Advanced Engineering Mathematics
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Lecture – 37
Finite Fourier Transforms

Hello friends, welcome to my lecture on finite Fourier transforms, we have seen that the Fourier transform technique can be applied to those problems which involved in finite or semi-infinite domains like you have a war of infinite length or you have a semi-infinite strip, okay so there we can apply Fourier transform techniques to solve the differential equation; partial differential equation.

Now but in many practical problems, we often come across finite intervals in boundary value problems therefore, we need to extend the Fourier transform method to problems where the range of independent variable is finite, there in versus can then we found by applying the theory of Fourier series.

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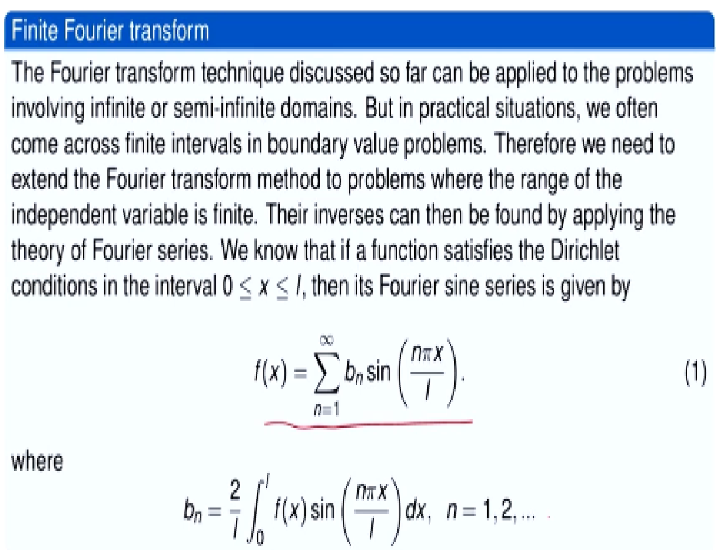

We know that if a function satisfies the Dirichlet conditions in the interval 0 < = x < l then it is Fourier sine series is given by Fx = sigma n = 1 to infinity bn sin n pi x over l, let us recall the half range expansions, okay, when the function is defined over the half range, 0< =x< =l, then we can extend this function over the other half that is –l< = x < = 0 by considering f to be an even or as in odd function.

If you consider f to be an odd function, then you will get the Fourier sine series, okay which is known as half range Fourier series, okay half range Fourier sine series, so there you get fx = sigma n = 1 to infinity bn sin n pi x over l and the value of the Fourier coefficients bn is 2 over l 0 to l fx sine n pi x over l dx, where n takes value 1, 2, 3 and so on. If you consider the even extension over the other half, okay over the interval –l to l, you take the function f to be an even function.

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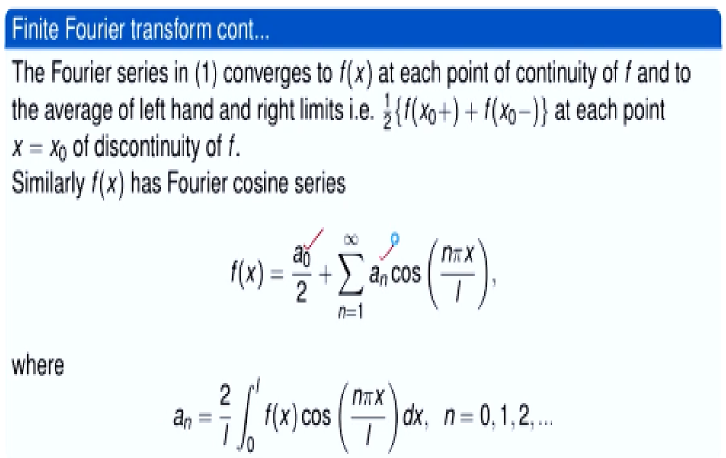

Then you get the Fourier sine series, okay and the Fourier sine series is given by fx = a0/2 + sigma an = 1 to infinity an cos n pi x over l, where the Fourier coefficients an’s are given by 2 over l 0 to l fx cos and pi x over l dx and takes values 0, 1, 2, 3 and so on. When you take n = 0, you get the value of this a0 and for n = 1 to 3, you get the values of these coefficients an’s and you also know that the Fourier series; half range Fourier series convergence to fx, okay, whether it is Fourier sine series or Fourier cosine series, okay, ait converges to fx at each point of continuing of f.

And to the average of left hand and right hand limits at each point of discontinuity of f, say the point of discontinuity is x0, then you have the some of the series; Fourier series as half of fx0 + + fx0-, okay, so we have to; we need this definitions here.

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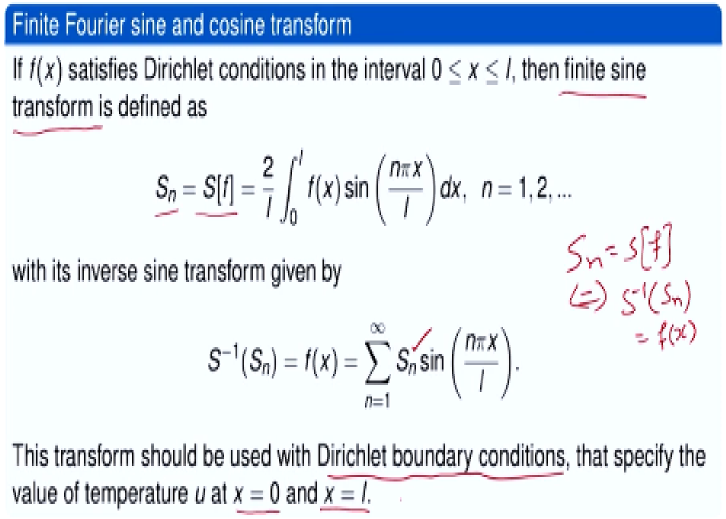

Now, let us consider finite Fourier sine and cosine transform, if fx satisfies Dirichlet conditions in the interval 0 < = x < = l, then finite sine transform, okay, the finite sine transform is defined as Sn = Sf = 2 over l 0 to l fx sin n pi over l dx, okay, the value of this okay, 2 over l 0 to l, no, not this 2 over l 0 to l fx sin n pi x over l dx, it is defined as finite sin transform, okay, we denoted by Sf or you can also write Sn, okay.

And its inverse sin transform is given by S inverse of Sn; S inverse of Sn = Sn is Sf, okay, so S inverse of Sn = fx, okay, so S inverse of Sn = fx = sigma n = 1 to infinity and then we this this Sn, okay, Sn sin n pi x over l, so this we denote by Sn, okay, then fx is given by sigma n = 1 to infinity Sn sin n pi x over l, okay. So, in the finite sine transform, okay the Fourier coefficient bn is replaced by Sn actually.

Sn is the Fourier sin transform of fx, okay and then the Fourier series; Fourier sine series, okay, half range Fourier series; Fourier sine series is gives the value of the function fx because now, we know the values of Sn's okay. So, this transform is applied when you are given the Dirichlet boundary conditions, remember this transform will applied only when we are given Dirichlet boundary conditions.

Because they satisfy; they specify the value of temperature u at x = 0 and x = l, this will be clear why we apply the Dirichlet boundary; why we apply Fourier sine transform in the case of Dirichlet boundary conditions, this will be clear, later when we consider the partial differential equations, there when we try to solve the partial differential equation will lead to take the transform of uxx.

And when you take the transform of uxx, there if you have Dirichlet boundary conditions, you will apply sin transform, if you have Dirichlet boundary condition, you will apply Fourier sine transform, okay finite cosine transform, so this is to be apply when we are given Dirichlet boundary conditions, Fourier sine transform we will apply then.

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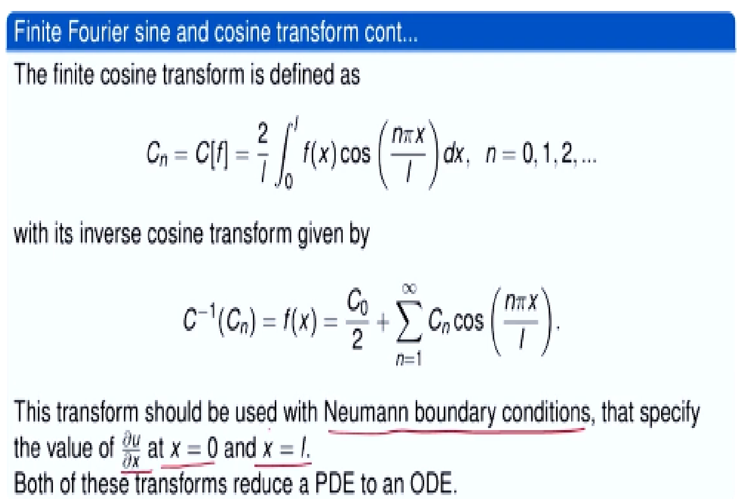

Now, finite cosine transform is defined as Cn = Cf 2 over l 0 to l fx cos n pi x over l, okay, so we have the value of an here, this an value is replaced by Cnr, you can say Cf, okay the this Cf gives the finite cosine transform of f and fx is then determined from this series, okay, we have the values of Cl's now, this is C0, an is nothing but Cn, so this C0/2 and then we have sigma n = 1 to infinity Cn cos n pi x over l.

So, putting the values of C0 and Cn here, we get the value of fx, okay, so Fourier cosine transform is to be used when we are given the Neumann boundary conditions, okay. In the Neumann boundary conditions, we have the value of the temperature gradient ux at x = 0 and x = l. Now, both of these transforms when we were apply to partial differential equations, they will reduce them to ordinary differential equations.

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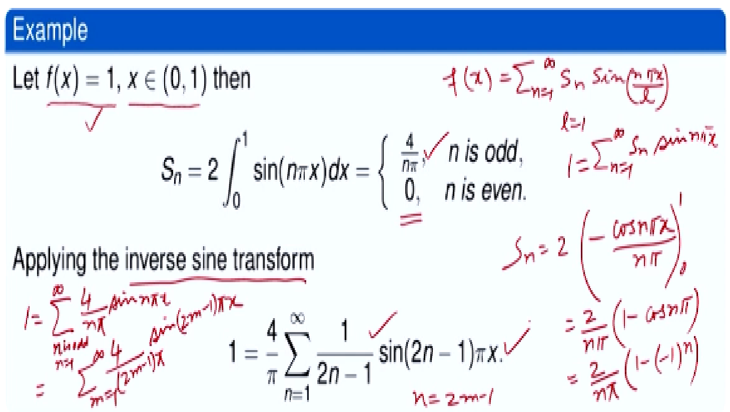

And now, let us see how we solve differential; partial differential equations using these finite sine and cosine transforms, so let us consider first example, suppose we are given fx = 1 over the open interval 0, 1, okay then the finite sine transform of f, okay, finite sine transform of f Sn is given by this expression, okay, since the domain of definition of f there is 0 to 1, l will be= 1, so we have 2 over 1, 0 to 1 fx sin and pi x over 1 dx, okay.

So, we have this, 2 over 1 0 to 1 fx, in place of fx, we put 1, so 1 * sin n pi over; sin n pi x dx, so Sn will be = 2 times, let us integrate, you get - cos n pi x divided by n pi, you put the values, you get 2 over n pi 1 – cos n pi, okay. Now, cos n pi is -1 to the power n, so you get 2 over n pi 1 - -1 to be power n, okay. Now, when n is even, you get this value 0, okay, so when n is even integer, the value is 0.

When n is an odd integer, we will have 4 over n pi, okay, so 4 over n pi, when n is an odd integer, so Sn = 4 over n pi, when n is an odd integer and 0 when n is an even integer, now let us apply the inverse sine transform okay, so inverse sin transform is obtained from fx = sigma n = 1 to infinity Sn sin n pi x over l, so we can get the; here now, fx is given to be = 1, so 1 = 4 over pi, okay 4 over pi, okay this is fx = sigma n = 1 to infinity Sn sin n pi x over l, okay.

So, l is = 1 here, okay, so we get sigma and fx = 1, fx is continuous over the whole interval 0, 1, okay, so 1 =; let us put fx = 1, sigma n = 1 to infinity Sn sin n pi x. now, Sn is 0, when n is even, okay, Sn is 4 over n pi, when n is odd, so we can say 1 = sigma n is odd integer, okay upto infinity, n = 1 to infinity but n is odd, okay, subject to the condition that n is odd and Sn is 4 over n pi * sin n pi x.

Now, if you take n = 2m -1 okay then when m is takes value 1, 2, 3 and so on, n will take odd integral values, so we will have n = 1 to infinity, we can write 4 over 2m -1 pi and then sin 2m -1 * pi x, okay, so replace now m by n, so you get 1 = 4 over pi we can write outside the summation, so 1 = 4 over pi, sigma n = 1 to infinity 1 over 2n - 1 sin 2n -1 pi x, so this is how we find the inverse sine transform, when Sn is known, okay.

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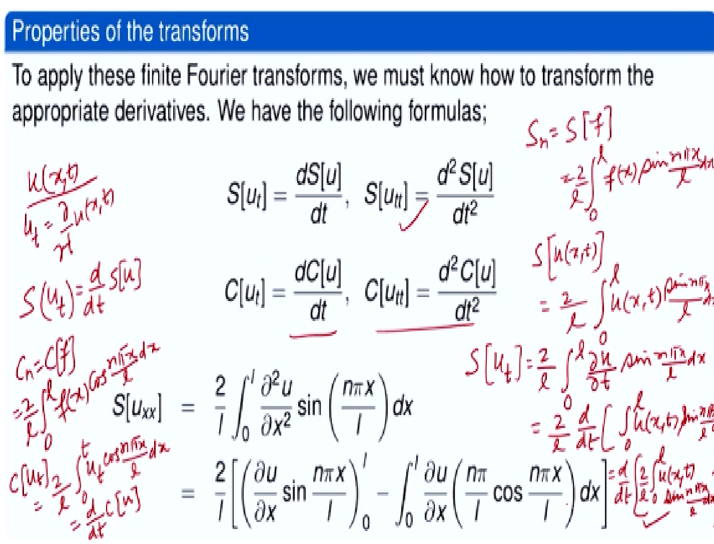

Now, let us see; we can solve the partial differential equations, when you solve the partial differential equations, we will need, suppose t the time variable, okay, x is the space variable, so we will need to find the sine transform of; sine or cosine transform of ut, utt, okay, ux, uxx and so on. So, sine transform of ut, okay, sine transform of ut let us recall that sine transform of Sn = S of f we have written.

S of f =; you take 2 over l, 0 to l fx sine n pi x over l, this is how we have defined, okay, so here we need the sine transform of ut; u is a function of x and t, okay, u is a function of x and t, so S of; in place of f, we shall write uxt, okay, this will be 2 over l 0 to l, in place of fx, we shall write uxt sin n pi x over l dx, okay. Now, if you want the sine transform of ut; ut is partial derivative of u xt with respect to t, okay.

So, this will be S of ur, okay, ut = 2 over l 0 to l, in place of uxt, now we will have ut, so partial derivative of u with respect to t * sin n pi x over l dx since t and x are independent of each other, we can write it is 2 over l d over dt of 0 to l uxt sin and pi x over l dx, okay and this 2 over l we can take inside, so we have d over dt of 2 over l 0 to l uxt sin n pi x over l dx, meaning that S ut = d over dt of Su okay, this is d over dt of Su, okay.

Similarly, if you have S utt, okay, in place of ut, you have utt here, then you will have utt here, you can write d square over dt square outside, then S utt will be = d square over dt square of Su in a similar manner, okay, C ut; how we define Cf? Cn =; Cf we defined as 2 over l 0 to l fx cos n pi x over l dx, okay. So, C ut, we can similarly write, C ut will be = 2 over l 0 to l ut cos n pi x over l dx and this will be = d over dt of Cu as we have seen here, okay.

So, we get the same thing and then here we also similarly get C utt = d square over dt square Cu, okay, now let us find Cs of Uxx, okay, S of Uxx means 2 over l 0 to l second order partial derivative of U with respect to x, okay, as I said U is a function of x and t, so partial derivative; second order partial derivative U with respect to x * sin n pi x over l dx. Now what we do; we integrate it by part, okay.

So, integration of by parts gives you 2 over l, integral of second derivative of U with respect to x, when you do, you get first order partial derivative of U with respect to x, so Ux * sin n pi x over l, 0 to l, okay, -0 to l Ux, okay, then derivative of sin and pi x over l, so that is n pi over l cos n pi x over l dx. Now, you can see here when you put x as l, sin n pi is 0, when you put x as 0, sin n pi is again 0, okay and ux is finite quantity, so this is 0, okay, this part becomes 0.

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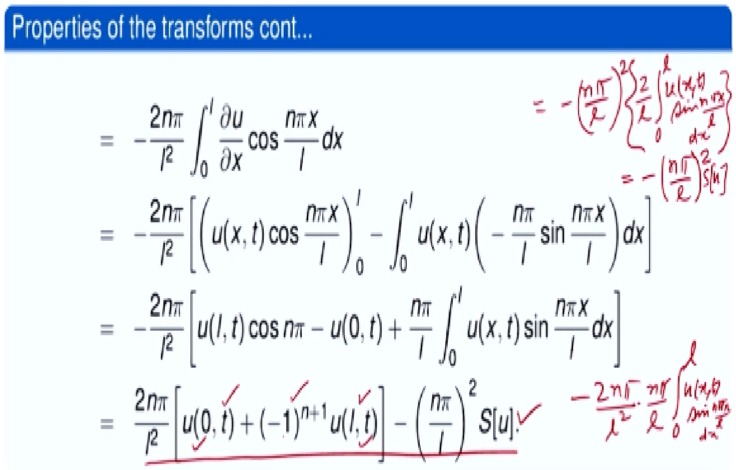

And therefore, we have -2 over l, so -2n pi over l square 0 to 1 Ux cos n pi over l dx, okay, so we get this, -2 n pi over l square 0 to 1, ux cos n pi x_over l dx, again we integrate by parts, okay, so integration of Ux gives you U xt, okay, U xt cos n pi x over l 0 to l, -0 to l U xt, then derivative of cos n pi x over l, so –n pi over l sin n pi x over l dx, okay. Now, put x as l, okay, so U lt cos n pi –U0 t cos 0; cos 0 is 1, okay.

Then, here we have minus, minus; plus okay, so + n pi over l 0 to l U xt sin pi x over l dx, okay, now let us recall that cos n pi is -1 to the power n; -1 to the power n is the value of cos n pi, so you multiply by this -1 inside, so 2n pi/ l square, you get when you multiply by -1, you get U0 t here, then -1 to the power n *-1, -1 to the power n + 1 * U lt, okay, so this we multiply to this and then this -2n pi over l square, when you multiply here, what you get?

You get -2 n pi over l square, we multiply to this, so n pi over l 0 to l U xt sin n pi x over l dx, okay, if you can write; this you can write as –n pi over l whole square 2 over l 0 to l U xt sin n pi x over l dx, okay, so this is nothing but finite sine transform of U, okay, so we can write it as – n pi over l whole square * Su, so you can see, this is –n pi over l whole square Su. Now, you can see when you finite sine transform of Uxx, when you find, it comes out to be in this form, okay this form, this one, okay.

So, here you need to know the value of U0 t and U lt, the value of temperature at x = 0 and x = l at the 2 boundary points of the problem, so we are given Dirichlet boundary conditions therefore, whenever Dirichlet boundary conditions are given, we use the finite sine transform to solve that problem, okay. Now, let us find the Fourier; finite sine transform of Uxx, okay.

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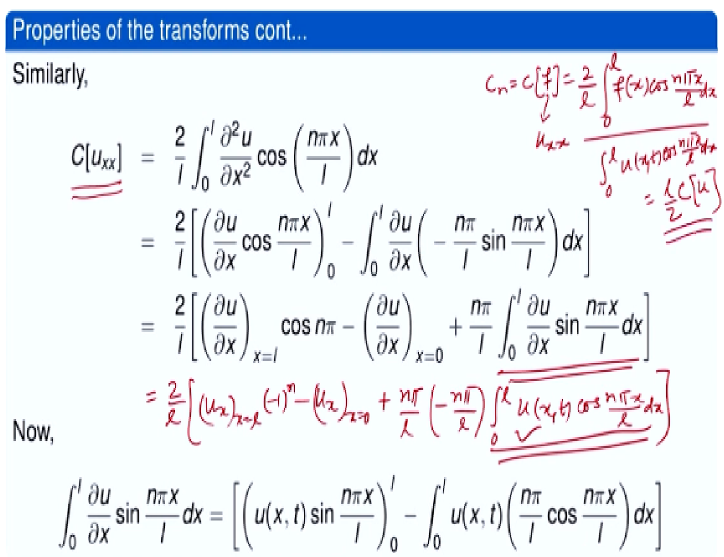

So, finite cosine transform of U xx, if you want to find then you, we know that cosine transform is given by Cn = Cf = 2over l 0 top l fx cos n pi x over l dx, okay, this is finite cosine transform of a; so cosine transform of U xx, so in place of x, we are now having U xx, okay, so 2 over l 0 to l Uxx cos n pi x over l dx integration by parts gives us 2 over l Ux cos n pi x over l 0 to l, -0 to l Ux derivative of cos n pi x over l will gives –n pi over l sin n pi x over l dx.

Then let us put the limits, so 2 over l Ux at x = l * cos n pi – Ux at x = 0, cos 0 is 1 and then this is +n pi over l 0 to 1 Ux sin n pi x over l dx. Now, this integral, okay 0 to 1 Ux sin n pi x over l, let us differentiate by parts separately okay, so 0 to l Ux sin n pi x over l, when we integrate by parts we get Ux t sin n pi x over l 0 to l, -0 to l U xt, derivative of sin n pi x over l gives n pi over l cos n pi x over l dx, okay.

Now, you put x as l here, U is finite, so sin n pi = 0, gives 0 value and when you put x = 0, again sin 0 is 0 and Uxt is finite, so we get 0, so this part becomes 0 and what we get here; we get – n pi over l 0 to l Ux t cos n pi x over l which can be related to the finite cosine transform of U, okay, so what you get here; this is 2 over l Ux at x = l cos n pi is -1 to the power n - Ux at x = 0 and n pi over l here we have and from here, what we get; - n pi over l, okay.

And 0 to l U xt, okay cos n pi x over l dx, okay, so and we know that from here, okay, 2 over l, 0 to l fx cos n pi x over l is dx = Cn, okay, so 0 to l U xt cos n pi x over l dx = l/2, so this l/2 we shall put for l/2 C uxx, okay, yeah, yeah, so we will write l/2 C Uxx, no, Cu, sorry, l/2 Cu okay, so we will get that so, this U put here, for this you write l/2 Cu, okay, so let us see what we get then.

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Properties of the transforms cont...

$$= -\frac{n\pi}{l}\int_0^l u(x,t)\cos\frac{n\pi x}{l}\,dx$$

Hence,

$$C[u_{xx}] = \left[\frac{2}{l}\left\{\left(\frac{\partial u}{\partial x}\right)_{x=l}(-1)^n - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right\} - \frac{n^2\pi^2}{l^2}C[u]\right]$$

$$= \left(-\frac{n\pi}{l}\right)^2 C[u] - \frac{2}{l}\left[u_x(0,t) + (-1)^{n+1}u_x(l,t)\right].$$

$$-\left(\frac{n\pi}{l}\right)^2 C[u] - \frac{2}{l}\left[u_x(0,t) + (-1)^{n+1}u_x(l,t)\right]$$

So, 2 over l, okay, we have 2 over l Ux at x = l -1 to the power n, in minus, yeah, so 2 over lis multiply to this as well as to this, so 2 over l –Ux at x = 0, so this is what we have okay and then – n square pi square over l square, yeah, okay, so this 2 over l, when you combine with 0 to l, this 2 over l, when you combine with 2 to l, U xt cos n pi x over l gives you this one, Cu, okay, so what you get is this.

So, -n square pi square is square Cu, okay, so this I can write – n pi over l whole square – n pi, this – should have been here, this is actually – n pi over l whole square, okay, yeah, - n pi over l whole square Cu – 2 over l, yeah, -2 over l, right, we have, we can – we have taken common, so -2 over l and then U x0 t, okay, -1 to the power n, yeah, -1 to the power n +1, yeah, okay, ux, okay.

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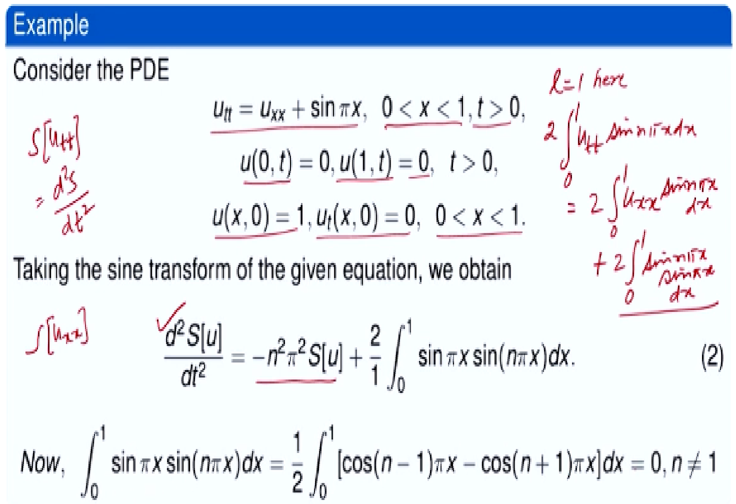

So, Cu xx is actually = this, okay, now consider the PDE, okay, second order PDE, so U tt +; Utt= U xx + sin pi x, x varies from 0 to 1, t is > 0, so we have a finite length, okay, x varies from 0 to 1, we are given boundary conditions x = 0, at x = 0, u 0, t is, u xt is 0, at x = 1, u xt is 0, so we are given the digital boundary conditions, okay and when t = 0, ux 0 is 1 and the derivative of u; partial derivative of u with respect to t, okay, ut at t = 0 is given to be 0, okay, for all x between 0 and 1.

So, what we do; let us take the finite transform of the given PDE, okay, taking finite transform of the given PDE means, you multiply the PDE by sin n pi x over l and then integrate with respect to x over the interval 0 to l and multiply by 2 over l, here we have 0 < x< 1 that means l = 1 here, okay, so we multiply both sides by sin n pi x and then integrate with respect to x and multiply by 2 over l that is 2 over 1, so 2, okay.

So, 2 times 0 to 1 U tt sin n pi x dx, okay = 2 times 0 to 1 U xx sin n pi x dx + 2 times 0 to 1 sin n pi x * sin x sin pi x * dx, okay, we multiply both sides by sin n pi x, integrate with respect to x over the interval 0 to 1 and multiply by 2 pi, so that will give us the Fourier's sine transform of the given equation, okay. Now, we know that S of Utt = d square S over dt square, we have seen it, okay.

So, left side becomes d square Su over dt square and then S of U xx, okay, S of U xx we have already found, okay, let us see what is that. So, this is S of Uxx, okay, S of Uxx = this one, U0 t is given to be 0, U1 t is given to be 0, okay, l = 1, okay, so this is 0, this is 0 and l = 1 means we get n square pi square Su, okay, so we get here, - n square pi square Su, okay and then this is the last term, 2 times 0 to 1 sin pi x sin n pi x dx.

Now, we know the property of sin pi x; sin pi n x, okay, they are mutually orthogonal functions, okay, when m and n are not same, their integral 0 to 1 will be 0, okay, so here you can see 0 to 1 sin pi x * sin n pi x dx, this is 0, whenever n is != 1, okay and when n = 1, we will get 0 to 1 sin square pi x dx that value we can calculate separately, okay, so this is 0, this integral is 0, when n is != 1.

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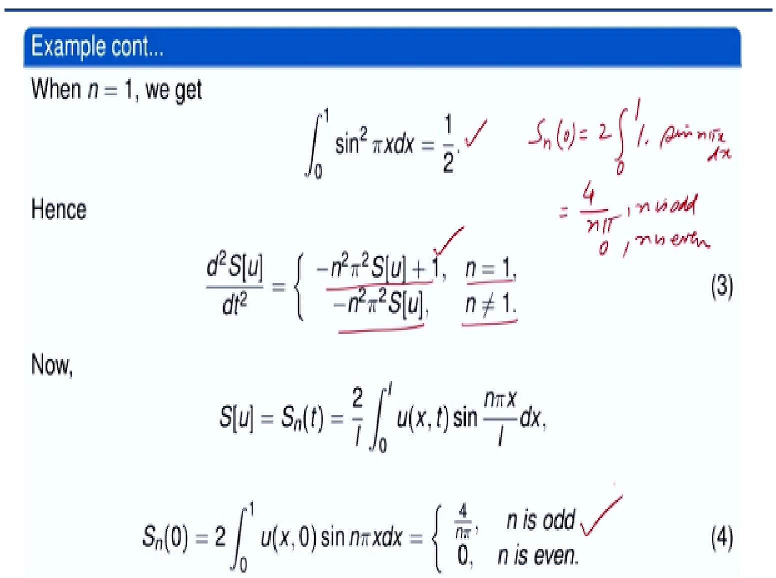

And when n = 1, we get 0 to 1, sin square pi x dx, we can easily integrate sin square pi x, put the limits and see that it is 1/2 okay, so what do we notice now; we noticed that this quantity becomes 0 when is != 1, okay and when n = 1, this becomes 1/2 okay, so we have 2 cases, this equation for n = 1 and this equation for n ! = 1, okay. So, let us first, so let us see, d square Su over dt square = - pi square Su, when n = 1, okay d square Su over dt square will be - pi square Su + 2 times 1 over 2.

So, we will get d square Su over dt square = - pi square + 1, okay, so – pi square Su +1, okay, so this is for n = 1, okay and then for n!= 1, this part becomes 0, so we get d square Su over dt square = - n square pi square Su, okay, so this is the value when n ! =1, this value we have when n =1, okay. Now, by our definition of sin transform, pi Su is nothing but Sn t and this is 2 over l, 0 to l U xt sin n pi x over l dx, okay.

So, from here you can see, if put you t = 0, okay if you put t =0, what you get; l = 1 in our case, so 2 times 0 to 1 Ux0 sin n pi x dx, okay and what we are given Ux0 as Ux0 = 1, okay, so what we have; Sn0 = 2 times 0 to 1, okay, Ux0 is 1, sin n pi x and this we have already found earlier, here this one, 2 times 0 to 1 sin n pi x over dx, so this is 4 over n pi, when n is odd and 0 when n is even, okay, so this is 4 over n pi, when n is an odd integer and 0 when n is even, okay.

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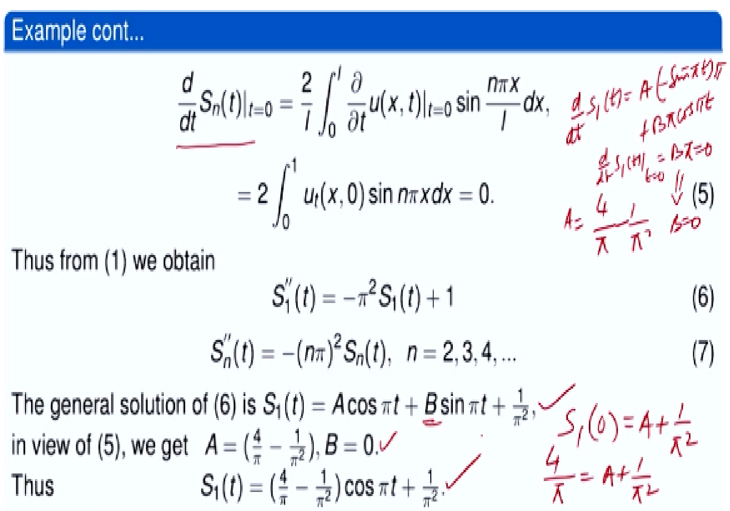

So, this value we know right, now d over dt of Sn t, when you differentiate Sn t with respect to t, since t and x are independent, we have 2 over 0 to l Ut xt, okay U txt and then you put t =0, okay, so d over dt Sn t at t =0 is 2 over l Ut xt at t =0 * sin n pi x over l dx and U txt let us see, U txt is given to be 0, so for all t, so, no U txt is given to be 0 at t =0, so we have this, this quantity becomes 0.

So, we get d over dt of S nt at t = 0 as 0, now then what do we get; okay, this double dash means, this dash actually denotes the derivative with respect to t, now we have ordinary differential equations, this ordinary differential equation of second order, d square S over dt square = -n square pi square Su + 1 or you have d square S over dt square = -n square pi square * Su, okay they are second order differential equations, okay.

So, what we have; S1 double dash t = -pi square S1 t + 1, this is for the case n = 1, okay, so we have this and the for n = 2, 3, 4 and so on, when n is ! = 1, we have S double dash = -n square pi square Su, okay, so S double dash = -n square pi square Sn t okay. Now, general solution of this equation, this if you want to simplify, let us find the general solution of this.

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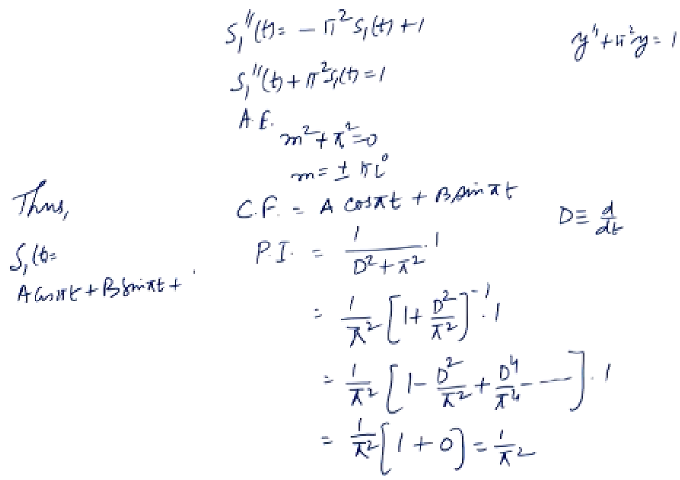

We have S1 double dash okay t = -pi square S1 t +1, okay, so I can write it as S1 double dash t + pi square S1 t = 1, okay, so this is of the form y double dash + pi square * y = 1, okay and we know how to solve this second order ODE with constant coefficients, okay, we write auxiliary equations, auxiliary equation is m square + pi square = 0, okay, so m = +- pi, okay, we have complex roots, okay.

So, complementary function is A times cos pi x + B sin pi x, okay, particular integral let us find, particular integral we can find by 1 over D square + pi square operating on 1, D is d over dt, so I can write it as 1 over pi square times 1 + D square over pi square -1 operating on 1, now this is 1 over pi square, we can write the infinite series expansion of this, d square over pi square + D 4 over pi 4 and so on, okay operating on 1.

And these derivative with respect to t, okay, so 1 over pi square, 1 * 1 is 1, all of these derivative; these second order derivative is fourth other derivatives, so they are all zeros, okay, so we have 1 over pi square, so thus S1 t, okay, the general solution S1 t will be = A cos pi t, this will be t not x because we have derivative with respect to t, okay, so A cos pi t + B sin pi t +1 over pi square, okay, so we have this.

Sn t =; S1 t = A cos pi t + B sin pi t + 1/ pi square, now here let us see, when you put t =0, okay, when you put t =0 S10 is how much; A + 1 upon pi square, S 10 we found earlier, yeah from here S 10, n = 1, okay, S10 is 4 over pi, okay, so this 4 over pi, S10 is 4 over pi = A + 1 upon pi square, so A = 4 over pi -1 over pi square, okay and so we have found the value of A, to obtain the value of B, let us differentiate this equation with respect to t.

So, d over dt of S1 t =; A is A times cos pi t, so A * -sin pi t * pi, okay + B pi cos pi t, okay, so when you put t = 0, what we get; d over dt of S1 t at t= 0 gives you this is 0 4B pi but d over dt of Sn t at t= 0 is 0, this is valid for all n, okay, so d over dt of Sn t at t= 0 is 0, so we get this, so with these gives you B = 0, okay, so B is 0, okay, A is 4 over pi – 1 over pi square, so we get the value of S1 t; S1 t is 4 over pi – 1 over pi square cos pi t + 1 over pi square, okay.

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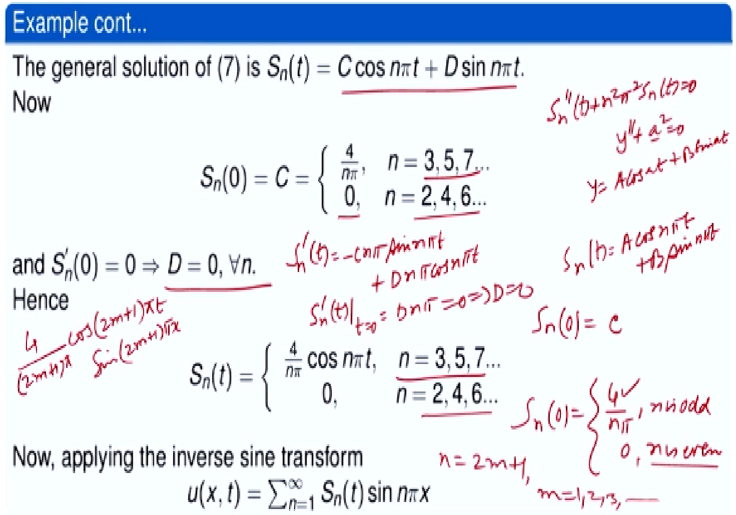

Now, let us find the general solution of the partial differential equation for the other values of n, n = 1 we have solved, let us find the value for a solution for other values of n, so Sn double dash t = - n square pi square Sn t, okay, so we have Sn double dash t + n square pi square Sn t =0 , so this is of the form y double dash + A square = 0, okay, so we know that the general solution is A cos at + B sin at, okay in this case because right side is 0, so particular integral is 0, okay, so here Snt will be = A cos; A = n pi, okay, so n pi t + B sin n pi t, okay, this.

Now, let us see we have Sn0 =; if you put n0 here, Sn0 = c, okay, this is 1, so C and this is 0, Sn 0 = c and c =Sn0 we have already found, Sn0 = 4 over n pi when n is odd, okay and 0 whe n is even, okay, so Sn0 =; let us recall for 4 over n pi when n is odd and 0 when n is even, okay, n = 1 case we have already considered, so when n takes value 3, 5, 7, Sn0 is 4 over n pi and for n even that is 2, 4, 6 Sn0 is 0, okay.

So, this value is known to us, so this gives you the value of c, c is 4 over n pi for n = 3, 5, 7; 0 for n = 2, 4, 6, Sn dash t will be = what; - c pi sin n pi t, n also will come, so this will be – cn pi; cn pi sin n pi t + Dn pi cos n pi t, now t = 0, so Sn dash t, t = 0 = dn pi and Sn dash t at t= 0 is 0, so we get d = 0, okay, so d = 0 for all n, all right. Now, Sn t = then so put the value of c 4 over n pi cos n pi t d0 and this is the value, when n is 3, 5, 7 and 0 when n is 2, 4, 6, okay.

Now, apply the inverse sine transform, okay, when you apply the inverse sine transform, you get U xt =; okay, let us see that one, inverse sine transform, this is inverse sine transform, okay, fx = sigma n = 1 to infinity Sn sin n pi x over l, l = 1 here okay, fx is U xt, so we have U xt = sigma n = 1 to infinity Snt sin n pi x over l, Sn t is 4 over n pi cos n pi t, we will write the value of Sn t for n-1 separately, okay because that is separate case, this one, okay.

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Example cont...

$$u(x,t) = \left[\left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos \pi t + \frac{1}{\pi^2}\right]\sin \pi x$$
$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos\{(2n+1)\pi t\}\sin\{(2n+1)\pi x\}.$$

So, let us write that so, ux t =; S1 t is this one, 4 over pi – 1 over pi square cos pi t + 1 over pi square, okay, this one, okay, this is S1 t * sin pi x and then we write for odd values of n, okay, this one, for n =3, 5, 7, we put n = 2m +1, so then m will take values of 1, 2, 3, 4, 5, 6 and so on, okay, so this will become 4 over 2m + 1 8 pi cos 2m + 1 pi t, okay and we multiply by sin n pi x, so sin 2m + 1 pi x, okay.

And summation will run over m from 1, 2, 3, 4 and so on up to infinity, so we get sigma, 4 over pi we can write outside, okay 4 over pi and then sigma n =; so in place of m, now we have written n, so 1 over 2 n +1 cos 2n +1 pi t sin 2n + 1 pi x, okay, so this is how we solve the second order PDE by finite sine transform, okay.

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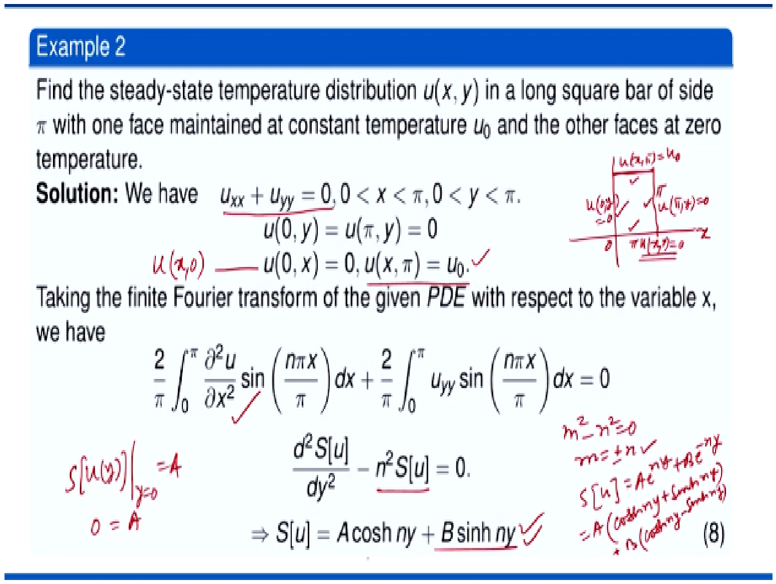

Now, let us look at one more problem; find the steady state temperature distribution u xy in a long square bar of side pi, okay, so we have a long square bar of side pi, okay this is side pi, okay, long square bar of side pi with one face maintained at constant temperature u0, u xy = u0, we have taken this, this face, this is face u xy okay, this face u xy is taken as u0, while other faces, this is u0 y, okay, u0 y is taken 0, u pi y is taken 0 here.

Here u x0 = 0, okay, so this side, this side, this side, these 3 sides are taken at 0 temperature, this 4 side we take at temperature u0, okay, so one face maintained at temperature u0, other faces are taken as temperature 0, so we have this, in the steady state in 2 dimensions, we have this Laplace equation, so u xx + u yy = 0, x varies from 0 to pi, y varies from 0 to pi. When x is 0; x is 0 means this y axis, okay.

So, u0 y = 0, when x is pi, this face okay, parallel to y axis, so this is 0 and then u0 x that is the segment on x axis, so this u0 x = 0 and then this face is taken a temperature u0. Now, taking the finite sine transform of the given equation because here also we have Dirichlet boundary conditions, so we multiply the given equation by sin n pi x over l, we take as y as we are treating as t, okay here.

So, we multiply the given equation by sin n pi x over l; l is = pi here, so sin n pi x over pi and integrate with respect to x and multiply by 2 over l that is 2 over pi, so 2 over pi 0 to pi, u xx sin n pi x over pi dx, then 2 over pi, 0 to pi u yy sin n pi x over pi dx = right side is 0. Now, this is what since we have this Su xx, okay, Su xx, let us look at the formula for that as uxx, so this is u xx, okay.

We are treating that y as t, this t here is treated as y there, so at x =0, temperature is 0, at x = pi, temperature is again 0, so this is 0, this is 0, so we have; so this means that Su xx = -n square pi square over pi square okay, l is pi, so this is –n square Su there, so we get –n square Su this one, okay, corresponding to this term and here this derivative with respect to y; second order derivative with respect to y, okay can be written outside the integral.

Because xn yn are independent variable and then we have d square Su over dy square because as I said, we are treating this y as t, okay, so d square over dy square Su – n square Su = 0 and this is equation of the ordinary differential equation of the second order with constant coefficients, so auxiliary equation is m square – n square = 0, so m = +- n, so we have 2 distinct real roots and therefore, Su will be = A cos hyperbolic ny + B sin hyperbolic ny.

Actually, we also write Su as Ae to the power ny + Be to the power - ny but we can convert it to cos hyperbolic sin hyperbolic functions writing as A times cos e to the power ny is cos hyperbolic ny + sin hyperbolic ny and B times e to the power –ny as cos hyperbolic ny – sin hyperbolic ny, so collecting the coefficients of cos hyperbolic ny and sin hyperbolic ny and using new arbitrary consonant, we can write Su as this, okay.

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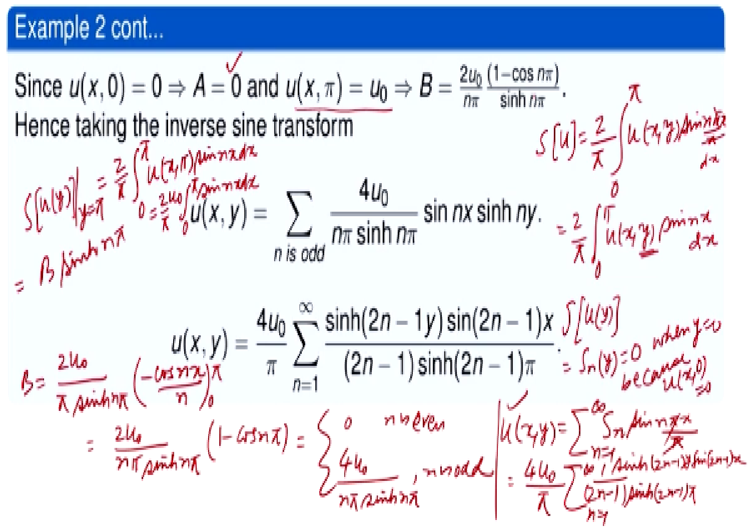

Now, since we are given, since here Su is A cos hyperbolic ny, we are giving uy = 0, okay, we are given ux 0, then y is 0, okay, when y is 0, ux 0, so this is x = 0, this is x = pi, this is; this will be ux0, okay, so ux0 = 0, okay. Now, what we have; Su of u, S of u = 2 over pi 0 to pi u xy sin n pi x over pi dx, okay, this pi is cancel with this pi, we have2 over pi 0 to pi u xy sin nx dx, okay. Now, put y = 0 here, okay, so then this is a function of y, okay.

Su is a function of y, so actually we have S of uy there, so S of uy actually we have, you can also write it as Sn y, so this = Sny, when you take y = 0, what we get; Sn y = 0, when y = 0 because u x0 = 0, when you put y =0 here, we are given that u x0 is 0, so Su y that is Sn y is 0 okay, so now this value put here, Su y, okay, this you put y= 0 here, sin hyperbolic ny is 0, so we get n cos hyperbolic ny is 1, so A = Su y at y = 0 which is Sn0, okay.

So, Su y at y = 0 = A, okay but this is 0, so we get A = 0, okay. Now we have the other boundary condition, y = pi, when y= pi for all x from 0 to pi, u x pi is u0, okay, so this will give us the value of B, okay, so here in this you put y as now pi, okay, A is 0, so S pi; S u y at y = pi, this given as u0 and this is = 2 over pi, okay 0 to; okay, no not this, we will have right side, okay, y = pi, so B sin hyperbolic n pi.

So, B = u0 over; here sin Su y at y = pi is not u0, actually it is 2 over pi 0 to pi okay, the here we have Su pi, sin n pi x over pi, so sin nx dx, okay, so this is actually u0, so 2 over pi u0 and we have 0 to pi sin n pi nx dx that we have to evaluate, so B is 2 over pi 2 u0 upon pi times sin hyperbolic n pi and then we integrate sin nx, so –cos nx divided by n, 0 to pi, okay, so this will give you 2y0 upon n pi sin hyperbolic n pi 1 – cos n pi, okay.

So, this will be 0, when n is even, okay because cos n pi will be 1 and this will be 4u0 upon n pi sin hyperbolic n pi, when n is even; odd, okay because cos n p pi will be -1, so you replace n by 2n – 1, okay, then n will run from 1 to infinity, so u xy will be =; we take inverse finite transform, okay to write u xy, uxy = sigma n = 1 to infinity Sn sin n pi x over l dx, okay u xt, u xy, sorry uxy = sigma n =1 to infinity Sn sin n pi x over pi d; n pi x over pi, so sin nx, okay.

Sn is this, Sn is this value okay, so Sn is B cos sin hyperbolic n pi and B we have found here, so this 4u0 divided by pi, okay, summation n is replaced by 2n – 1, so that n runs from 1 to infinity, so 1 over 2n – 1, oaky sin hyperbolic 2n – 1 pi and then we have sin hyperbolic 2n -1 y * 2n – 1 x, okay, oh here, so we have found value of B and then from B, we find the value of Sn, okay, Sn is found from here after putting the values of p.

Once we have Sn here, we can then write inverse finite sine transform by uxy = sigma n = 1 to infinity Sn nx; sin nx and Su y as I said Su y is integral 2 over pi 0 to pi u xy u x pi sin nx dx, okay that is Sn y, okay, this Sny okay. So, from here we get the value of Sn u; Sn u is Sn y, when we want to calculate Sny, okay in order to get the value of B, we get the value of B is found from here by putting y as pi, okay.

So, this is the value of S, then we get the value of Sn, okay, Sn = this, B= yeah, right, so this is how we find the value of this. Now, this is how we solve the second partial differential equation; Laplace equation in the case of square plate, okay whose one is at u0 temperature, one face is at u0 temperature and the other faces are at 0 temperature, so this is how we solve the equation. Thank you very much for your attention.