Advanced Engineering Mathematics

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Lecture – 38
Fourier Integral and Fourier Transforms

Hello friends lets welcome to my lecture on Fourier integral and Fourier transforms let us first discuss Fourier integral we have seen Fourier series are powerful tools in treating various problems involving periodic functions but in many practical problems we do not have periodic functions. Hence it is desirable to generalize the method of Fourier series to include non-periodic functions let us consider 2 simple examples of periodic functions of period T.

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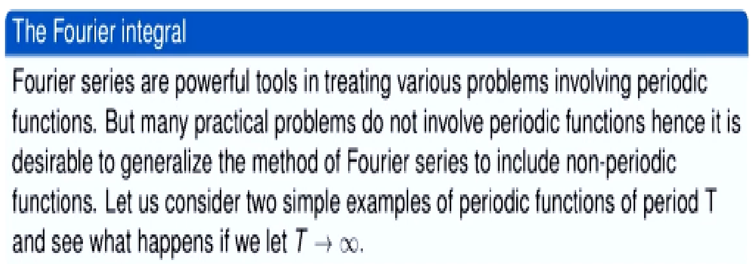

And see what happens if we let T go to infinity.

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Example 1

Consider the function

$$f_T(x) = \begin{cases} 0, & \text{when } -\frac{T}{2} < x < -1 \\ 1, & \text{when } -1 < x < 1 \\ 0, & \text{when } 1 < x < \frac{T}{2}. \end{cases}$$

having period $T > 2$.

So, consider this function ftx = 0 / interval- T/2-1 1 when -1 is x < 1 0 when 1< x < T/2 and let us assume that T >2.

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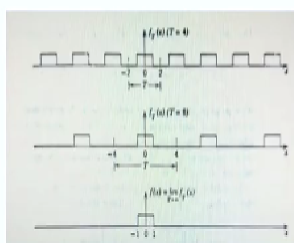
Example 1 cont...

When $T \to \infty$, we obtain a function $f(x)$ which is not periodic

$$f(x) = \lim_{T \to \infty} f_T(x) = \begin{cases} 1 & \text{when } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



So, we have this graph okay this is the graph of fTx when we take T=4 and we are taking this example for values of T more than 2 let us 1 st consider this special case when T=4 then the graph of the function fTx which is 0 when – x belongs to the open interval –T/2 – 1 when 1 x belongs to the open interval -1 1 and 0 when x belongs to open interval 1 T/2 is this 1. This is the graph of fTx when T=4 and this graph changes to this graph.

When we take T = 8 so this is the graph for T=8 now when T goes to infinity here when in this example when T goes to infinity what happens we see that fTx goes to let us say fx which is 0/ interval – infinity to -1/ the interval -1 to 1 and 0 when 1<x<infinity we get the definition of fT fx will be = limit of fTx goes to infinity it is 1 when -1<x<1 and 0 other wise so when we consider the limit of fTx as T goes to infinity.

The limit in function fx assumes value 1/ the open interval -1 1 else where it is 0 so this is the graph of the limiting function fx you can see. And you can also see that this function is not a periodic function so we get a non-periodic function when we let T go to infinity now let us consider another example.

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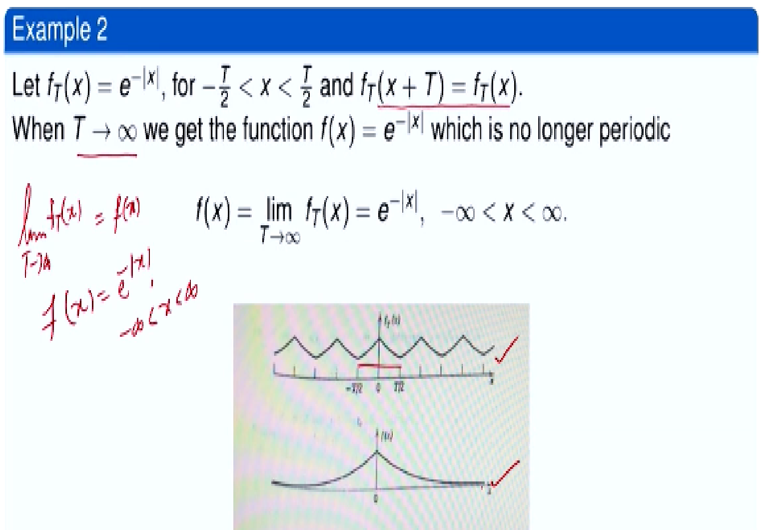

Suppose fTx = e to the power – mod of x for the interval –T/2<x<T/2 so this is the interval – T/2 and this is the graph of fTx this graph of fTx this one / the interval –T/2 T/2 and we are assuming it to be T periodic so by using T periodicity or with period T we can extend this graph / the whole real line and this is how the graph of fTx looks like for all real values of x now let us take T goes to infinity fTx let us say goes to fx limit of fTx as T goes to infinity.

We are taking s fx so then what will happen fx will be = e to the power – mod of x / the interval – infinity to infinity and this is how we get the graph of fx and you can see that this function fx is not a periodic function so we get a non-periodic function so these are 2 examples where we have tried to show that a periodic function fTx when T goes to infinity does not lead us to a periodic function. So, how we will treat non-periodic functions that we will see through Fourier integral

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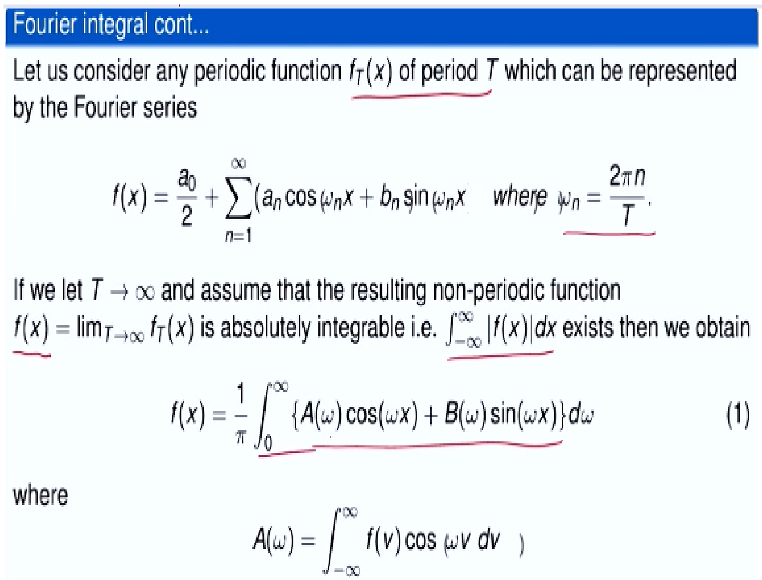

So, let us consider any periodic function fTx of period T which can be represented by this Fourier series fx = a0/2+ sigma n=1 to infinity an cos omega nx+bn sin omega nx where omega n we are writing as 2Pin/T because fTx has period T and in the case of period T we know that we get the terms in sin and sin x sin2Pi nx/T sin 2Pi nx/T so for convenience we are writing 2Pi n/T x omega n so we get this Fourier series of the function f Tx.

Now let us take T goes to infinity so if we take T goes to infinity and assume that the resulting non – periodic function fx is absolutely integrable so this condition we put on fx that fx is absolutely integrable meaning that integral / - infinity to infinity mod of fxdx exists then we obtain fx = 1/Pi integral 0 to infinity A omega cos omega x+ B omega sin omega x d omega where A omega is given by this integral /-infinity to infinity fV cos omega vdv.

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and

$$B(\omega) = \int_{-\infty}^{\infty} f(v) \sin \omega v \, dv \quad)$$

The integral on right hand side of (1) is called as the Fourier integral. The sufficient condition for the validity of (1) are given in the following result:

Theorem

If $f(x)$ is piecewise continuous in every finite interval and has a left and right hand derivatives at every point and f is absolutely integrable, then $f(x)$ can be represented by a Fourier integral. At a point of discontinuity, the value of the Fourier integral is equal to the average of the left and right hand limits at that point.

And B omega given by integral/ - infinity to infinity f B sin omega dv the integral on the right hand side of 1 this integral is known as the Fourier integral the sufficient condition from the validity of Fourier integral is given in the following theorem if fx is piecewise continuous in every finite interval and has a left and right hand derivative at every point and f is absolutely integrable then fx can be represented by a Fourier integral.

At a point of discontinuity the value of the Fourier integral is = to the average of the left hand and right hand limits at that point so here when we write this Fourier integral to = fx we are assuming that x is a continuity point of fx if x is the discontinuity point of f then this fx will be replaced by the average of left and right hand limits of f.

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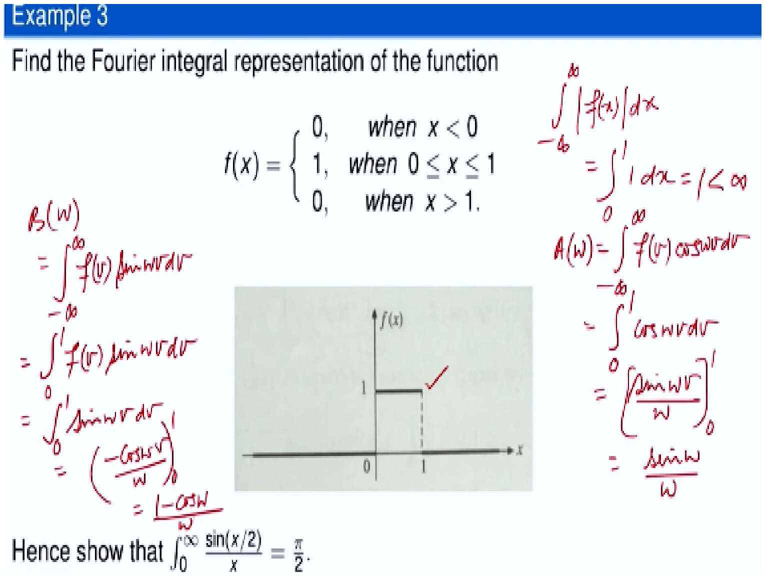

Now let us look at this example suppose if we have the function fx =0 when x is > than 0 1 1 0 is <=x<1<=1 and 0 when x >1 so this is the graph of this function we can let us try to find the Fourier integral representation of this we can see that it satisfies all the conditions which are stated here in the theorem it is piecewise condition in a every finite interval it has left hand and right hand derivative at every point and it is absolutely integrable.

We can see absolutely integrable integral /-infinity to infinity mod of fxdx will reduce to integral/0 to 1 because otherwise it is 0 and it is everywhere else other than the interval is 0 so 0 to 1 mod of fx is fx = 0 to 1 mod of 1 dx which is = 1 so the integral converges integral /- infinity to infinity mod of fx dx is< infinity so it is absolutely integrable so therefore we can find its Fourier integral representation.

In order to find the Fourier integral representation we need to calculate a omega and b omega here so let us find the value of A omega will be = let us go to the definition –infinity to infinity fv cos omega v db so using the definition of fx it is 0 to 1 fv = 1 we get cos omega v dv so this is sin omega v /omega let us put the limits and we get sin omega / omega let us find the value of B omega is integral / - infinity to infinity f v sin omega v dv.

So, this will reduce to integral 0 to 1fv sin omega v dv but fv = 1/ the interval 0 1 so 0 to 1 sin omega v dv with integrate with respect to v we get – cos omega v /omega 0 1 this is 1- cos omega /omega so we get the value of a omega and b omega we put in the right hand side. Here in this equation and get the Fourier integral.

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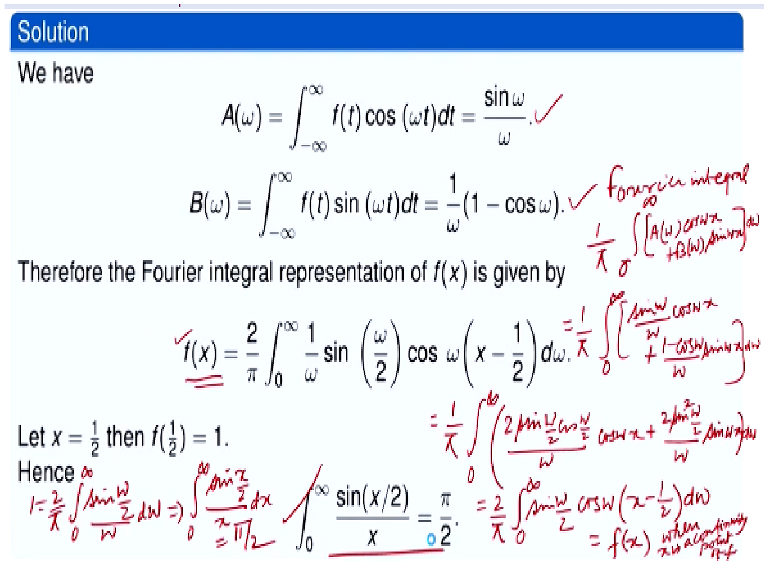

So, we get the following we get the value of A omega sin omega/ omega we get the value of B omega 1- cos omega/ omega then the Fourier integral representation of fx given by 1/Pi 0 to infinity so Fourier integral representation 1/Pi 0 to infinity A omega cos omega x+ B omega sin omega x d omega we have so this is 1/Pi 0 to infinity we get A omega = sin omega /omega cos omega x + B omega 1 – cos omega/ omega.

And then sin omega x d omega we can simplify this we can write sin omega as 2sin omega /2 cos omega/2 1- cos omega x2 sin square omega/2. We will have 1/Pi 0 to infinity we will have 2 sin omega /2cos omega/2/ omega cos omega x+2 sin square omega/2/ omega sin omega x d omega and this will be 2/Pi sin omega Pi/2 we can write outside so 2/Pi 0 to infinity sin omega/2 and we get then cos omega/2 cos omega x+ sin omega /2sin omega x.

That is cos omega x – 1/2d omega we get the Fourier integral representation of f and this will be =fx at each point of continuity of f this will be = fx when x is a continuity point of x when x is a continuity point of f and when you have taken any discontinuity point then it will be average of left hand and right hand limit now we have to find the value of this integral 0 to infinity sin x/2/ x so let us see we can consider x = 1/2 so let us take x = 1/2.

Then what we would notice x=1/2 is here and that1/2 fx is a continuous function so what we have x = 1/2 is the continuity point and f1/2 = 1 so we get 1=2/Pi 0 to infinity and here what do we get sin omega/2 x =1/2 is cos 0 is 1 so this divided by omega d omega v get so this implies integral 0 to infinity sin x/2 /x dx = Pi/2 and we get this result so integral 0 to infinity sinx/2 /xdx = Pi/2.

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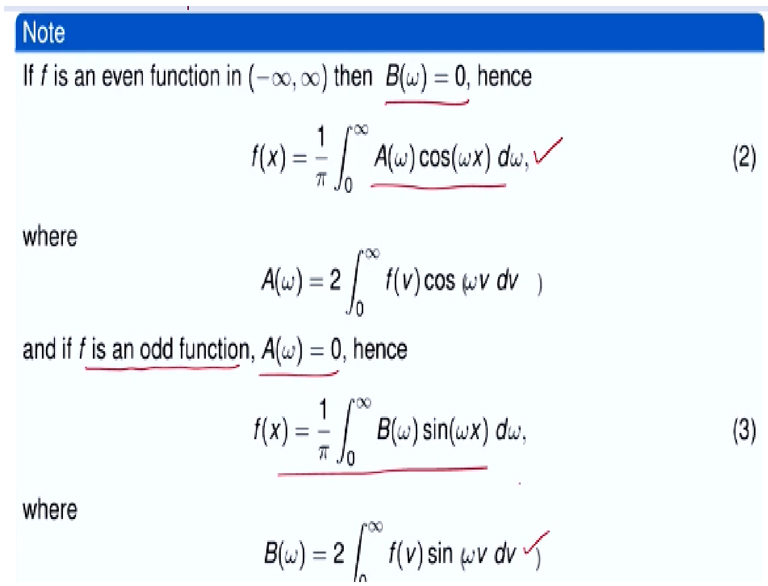

Now if f is an even function in the interval – infinity to infinity then you can see B omega will be = 0 because f is an even function and sin omega B is an odd function of B so their product is an odd function and therefore integral /- infinity to infinity fvs in omega v dv will be 0 so B omega is 0 and therefore fx = different integral reduces to fx = 1/Pi integral / 0 to infinity A omega cos omega x dx cos omega x d omega.

Where A omega now A omega will become twice integral 0 to infinity this is A omega so if f is even cos omega B is also even then this A omega will become 2 times integral 0 to infinity fv cos omega v dv so fx will be given by this integral where A omega will be given by twice integral 0 to infinity will be cos omega vdv if f is an odd function then B omega will become double B omega will become twice 0 to infinity fB sin omega vdv.

While A omega will become 0 because A omega is integral – infinity to infinity fv cos omega vdv so when f is odd cos omega is even their product will be odd and therefore their integral / - infinity to infinity will be 0 so fx is = 1/Pi integral 0 to infinity B omega sin omega x d omega where B omega is 1/ this twice 0 to infinity fv sin omega vdv.

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The integrals in (2) and (3) are called Fourier cosine and sine integrals respectively.

Remark

Just as in the case of half range Fourier series, a function $f(x)$ defined over the interval $(0, \infty)$ may be expressed as a Fourier sine or cosine integral. Further, we observe that the representation of a non-periodic function $f(x)$ given by (1) is similar to the representation of a function by a Fourier series except that the range is now $(-\infty, \infty)$ and the summation has been replaced by integration.

The integrals in 2 and 3 are called Fourier cosine and sine integral so this is called as Fourier cosine integral this 1 is called Fourier cosine integral this 1 is called Fourier sin integral now just as in the case of 1/2 range Fourier series a function defined over the interval suppose the function there in the 1/2 range Fourier series what we had we had function defined over the interval 0 to l so over the 1/2 interval - infinity to infinity 0.

We were free to define in any way so we define it by odd and even periodic extensions when we used odd periodic extension we found 1/2 range Fourier sine series when we define the function f as an even extension / the interval – l to l then we found Fourier sine series so just like that so just like that in the case of 1/2 Fourier series a function defined / the interval 0 to infinity may be expressed as a Fourier sine or cosine integral/ interval - infinity to 0.

We can define by taking even extension or by odd extension then the function will be even or odd and so we will get the corresponding Fourier cosine or Fourier sine integral so we observe that the presentation of a non-periodic function fx given by 1 is similar and now we can see here this integral you can see Fourier cosine integral this Fourier cosine integral is similar to the Fourier sine Fourier series so given by 1 is similar to the representation of a function.

By a Fourier series except that the range is now – infinity to infinity and the summation has been replaced by integration so there also we had similar situation we had an cos nx+ B n sin nx and so it is similar expression here there we had summation here we have integration and their we had the interval – Pi to Pi here we have the interval – infinity to infinity in the Fourier integral so it is similar to the Fourier series so this is what we have observed here

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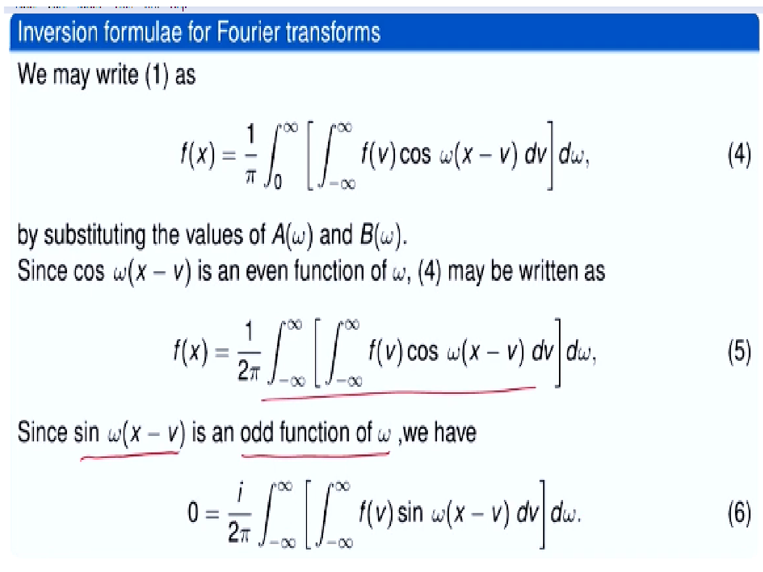

Let us now move to inversion formulae for Fourier transforms now this is equation 1 can be written in the following manner this equation fx = 1/ Pi mod 0 to infinity A omega cos omega x B omega sin omega x let us put the value of A omega and sin B omega here what we get 1/omega 1/Pi0 to infinity a omega is integral/- infinity to infinity fv cos omega vdv* cos omega x+ integral/-infinity to infinity fv sin omega vdv* sin omega xd omega.

So, I can write it as 1/Pi integral / 0 to infinity integral / - infinity to infinity so I can write it as 1/Pi integral/0 to infinity integral/- infinity to infinity fv* cos omega t –x v-x dv d omega so we can write like this now cos omega v – x is an even function now omega we can make it 1/2Piintegral/-infinity to infinity fvcos omega v-xdv d omega this is because cos omega v –x is an even function of omega so let us now go there so this is how.

We can use the fact that cos omega x – v is an even function of rare so we can write fx like this now sin omega x – v is an odd function of omega therefore integral /-infinity to infinity integral /- infinity to infinity fv sin omega x-v dv d omega will be = 0. I can multiply by i/2Pi it will be also 0 = i/2Pi integral/-infinity to infinity integral/-infinity to infinity fv sin omega s-v dv d omega now let us add this quantity to this equation to this equation. Let us say 5 and 6 what we will get.

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Inversion formulae for Fourier transforms cont...

Adding (5) and (6), we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}dv\right]d\omega. \qquad (7)$$

This is known as complex form of the Fourier integral. We may write (7) as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega, \qquad (8)$$

where

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv. \qquad (9)$$

The function $F(\omega)$ given by (9) is called the Fourier transform of the function $f(x)$. Also, the function $f(x)$, given by (8) is known as inverse Fourier transform of $F(\omega)$.

We will get fx = 1/2Pi integral/-infinity to infinity integral /-infinity to infinity and here cos of theta + i times sin of theta e to the power i theta you will write so e to the power i omega x –b dv d omega this is known as complex form of the Fourier integral. We may write this as 1/2Pi integral / - infinity to infinity e to the power i omega x e to the power – i omega v. I can write i omega x here and the integral /-infinity to infinity fv e to the power –i omega v dv d omega.

This I can write as 1/root 2 Pi integral/-infinity to infinity F omega this I write as 1/root Pi root 2Pi we take from this coefficient and define the integral – infinity to infinity fv e to the power – i omega vdv s root 2Pi* f omega so this is f omega actually so this f omega and then we get 1/ root 2 Pi integral/-infinity to infinity f omega e to the power i omega d omega now the function F omega given by this equation 9 is called the Fourier transform of the function fx.

The function F given by this equation fx = this is known as inverse Fourier transform of f omega some people take 1 here in f omega and while taking the inversion formulae they take 1/2Pi here we have taken 1/root 2Pi here so there is no problem because when we use this Fourier transform in the solution of partial differential equations their first we apply the transform and then we take the inverse Fourier transform so this coefficient is taken in to account there.

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Inversion formulae for Fourier transforms cont...

In a similar fashion, we obtain from equations (2) and (3)

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\omega) \cos (\omega x \, d\omega,) \qquad (10)$$

where

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos (\omega v \, dv,) \qquad (11)$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega) \sin (\omega x \, d\omega,) \qquad (12)$$

where

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin (\omega v \, dv,) \qquad (13)$$

So, we now in a similar manner let us see how we define the Fourier sine and cosine transforms in their inversion formulae let us see how we find so from equation 2 and 3 let us look at question 2 and 3 this is equation 2 and this is equation 3 you put the value of A omega here but you get 2/Pi integral /0 to infinity A omega is fv cos omega vdv 2/Pi integral /0 to infinity and let us write it as integral /0 to infinity here fv cos omega v dv cos omega x d omega.

So, this can be written as 2/Pi we have 2 integrals/ 0 to infinity fv cos omega v cos omega xdvd omega now let us see how we define Fourier sine and cosine transform so we can write it as integral root 2/Pi /0 to infinity Fc omega cos omega x d omega so we can write it like this this = root 2 /Pi0 to infinity fv cos v-x dv omega this is because cos omega v-x is an even function of omega okay.

So, let us now go there okay this is how we can use the fact that cos omega f-b is an even function. So, we can write fx like this. Now sin omega x- b is an odd function of omega okay therefore integral/-infinity to infinity fb sin omega x -v dv d omega will be =0. So I can multiply i/2pi it will be also 0. So, 0=i/2pi integral/-infinity to infinity integral/-infinity to infinity fb sin omega x- dv d omega.

Now. Let us add this quantity to this equation to this equation. Let us take 5 and 6 what we will get we will get fx=1/ 2pi integral/-infinity to infinity integral/-infinity to infinity and here cos of theta +i times sin of theta so e to the power I theta we will get you write e to the power i omega x -v dv d omega this is known as complex form of the Fourier integral. Okay we may write this as 1/ 2 pi integral/-infinity to infinity.

e to the power i omega x e to the power i omega v we have here. So, I can write it for i omega x here integral/-infinity to infinity fv e to the power – i omega v dv d omega okay. This I can write as 1/root 2 pi integral/-infinity to infinity f omega. This I write as 1/root 2 pi we take this from this co efficient and define the integral – infinity to infinity fv e to th power -i omega dv as root 2 pi * f omega.

So, this is f omega actually. Okay so this is f omega and then we get 1/root 2 pi integral/-infinity to infinity f omega e tot eh power i omega d omega. Now the function f omega given by this equation9 is called the Fourier transform of the function fx. The function f given by this equation. Fx=this okay this is known as inverse Fourier transforms of F omega okay. Some people take one here okay f omega.

And while taking the inversion formula they take 1/2 pi here we have taken 1/root 2 pi here 1 /root 2 pi here. So, there is no problem because when we use Fourier transform in the solution of partial differential equations there first, we apply the Fourier transform then we take the inverse Fourier transform. So, this coefficient is taken into account there. So, we now in a similar manner let us see how we define the Fourier sin transforms.

And their inverse formula. Let us see how we find okay. So, from equation 2 and 3 let us look at equation 2 and 3 okay this is equation 2 and tis is equation 3. So, put the value of a omega here what value you get 2/pi integral/0 to infinity a omega is fb cos omega v dv okay 2/pi integral/0 to infinity. And let us write it as integral/ 0 to infinity here b cos omega v dv cos omega x d omega okay so this can be written as 2/pi.

We have 2 integrals 0 to infinity 0 to infinity fb cos omega b soc omega x dv d omega. Now let us see how we define Fourier sin and cosine transforms. So, we can write it as root 2/pi 0 to infinity fc omega cos omega xd omega okay. So, we can write it like this this is = root 2/pi 0 to infinity fv cos we interchange the order of integration. We have Instead of dv d omega we have d omega dv.

Okay so cos omega v d omega we have and outside we have cos omega x d omega. Okay then we define it write like this.2 root 2/phi 0 to infinity fc omega cos omega x d omega. Where fc omega =root2/pi 0 to infinity fv cos omega v dv. Okay so this is defined as Fourier sin transform of f and this formula which gives the fx okay this formula fx= this. Okay fx= root 2/pi 0 to infinity fc omega cos omega x d omega is called the inverse formula for the Fourier cosine transform. Similarly, we have the formula for Fourier sin transform.

You put the value for v omega here 2 times 0 to infinity fv sin omega v dv interchange the order of integration. So, after inter changing order of integration root 2/pi integral/ 0 to infinity fv sin omega v dv sin omega v d omega. Okay sin omega v d v okay fs omega will be =root 2/pi 0 to infinity fv sin omega v dv this will be Fourier sin transforms and inverse formula will be fx= root 2/pi 0 to infinity fs omega sin omega x d omega in a similar manner we do it okay just like in the case of Fourier cosine transform.

We put the value of v omega from here in this integral and we get the Fourier sin transform and inverse formula for Fourier sin transform okay. So, this is formula for inversion formula for Fourier cosine transform this is inversion formula for Fourier sin transform this is Fourier formula for Fourier sin transform.

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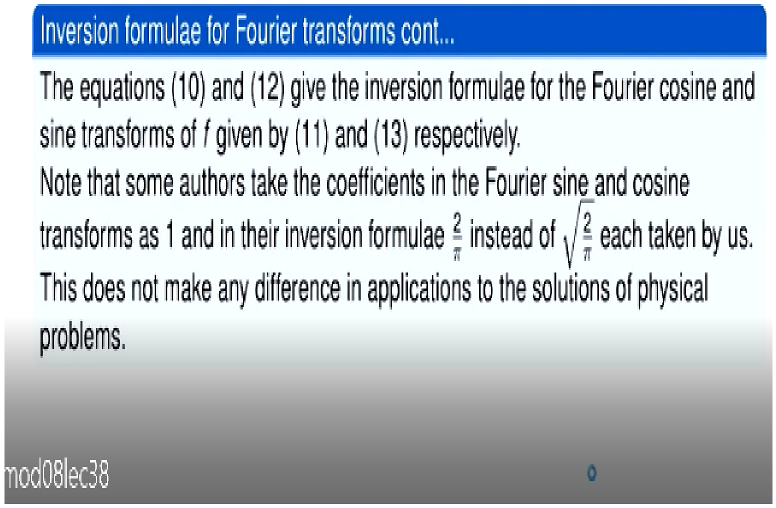

Now let us look at this the equations 10 and 12 give the inversion formulae for the Fourier cosine and sine transforms this one and this one they give the formulae for the inversion formula for the Fourier cosine and sine transforms of f which are given 11 and 13. Note that some authors take the coefficients in the Fourier Sine and cosine transform as 1 and in their inversion formulae 2/pi instead of root 2/pi each taken by us. This does not make any difference in applications to the solutions of physical problems.

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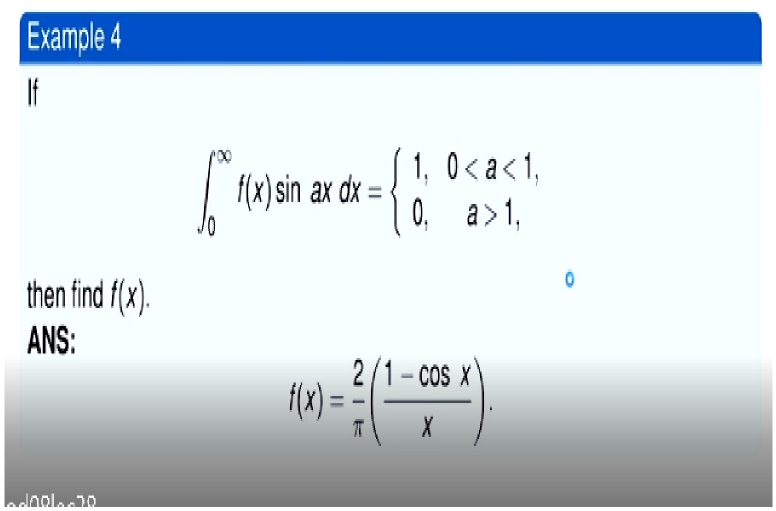

Now let us take this example suppose we have an integral equation because we have here an unknown function fx it occurs inside he integral sine. So, integral equations can also be solved using the Fourier cosine sine transform so 0 to infinity fx sin ax dx =1 when 0 is <a<1; 0 when a>1, we have to this unknown function fx okay. So, let us compare this with the Fourier sine transform okay.

So, this is the Fourier sine transform okay 0 to infinity fv is n omega dv* root 2/pi fs omega okay. So, we have given fs omega okay. So, here we are given Fourier sine instead of omega we have here a okay we do not have sin omega x we have sine x so we have fsa and root 2/pi I am taking as 1 instead of root 2/pi here for convenience I am taking it as this co efficient as 1. So, while taking this inverse Fourier sine transform, I will take here 2/pi.

So, fsa given to be 0 to infinity fx sin axdx=1 0<a<1 and 0 when a>1. Okay now by then inversion formula fx will be 2/pi 0 to infinity fs omega sin omega x dx d omega okay. Why I have to use sine inverse Fourier sine transform because we have sine function here. If here we have here cos as function ax the we will use Fourier cosine transform formula. So, this is =2/pi and over the interval 0 it is 1 otherwise 0.

So, 0 to 1 fs omega is 1 sin omega s d omega, so this is 2/pi -cos omega x x because we are integrating the two omega 0 to 1 okay. So, this is 2/pi 1-cos 0 so this is 1. We have 0 1 interval okay 1-cos x/dx because these are the limits for omega. So, 1- cos x/x so we have 2/pi *1- cos/x cos x/x.

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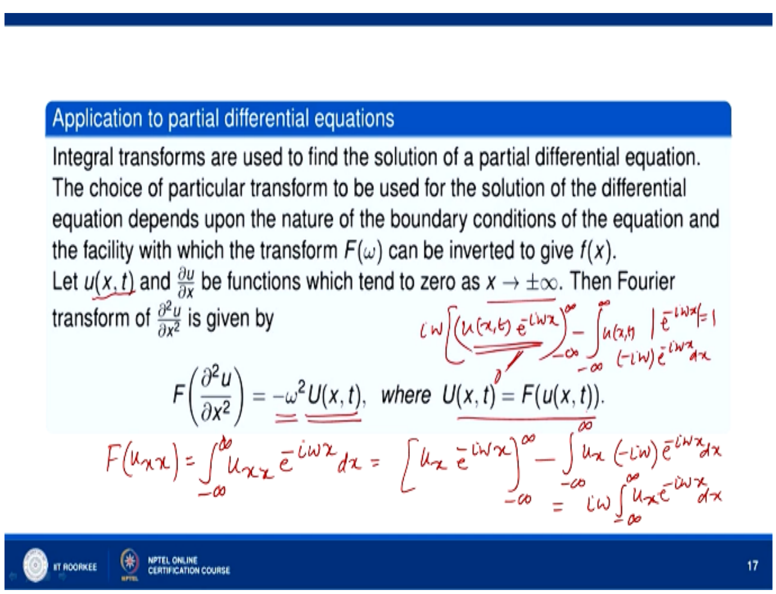

Now application to partial differential equations integral transforms can be used to find the solution of a partial differential equation. The Choice of a particular transform to be used for the solution of the differential equation depends upon the nature of the boundary conditions of the equation and the facility with which the transform F omega can be inverted to give fx let uxt and ux be functions which tend to 0 sa x goes to +-infinity.

Then the Fourier transform of uxx is given by F uxx=-omega square uxt we can see this. Fourier transform of uxx okay let us go to the formula for Fourier transform so Fourier transform is 1/root 2 pi I will take here 1 and then while we take the inversion formula, we will take 1/2 pi here, so f omega is -infinity to infinity fv e to the power -I omega v dv okay let us use this formula.

So, this will be integral -infinity to infinity uxx e to the power- i omega x dx okay. This will be now let us integrate it by parts so we will have ux when we integrate with respect to x ux e to the power -i omega x- infinity to infinity – infinity to infinity ux -i omega e to the power -i omega x okay now ux goes to 0 we have assumed ux goes to 0 as x goes to +-infinity. So, this will go to 0 when x goes to +infinity.

And this will also go to 0 when x goes to -infinity. E to the power -i omega x is a bounded quantity we get mod of e to the power -i omega x=1. Mod of e to the power omega x is under toot cos square omega x+sin square omega x. So, that is 1 so bounded quantity*0 so we get 0 here and this becomes i omega integra1/-infinity to infinity ux e to the power -i omega x dx. Okay and we integrate it again so when you integrate it again.

What do you get I omega integral of ux is now x so uxt * e to the power -i omega x -integral/-infinity to infinity uxt *-i omega e to the power – i omega x dx okay so this will again go to 0 because we have assumed that uxt goes to 0 as x goes to + or – infinity and e to the power -i omega x is a bounded quantity it is bounded by 1. So, this will go to 0 and we will have then i omega* i omega i square omega square i square is -1.

So, we will get – omega square and we will get integral/-infinity to infinity uxt e to the power – i omega x dx which is Fourier transform of uxt and we are denoting it by uxt Fourier transform of uxt we are denoting by you can see here. Fourier transform of uxt we are denoting by uxt. So, Fourier transform of uxx is – omega square uxt.

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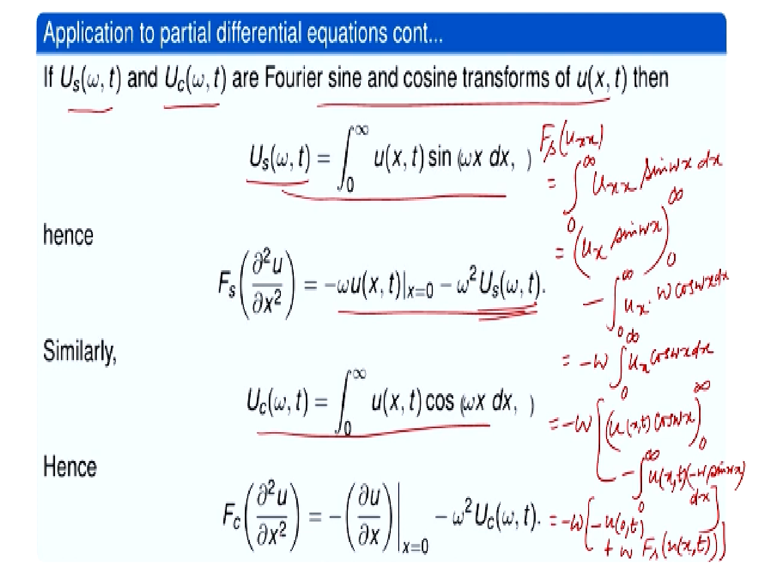

If us omega t and uc omega t are Fourier Sine and Cosine transform of uxt then us omega t= integral 0 to infinity uxt sin omega x. This is how we denote the Fourier by this notation we are denoting the Fourier sine transform of uxt okay Fourier sine transform of uxx then will be given by this formula we can easily see that. So, Fourier sine transform of uxx will be then uxx will be integral 0 to infinity uxx sine omega x dx.

So, integrate by parts so ux sin omega x 0 to infinity -u we are integrating with respect to x. So, omega times cos omega x dx 0 to infinity. Now ux goes to 0 when x goes to infinity and sin omega x is a bounded quantity. It is bounded by 1 so it is 0. So, when x is 0 sine omega x is 0. So, this quantity becomes 0 and what we get -omega integral 0 to infinity ux cos omega x dx. Let us integrate by parts again so we get uxt function.

Now integral of u is uxt cos omega x 0 to infinity – 0 to infinity uxt*-omega sine omega x dx. Okay when x goes to infinity, we have assumed u goes to 0 so this goes to 0 when x goes to 0 this becomes u-u0t cos 0beocmes 1 and here we get +omega Fourier sine transform of uxt. Okay so because we are getting this omega times 0 to infinity uxt sine omega x dx okay so what we get.

Let us see we get -omega we got outside and then we get here – omega square okay -omega square fs u xt fs u xt. We denote by us omega t okay us omega t and this will be omega uxt at x=0. Okay so omega uxt at x=0- omega square fs uxt which is us omega t. In a similar manner we can derive for the Fourier Cosine transform the formula fc uxx =-ux at x=0- omega square uc omega t.

Now when we apply these sin transform, or Cosine transform to the partial differential equation we have to see the boundary condition. If the boundary condition is given at x=0 we are given u0t then we apply Fourier Sine transform. If we are given the gradient t x=0 that is at uxx=0 we apply Fourier Cosine transform. So, we can decide from the boundary condition which transform we have to apply in case of a semi-infinite domain.

Okay here we are using integral/-infinity -integral/0 to infinity if we use integral/-infinity. If we are given the problem over-infinity<x<infinity, then we will need to use Fourier transform. So, let us apply the Fourier transforms to solve partial differential equation in one dimensional problem the pd is transformed in to an of by applying a suitable transform if In a problem uxt at x=0 is given then we use Fourier sine transform to remove uxx.

In case, ux at x=0 is given then the Fourier Cosine transform is applied to remove uxx.
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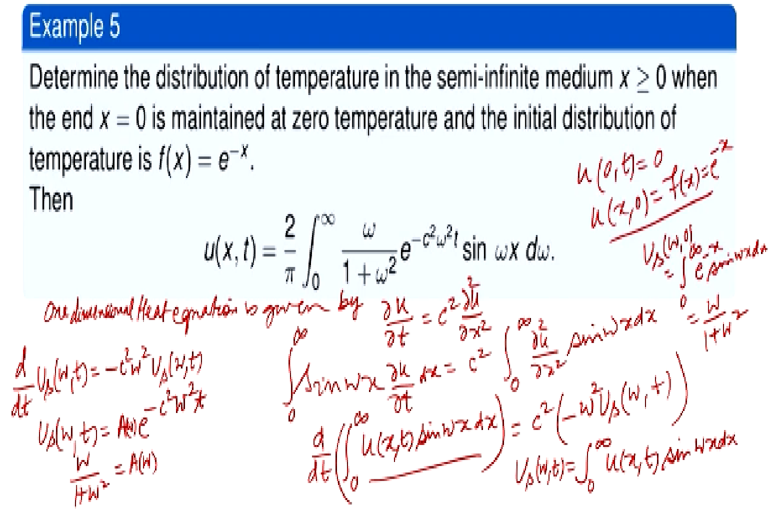

Let us cosine determine the distribution of temperature in the semi-infinite medium x>=0 when the end x=0 is maintained at 0 temperature so we are given u 0t x=0 t is given as 0 okay u0t is given as 0 and initial distribution of temperature is e to the power -x. So, we have ux0=fx=e to the power -x okay now we know that the heat equation in 1 dimensional heat equation is given by 1 dimensional heat equation is given by ut=c square uxx.

Now we are given uxt at x=0 we will use Fourier sine transform. So, we will apply Fourier Sine transform here okay we multiply this equation by Fourier Sine transform we have to apply. Okay so we multiply this equation by sin omega x and integrate with respect to x okay. So, let us multiply by sin omega x and integrate with respect to x over the interval 0 to infinity okay. so, then x and t are independent of each other.

I can write it as d/dt of okay derivative of with respect to t of integral to infinity uxt sin omega dx and this is c square times 4a sine transform of uxx. So we have seen Fourier sine transform of uxx is given by omega times at txt at x=0-omega square us omega t uxt=0 when x=0 so it reduces to – omega square us omega t. So, we get – omega square us omega t okay. So, left hand side is this is us omega t okay.

So, we get d/dt us omega t = -c square omega square us omega t. Okay so this is dy/dt=- c square w square/ so we can integrate it easily now okay so we get us omega t e= some constant a times e to the power -c square omega square t we can integrate with respect to t keeping omega as constant. So, this a will depend on omega a will be a function of omega. Now let us taken let us use this boundary condition.

Okay so we have given that integral 0 to infinity uxt sin omega x dx this is = us omega t this is what we have assumed. So, put t=0 in this so us omega 0 will be = integral 0 to infinity e to the power -x because uxt is ux 0 which is e to the power -x sin omega x dx and its value is omega/1+ omega square okay. So, put t=0 here so us omega 0 becomes omega/1+ omega square and this is A omega, and this is 1. Okay we got the value of A omega okay thus we get.

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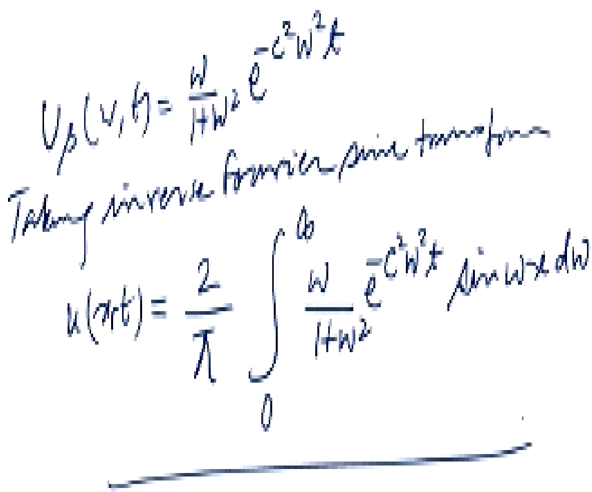

us omega t=omega/1+omega square e raised to the power -c square omega square t. Now we take the inversion formula use the inversion formula, so we take the inverse Fourier sine transform of this. Taking inverse Fourier sine transform we get uxt= we multiply by sine omega x and integrate with respect to omega okay. Now we have taken while taking Fourier Sine transform, we have taken the Fourier co efficient as 1.

So, here we will take the co efficient 2/pi. So, 0 to infinity omega/1+omega square e to the power – c square omega square t* sine omega x d omega. Okay so this is the temperature distribution in the semi-infinite medium for x>0 t >0. Okay so this is what we have 2/mega 0 to infinity omega/1+o mega square e to the power -c square omega square t sin omega x d omega. With this I have come to the end of this lecture thank you very much for your attention.