pi x cos kx dx, k= 0 1 2 3 and son and bk will become by 1/ pi integral/- pi to pi phi x sin kx dx k to x value 1 2 3 and so on. Now we have assumed that x= 2 pi T/t. From here we can see x= 2 p t/T okay, so x is 2 pi/T and therefore dx=2 pi/T*dt let us make that substitution.

So, when we will change x to t okay f of tx/ 2 pi will change to ft function okay. So, this f of tx/ pi will become ft here we will have in this expression for ak we will have cos of 2 pi kt/T because x is 2 pi/T so cos 2 pi kt/T dx will be 2 pi/T dt okay. So, we will have ak=1/ pi the limit of integral will become – pi will go to -T/2 pi will go to T/2 and we will have here ft this will become ft and cos 2 pi kt /T dx will be 2 pi/ dt okay.

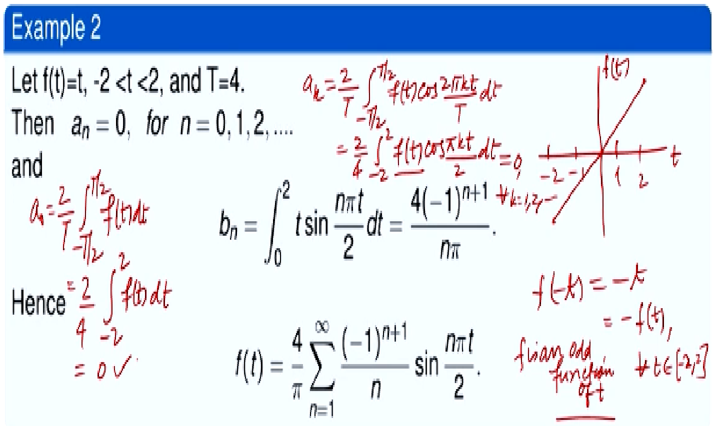
So, this pi will cancel with this pi 2/T is a constant we can write outside so 2/T -T/2 to T/2 ft cos 2 pi kt/T dt so we get this value of ak. Similarly, we can write vk=2/T inetgra1/-T/2 to T/2 ft*sin 2 pi kt/T*dt okay that is the value of bk.

(Refer Slide Time: 34:29)



And the Fourier series will become a0/2 sigma n=1 to infinity an cos nx is 2 pi t/T cos 2 pi n t/T bn sin 2 pi nx. So, nx will be replaced by 2 pi nt/T so we may have bn sin 2 pi nt/t. So, this is the Fourier series for the function f if it is of period t. The interval of integration may be replaced here because f is T periodic and 2in 2 pi kt/T is also T periodic. So, we can replace the integral of integration -T/2 to T/2 by 0 to T okay.

(Refer Slide Time: 35:09)



Now let us take or example ft=t over the interval -2t 2 so this is your -1 this is -2 so we have this by =x function over the integral -2 to 2 okay ft=t okay so this is t this is ft. Now we have given that f is periodic with period 4 okay. So, the graph of f over the integral -2 to 2 is then repeated over the intervals of length 4 say that is from 2 to 6 or from -6 to -2 okay. So, we can just repeat this graph okay now since you can see here that f -t.

If you put f of in place of t you put -t then f of -t is -t okay so I can write it as -ft okay this is valid for all t belonging to -2 2. Okay because of the definition of t so this means that f is an odd function of t. Okay so integral to the value of a0 a0= we can find the value of a0 from here 2/T -T/2 to T/2 ft and k is 0. So, ft dt okay so we have 2/T integral/-T/2 to T/2 ftdt. Okay T=4 here so 2/4 integral -2 to 2 ft dt. Now f is an odd function so integral /-2 to 2 ft dt=0.

Now similarly if you find ak okay ak=2/T integral/_t/2 to T/2 ft cos 2 pi kt/dt then it will become 2/4 integral/ -T/2 to T/2 ft T is 4 so 2 pi kt/4 means pi kt/2 dt okay. Now ft is an odd function cos pi kt/2 is an even function product of an even and odd function is an odd function, so the integral is an odd function and therefore it is integral/-2 to 2 is 0 okay for all k =1 2 3 and so on okay. So, a0 is 0 ak is 0 for all k 1 2 3 and so on so we just had to calculate bk.

(Refer Slide Time: 38:22)

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi k t}{T}\,dt$$

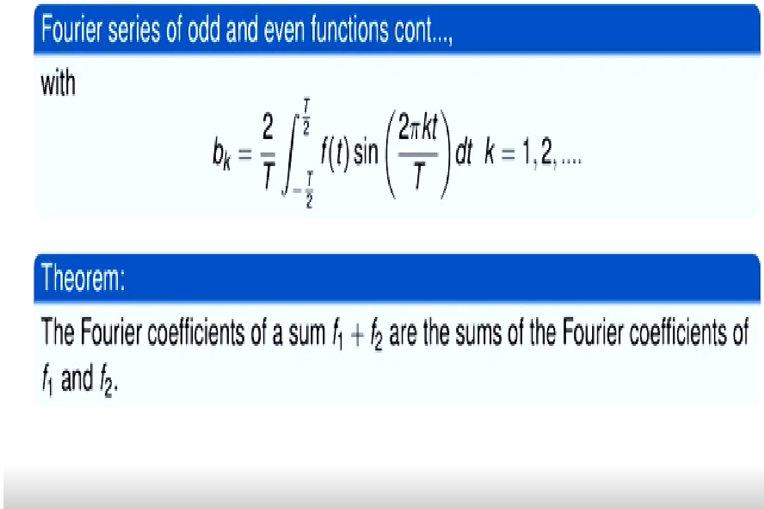
$$= \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{\pi k t}{2}\,dt$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_{0}^{2} t\sin\frac{\pi k t}{2}\,dt$$

So, bk is given by bk=2/T integral -T/2 to T/2 ft sin 2 pi kt/T dt okay t is 4 so we have 2/4 that is 1/2-2 to 2 ft is * sin pi kt/2 dt. Now ft is an odd function of T sin pi kt/2 is also odd function of T therefore product of odd function and the odd function is an even function. So, we have integral/-2 to 2 we can replace by 2 times integral/0 to 2 so 2 times 2/2 integral/ 0 to 2 ft is T okay sin pi kt/2 dt we get okay.

So, we get the value of we can integrate this by parts and evaluate the value of bk comes out to be 4 times -1 to the power n+1/n pi and then we have the Fourier sine series because the cosine terms vanish and now also the cos term vanishes okay. So, ft =sigma n= 1 to infinity b and sin 2 pi nt/T T is 4 so pi nt/2 okay so sin pi nt/2 here we have and then we put the value of bn that is 4/pi -1 to the power n+1/n.

(Refer Slide Time: 40:02)



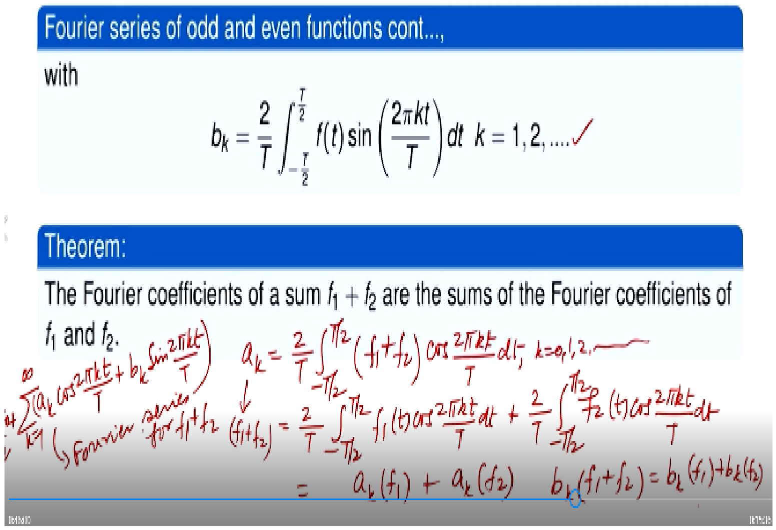
Okay now let us see the Fourier series of an even function ft of period T is a Fourier cosine series. As I said just now an even function okay if f is an even function then will have ak=2/T integral over -T/2 to T/2 ft cos 2 pi kt/ T dt okay f is an even function cosine function is also even. So, their product is even so we get 2 times integral over -T/2 to T/2 becomes 2 times integral/ 0 to T/2 so we get 4/ T integral/0 to T/2 ft cos 2 pi kt/T dt while bk this will be 2 for all k.

K= 0 1 2 and so on okay an vk will be 2/T integral/-T/2 to T/2 ft sin 2 pi kt/T dt okay f is an even function sin 2 pi kt/ T is an odd function. So, integral/-T/2 to T/2 is 0 this is 0 for all k. Okay and so the Fourier series will reduce to a0/2+sigma k=1 to infinity ak cos 2 pi kt/T okay that is this okay so where the value of ak will be given this integral this integral can be written by 4/T integral/0 to T/2 ft cos 2 pi kt/T dt.

Now if we have f is an odd function okay f is an odd function then just now, we have seen the example of ft=T. In the case of an odd function the ak is 0 for all values of k k=0 1 2 3 and so on up to infinity and vk becomes 4/T bk becomes 4/T because the integral will become even in the case of bk so 0 to T/2 ft sin 2 pi kt/T dt. Okay and this is the value of bk for k= 1 2 3 and so on okay.

And the Fourier series will become sigma k=1 to infinity bk sin 2 pi kt/T okay so this is the situation.

(Refer Slide Time: 42:58)



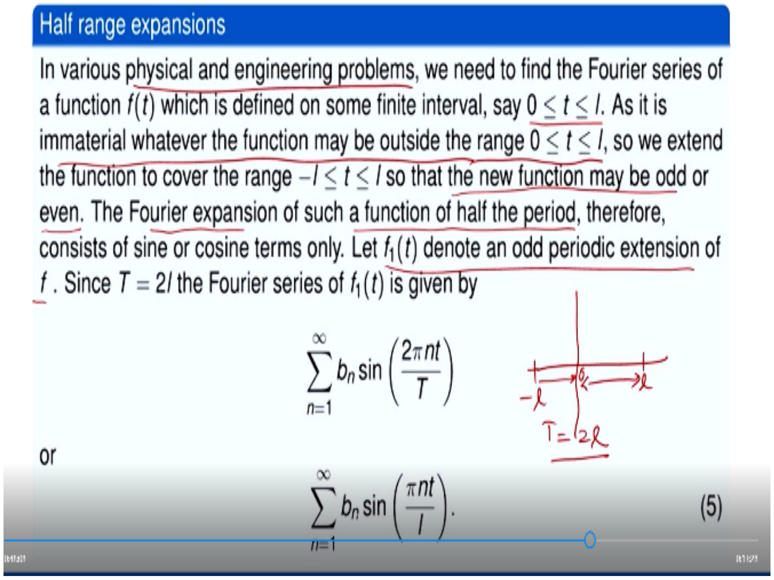
Now if we have two functions f1 and f2 okay then the Fourier co efficient of a sum f1+f2 are the sums of the Fourier co efficient of f1 and f2 this is very easy to see you can see that if we have some function f1+f2 then for this sum function f1+f2 ak will be=2/T integral/-T/2 to T/2 f1+f2 cos 2 pi kt/T dt okay and this is valid for k=0 1 2 and so on. Okay and this I can write as 2/T integral/ -T/2 to T/2 f1 t cos 2 pj kt/T dt+2 /T integral/-T/2 to T/2 f2 t cos 2 pi kt/T dt.

So, for the Fourier series of the function f1+f2 we see that ak which is the co-efficient of cos 2 pi kt/T is the sum of the Fourier co efficient of f1 and f2 which are the Fourier co efficient of the cos 2 pi kt terms cos 2 pi kt/T terms in the case of functions f1 and f2 this is let us say f1+f2 we write as Fourier series as a0/2+sigma k=1 to infinity ak cos 2 pi kt/T+bk sin 2 pi kt/T okay this is the Fourier series for the function f1+f2.

Then this is the Fourier co efficient ak for f1 function and this one is ak for f2 function this ak is for f1+f2 function okay this one. Okay so Fourier co efficient of ak for f1+f2 is sum of the Fourier co efficient of f1+f2. The Corresponding co efficient of f1 and f2 similarly bk you can see bk for the function f1+f2 is sum of bk for f1+ bk for f2 so Fourier series for the function f1+f2 can be found by the Fourier series for the function f1.

And the Fourier series for the function f2. We can separately find and take their we can separately find the Fourier co efficient for f1 and f2 and then add them to get the Fourier co efficient for f1++f2.

(Refer Slide Time: 46:04)



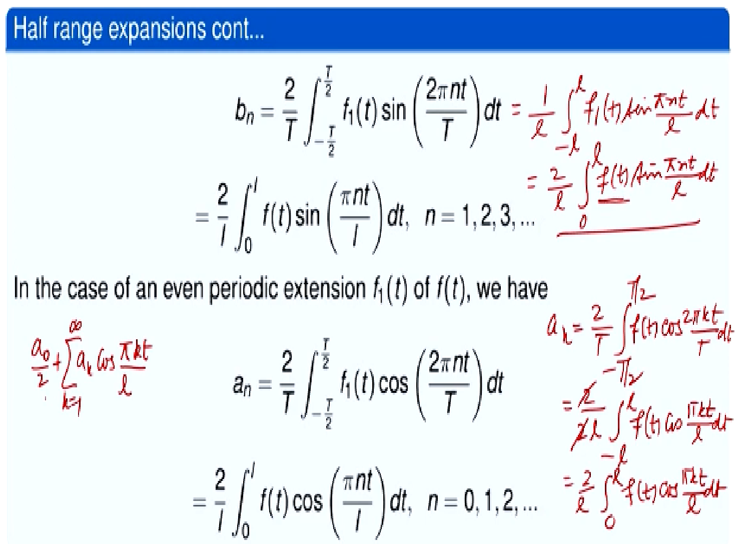
Let us consider through half range expansions in various physical and engineering problems we need to find the Fourier series of a function ft which is defined on some finite interval, say 0<=t<=l. So, it is defined over the interval 0 to l over this interval. Okay as it is immaterial whatever be the function outside the range 0<=t<=l. So, we extend the function to cover the range -l to 0 l.

So, we define the function from our side on the other half of the interval that is -l to 0 okay now the function will be defined over the interval -l to l now we define the function over the interval -l to l either as an even function or as an odd function. Okay so the new function will be either odd or it will be even function. The Fourier expansion of such a function of half the period therefore consists of Sine or Cosine terms only.

Half the period means we denote the function only over the half period 0 to l okay we are defining over the other half period -l to 0 by either an odd extension or by an even extension okay. So, the resulting function f 1you can call the resulting function a new function f1 is either even or it is an odd function. Okay f1 t denote an odd periodic extension of f. Okay then now T=2l okay the function is defined over the interval -l to l so since T=2l the Fourier series of f1t.

Now we are defining f1 as an odd periodic odd function therefore the Fourier series will consist of only sine terms okay. So, Fourier series will be sigma n=1 bn sin*2 pi nt/T now T=2l okay T=2l let us put the value here we get sigma n=1 to infinity bn sin pi nt/l.

(Refer Slide Time: 48:04)

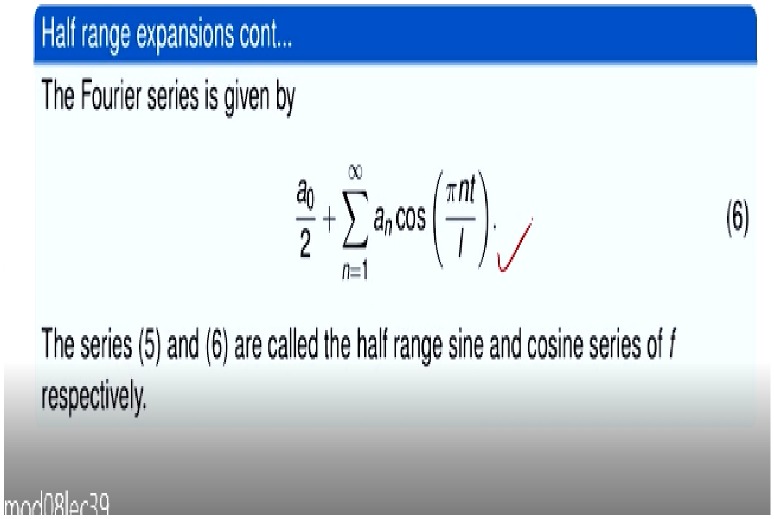


The bns are given by 2/T integral/-T/2 to T/2 f1t sin 2 pi nt/T dt let us put T=2 l here okay. So, we get integral/-l to l this is 2/l integral/-l to l f1 t sin pi nt/ldt now f1t is an odd function sin pi nt/ l is also an odd function we can write it as this is 1/l this 2/2l means 1/l okay so using the fact that f1t*sin pi nt/l is an even function we can write 2/l 0 to l f1t now becomes ft because over the interval 0 to l f1 t coincides with ft sin pi nt/l dt.

So, this is how if we function over the interval 0 to l by defining over the other half that is -l to 0 By taking odd periodic expansion we can find the Fourier Sine series for the function whose Fourier co efficient bn is given by 2/l 0 to l ft sin pi nt/l dt. You can see here that the function values/ 0 to l only are being used. Okay in the case of an even periodic extension f1t of ft okay f1t will be an even function so we will have Fourier Cosine series okay ak will be given by 2/T integral/-T/2 to T/2 ft cos 2 pi kt/ T dt okay.

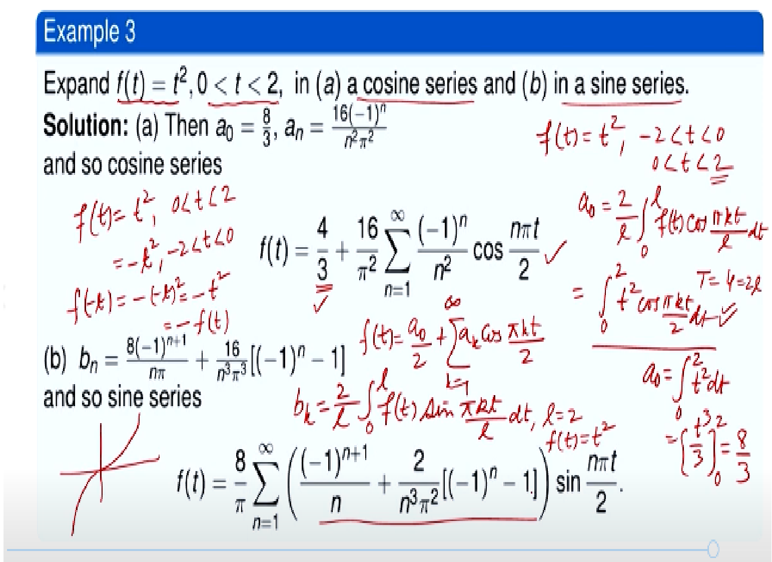
So, this is 2/2l okay T is 2l and limits are -l to l because T is 2l so integral/-l to l and ft cos pi kt/l dt. Okay now f is even cos pi kt/l is also even. So, you can write 2/l 0 to l ft cos pi kt/ldt and we have the Fourier Cosine series will be a0/2+sigma k= to infinity ak cos pi kt/l where a0 and aks are given by this integral k=0 1 2 and so on okay. So, this is the formula.

(Refer Slide Time: 51:14)



These are called half range sine and cosine series of the function f.

(Refer Slide Time: 51:20)



Now let us say for example ft= t square okay ft = t square it is defined over the interval 0 to t, 0 to 2 we want to find Fourier Cosine series and a Fourier Sine series if we want Fourier Cosine series, we will extend the function ft over the other half interval -2 to 0 by taking even periodic expansion. So, we define the function ft as an even function okay over the other half okay that is ft = t square when -2<t<0.

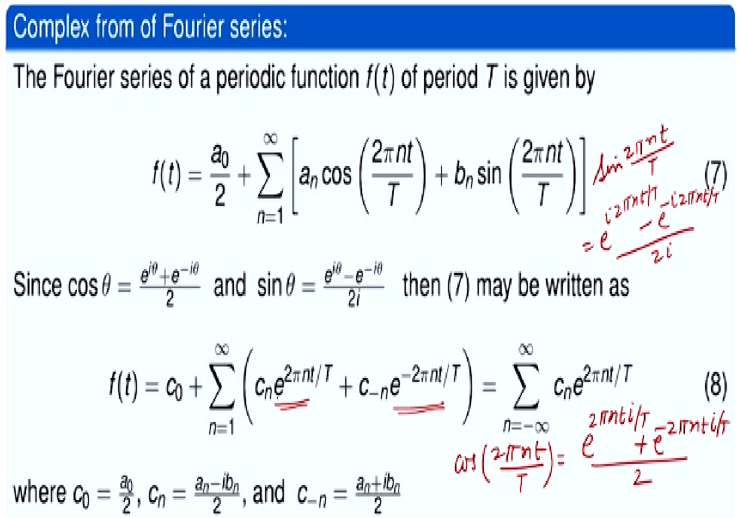
So, if we define ft= T square it will be an even function okay and then we will have its Fourier Cosine Series so a0 will be given by 2/l 0 to l ft cos pi kt/l dt okay so 2/l 0 to l ft cos pi kt/l dt okay l=2 here okay T=4 right T=4 nt=2l so l=2 this is 2/2 that is 1 so we get 0 to 2 ft is T square T square cos pi kt/2 dt okay. We integrate by parts substitute the limits okay when you take when you want a0 put k=0.

So, a0 will be =0 to 2 t square dt and what do we get here t cube/3 so 0 to 2 so we get 8/3 okay but here in the Fourier Cosine series we have a0/2 here okay ft is =a0/2+sigma k=1 to infinity ak cos pi kt/2 okay so a0 will become 4/3 we will get this term and then we can get the value of ak for k= 1 2 3 and so on by integrating by parts the function t square cos kt/2 and putting the limits we will get this 16/ pi square -1 to the power n/n square.

Okay this is the Fourier cosine series in this case when we want Fourier Sine series, we will definite the function ft=t square/0 to t 0<t<2 and – t square/-2 <t<0 okay then f of -t okay suppose t belongs to 0 to 2 interval okay so f of -t will belong to them -2 to 0 okay and therefore we will get – of -t whole square okay so we will get here – t square okay which is =-ft. Okay so if we want odd periodic function of the expansion ft.

Then we define ft=-t square over the integral -2 to 0 okay then a0ak will be 0 for all k okay we will get the value of bk will be given by 2/l 0 to l ft sin pi kt/l dt okay l=2 and ft=t square okay we can evaluate the integral and we get the value of bk for all k. The value of bk will be 8/pi – 1 to the power n+1/k 2/k cube *pi square -1 to the power k over -1 by integrating by parts So, that we can easily do.

(Refer Slide Time: 55:45)

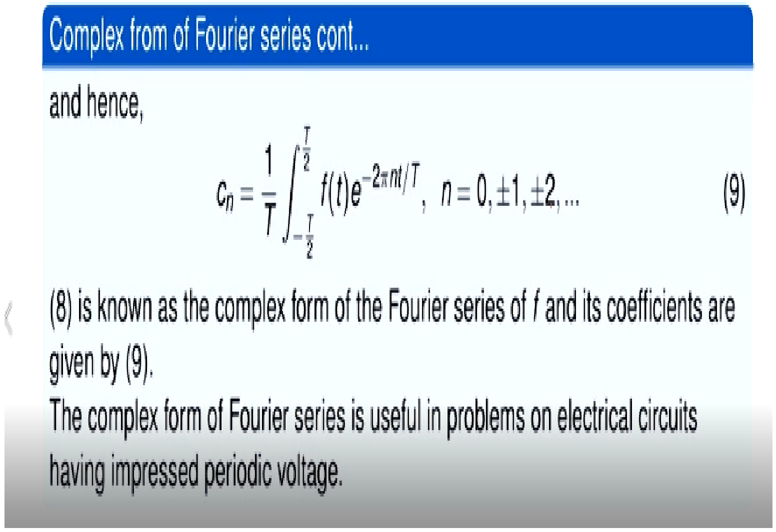


Now let us come to the complex form of the Fourier Series the Fourier series of a periodic function of ft of period T is given by ft=a0/2+sigma n=infinity an cos 2 pi nt/T+bn sin 2 pi nt/T. Let us use the Eulers formula for cos theta and sin theta cos theta is e to the power i theta +e to the power – i theta/2sin theta is e to the power i theta—e to the power -i theta /2. Let us put these expressions for cos 2 pi nt/t and sin 2 pi nt/T.

Okay then what we will get cos we will have cos theta okay so cos 2 pi nt/ T okay will be = e to the power 2 pi i times 2 pi nt i/T + e raised to the power – 2 pi nti/T /2 and similarly for sin 2 pi nt/T sin 2 p pi nt/T will be = e raised to the power 2 pi nt/ T-e raised to the power -i 2 pi nt/T/2 i okay we put the values and collect the co efficient of e to the power 2 pi nt i/T and e to the power -2 pi nti/T.

Okay then you will see that the co efficient are given by co efficient of e to the power 2 pi nti/T will be given by an- ibn/2 and the co efficient of e to the power -2 pi nt/T will be given by an+ibn/2. If that is dn is nothing but C of -n okay C0/T will be given by an+ibn/2 if that is dn it is nothing but of C of -n okay. So, C0=a0/2 okay and the co efficient cn and c-n are given by an-ibn/2 c- an-ibn/2.

(Refer Slide Time: 58:05)

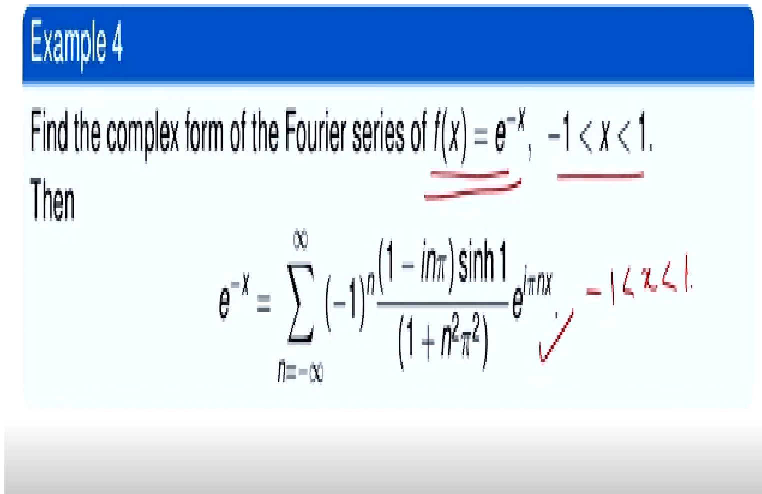


And then we can write Cn as the value of Cn okay Cn can then be written as this 1/T – T integral/-T/2 to T/2 ft e raised to the power- 2 pi nt i/T dt. So, n takes value 0 +-1 +-2 and so on. this Cn value can be obtained by an-ibn.2 we know the value of an we know the value of bn okay we put the values of an and bn to put the expressions for cn and I can combine this okay by the fact that.

This can be written as sigma integral/-infinity to infinity Cn e to the power 2 pi i nt/T okay. So, because C-n it replace -n/ replace here -n some integer n -n/n then n will run for this term from -infinity to -1. So, we can write this series in the abbreviated form like this. Okay so we get Cn like this and this is known as Cn e to the power sigma n=-infinity to infinity Cn e to the power – 2 pi nt i e to the power 2 pi nt i/t as the Fourier complex form of the Fourier series of f okay.

And its co efficient are given by this. Now the complex form of Fourier series is useful in problems on electrical circuits having impressed periodic voltage.

(Refer Slide Time: 59:54)



We can take an example f= e to the power -x okay you are given the interval -1 to 1 over which it is defined okay. We can find the value okay you can put here T=2 so you will get the integral Cn will be ½ integral/-T/2 to T/2 ft e to the power -2 pi nt/T T=2 okay dt and we can evaluate ft we know ft =e to the power -t we can find the value of Cn and put the value of Cn in this complex form of the Fourier series.

We get this Fourier complex Fourier series of the function e to the power -x valid in the interval -1<X<1 okay. So, that is all we have to do in this lecture in the next lecture on Fourier series we shall consider Fourier integral and Fourier Sine Cosine Transforms. Thank you very much for your attention.