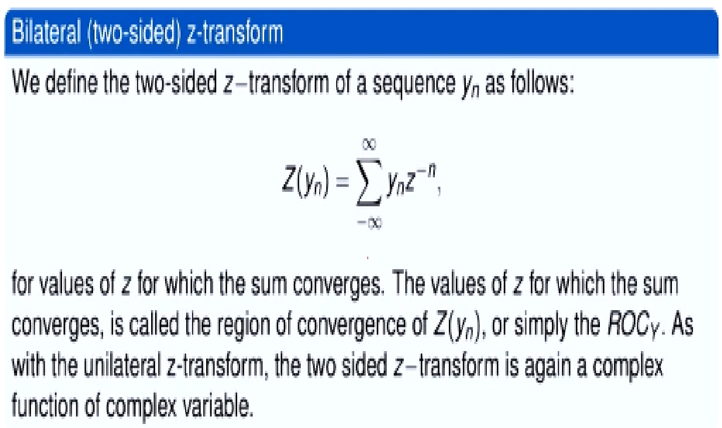
Advanced Engineering Mathematics
Prof. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 36
Review of Bilateral Z-Transforms

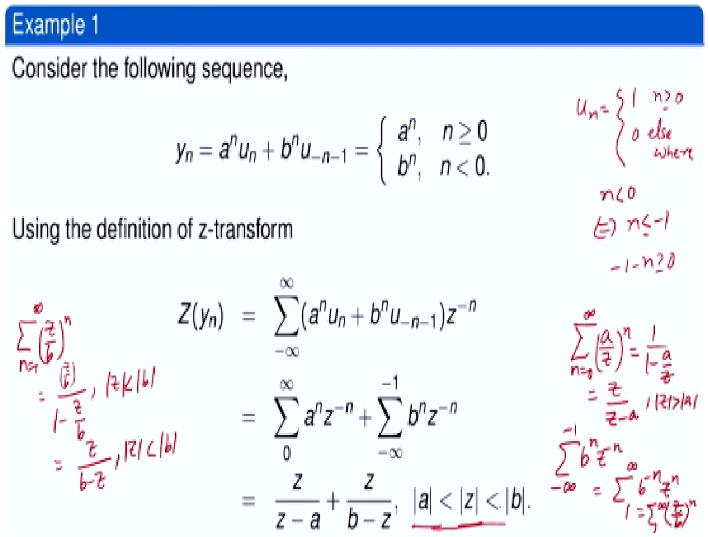
Hello friends, welcome to my lecture on review of bilateral Z transforms, the 2 sided Z transform of a sequence yn is defined as z yn = sigma n = - infinity to infinity z to the power – n.

(Refer Slide Time: 00:45)



And this is defined for those values of z for which these series on the right side is convergent, the values of z for which some on the right side convergence is called the region of convergence of z yn are simply the ROCy; ROCy means region of convergence for yz, this is yz, z of yn is yz, as with the bilateral z transform, the 2 sided z transform is again a complex function of a complex variable, z is the complex variable.

(Refer Slide Time: 01:18)



And when we take the z transform, we get again a complex number, so let us consider the following sequence; yn = a to the power n un + bn u to the; u – n – 1, we know that un is defined as 1, when n is > = 0 and 0 elsewhere, okay, so here when n is > = 0, we get yn as a to the power n and when n is < 0, okay n is < 0, then what will happen; u – n – 1, so n is < 0 means, n is taking values -1, -2 and so on.

N is < 0 means, n is < = -1, so we can say that -1, –n is > = 0, so when - 1-n is > = 0 means n is < = -n, so when n is < = -1, u – n -1 will be = 1 and this is un will be 0, so yn will be b to the power n, okay. So, yn = a to the power n, when n is > = 0, b to the power n, when n is < 0. Now, let us take the z transform of the sequence, yn, so z transform yn is sigma n = - infinity to infinity an un + bn u – n – 1 z to the power –n.

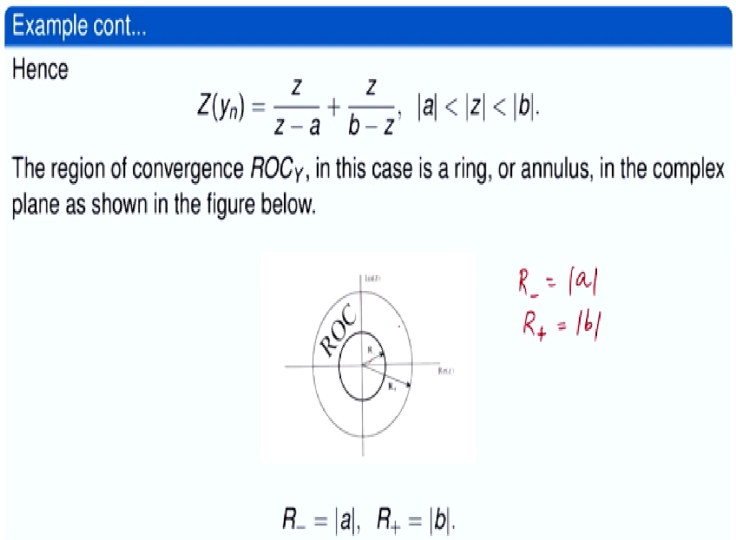
Now, when n is > = 0, un is 1, so this is sigma 0 to infinity an z to the power –n and when n is < = - 1, un is 0, u – n - 1 is 1, so – infinity to – 1 b to the power n z to the power –n. now, we know that sigma n = 0 to infinity to a to the power n z to the power –n, this is the z transform of a to the power n, this is n = 0 to infinity a/z to the power n, so this is = z over z – a because it is a geometric series 1 over 1 – a yz provided mod of z is > mod of a.

Here, we have sigma n = - infinity to – 1 bn z to the power –n, okay, we can also express this as sigma 1 to infinity replacing n by –n, b to the power –n z to the power n, okay, so that this is

=sigma 1 to infinity z/b to the power n, okay, these again a geometric series and will be convergent if mod of z over b is < 1 okay, sigma n = 1 to infinity z over b raised to the power n is z over b over 1 – z over b provided mod of z is < mod of b.

So, this is z over b – z, okay, so if mod of a is < mod of z, then, we have z over z – a, if mod of z is < mod of b, then we have the some of the other series z over b – z, so both these series are convergent in the annular region given by mod of z > mod of a but < mod of b.

(Refer Slide Time: 04:57)



Now, we can say that z transform of the yn sequence is z over z – a + z over b – z, where mod of a is < mod of z < mod of b, the region of convergence of the sequence yn of the z transform of yn is given by ROCy, then ROCy here is a ring or we can call it an annular region, in the complex plane as shown here, you see we have mod of a, mod of z, so this R – is mod of a and R+ is mod of b, okay.

So, taking R – n mod of a, R + n md of b, we see that region of convergence lies between 2 concentric circles, one is mod of z = mod of a and the other one is mod of z = mod of b.

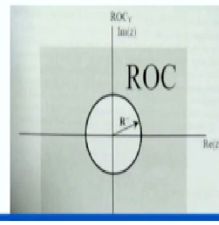
(Refer Slide Time: 05:49)

Example cont...

R_- is determined by y_n, $n \geq 0$ and R_+ is determined by y_n, $n < 0$. If $y_n = 0$ for n<0, then we have

$$Z(y_n) = \sum_{n=0}^{\infty} y_n z^{-n},$$

and $R_+ = \infty$, which is essentially one sided (unilateral) z−transform. Consequently, the region of convergence corresponds to $|z| > R_-$ as shown in the figure below:



$|z| > |a|$

Now, R – 1; R – you see we have seen that R - is determined by yn when n is > = 0, okay, this is R-, okay, R - is determined by this sequence okay, this series, okay, R + is determined by this one, okay, so R – is determined by yn when n is > = 0, R + is determined by yn when n is < 0. If yn = 0 for n < 0, suppose we take yn = 0 for n < 0, then what will happen; yn = 0 for n < 0, we take that means we will only have yn = a to the power n, when n is > 0.

And then the region of convergence will be mod of z > mod of a because this part will vanish, okay, so we will have when yn is 0 for n < 0, then we have z yn = this, okay and R + is infinity, so we have the region of convergence as mod z > mod of a that is outside of the circular disc mod of z = mod of < = mod of a, this is the circular disc which centre at origin radius mod of a, so the region outside the circular disc will be the region of convergence.

And which is actually the one sided z transform, now conjugately the region of convergence corresponds to mod of z > R- as shown in this figure.

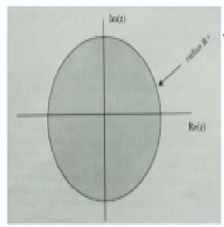
(Refer Slide Time: 07:24)

Example cont...

If $y_n = 0$ for $n > 0$, then we have that

$$Z(y_n) = \sum_{n=-\infty}^{0} y_n z^{-n},$$

and $R_- = 0$ which implies that the region of convergence corresponds to a solid disk in the complex plane i.e. we have $|z| < R$ as in figure.



$|z| < R_+ = 16$

Now, if yn = 0 for n > 0, okay, then we will have z yn = - infinity to -1 or you can say – infinity to 0 yn z to the power –n, okay and R – then will be 0, okay, R - will be = 0, so this will be this one, this region, okay, the region is mod of z < b, okay mod of z < b means circular disc, so when yn is 0, for n > 0, then we have z yn = sigma n = - infinity to 0 yn z to the power - n and the region of convergence will be mod of z < mod of b that is the circular disc, this circular disc, okay.

Region of convergence corresponds to a solid disk in the complex plane that is we have mod of z < R+, okay, R+ is mod of b.

(Refer Slide Time: 08:28)

Thus the two-sided $z-$transform can accommodate a wider range of signal behaviors, since they can be left-sided, right-sided, or two-sided and still have a bilateral $z-$transform.

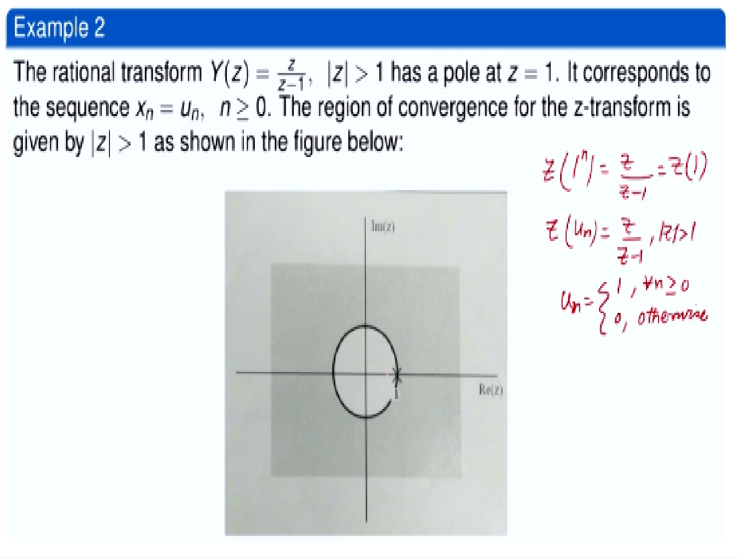
Poles and zeros

Let the z-transform of a sequence $\{x_n\}_{n=-\infty}^{\infty}$ be given by $X(z) = \frac{B(z)}{A(z)}$, $z \in ROC_X$. Then the zeros and poles of $X(z)$ are given by the sets $\{z : B(z) = 0\}$ and $\{z : A(z) = 0\}$ respectively. In the case of a rational z-transform, the ROC is always bounded by poles. Therefore, ROC is either a disc, an annulus or the entire plane minus a disc with the possible exclusion of zero and infinity.

Now, thus the two sided z transform can accommodate a wider range of signal behaviours, since they can be left sided, right sided or two sided and still have a bilateral z transform. Now, let us say the z transform of a sequence xn n = - infinity to infinity be given by Xz = Bz over Az, where z belongs to ROCx, region of convergence of Xz, then the zeros and poles of Xz are given by the sets, zeros of Xz; this Xz are given by the set of all z such that Bz = 0.

And the pole sof the Xz are given the set z, set of all z such that Az = 0, in the case of a rational z transform, the ROC is always bounded by poles, okay, therefore, ROC is either a disk or an annulus or the entire plane minis a disc with the possible exclusion of zero and infinity.

(Refer Slide Time: 09:25)

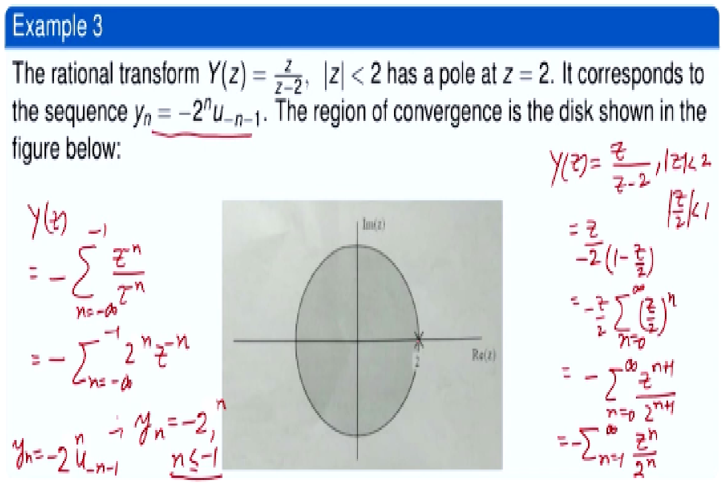


Now, let us consider the yz = z over z – 1, so you can see yz over z; yz = z over z – 1 has a pole at z = 1 and the region of convergence is mod of z > 1, it corresponds to the sequence you know that z over z -1 is the z transform of un sequence that is 1 to the power n, z transform of 1 to the power n is z over z -1 okay and 1 to the power n is also written as un because un = 1 for all n > = 0 and 0 otherwise, okay.

So, z transform 1 to the power n are also we write z transform of 1, okay, so this is the z transform of the sequence un and > = 0, the region of convergence for the z transform are given by mod of z > 1, okay, the region outside this circle; mod z = 1 and you can see this region is bounded by this circle which has got a pole, with the pole at z = 1, okay. So, what we have said here was the region of convergence is either a disk or an annulus or the entire plane minus a disc.

So, here the region of convergence is the entire plane minus this disc, mod of z < = 1 and you have; we have also seen that region of convergence bounded by poles you can see the pole is occurring here on the boundary of the region of convergence.

(Refer Slide Time: 11:07)



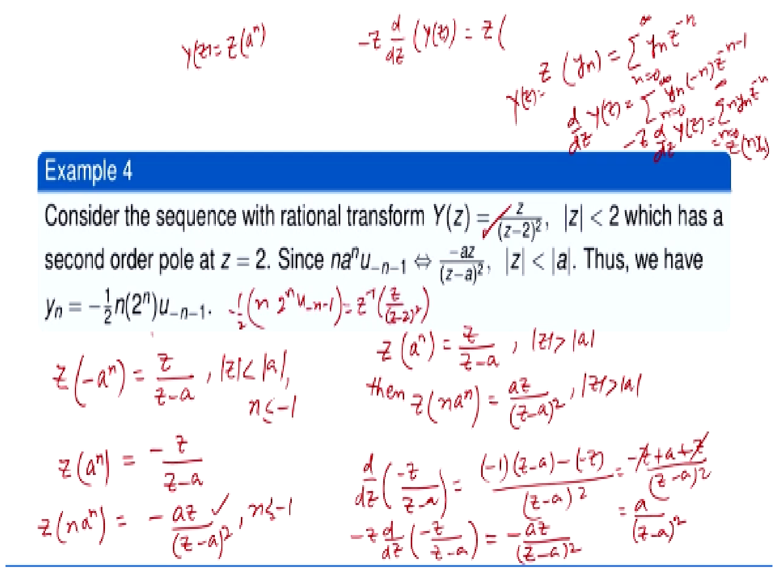
Now, rational transform yz = z over z -2, when mod of z is < 2, okay, let us take this one, so here we have a pole at z = 2, you can see yz = z over z -2 and you can see here mod of z is < 2, it means that the region of convergence is a solid disc, okay, region of convergence is a solid disc means this case will occur when your yn is 0 for n > 0, okay so, let us say, what is the sequence yn in this case?

We can write it as; we have mod of z over 2 < 1, okay, so this means that I will write it as z over -2 times 1 – z over 2, okay, then this will be - z upon 2, sigma n = 0 to infinity – z/2, so we get z/2 raised to the power n, okay, so this will be = - sigma n = 0 to infinity z to the power n +1 divided by 2 to the power n +1, okay, so this we can say, this is nothing but sigma, - sigma n = 1 to infinity z to the power n divided by 2 to the power n.

Or I can write it as yz = - sigma n = - infinity to -1 z to the power, replacing n by –n, okay, z to the power – n 2 to the power - n and we get - sigma n = - infinity to -1 2 to the power n z to the power – n, okay, so yn is you see -2 raised to the power n when n is < = -1, okay and this is; this can also be expressed as n < = -1 can be replaced by this expression yn = -2 raised to the power n u – n -1, okay.

When you have u –n -1, it will have value 1, when n is < = -1, so this condition on n that n is < or = -1 can be replaced by taking u - n -1 together with -2 raised to the power – n, so this is the sequence here, region of convergence you can see is the disc; this disc, mod z < 2 and its boundary has a pole at z =2.

(Refer Slide Time: 14:03)



Now, consider the sequence with rational transform yz = z over z – 2 whole square, when mod of z is < 2, we can see that it has a pole of order 2 at z = 2, okay and we know that z transform of a to the power n, az over z – a, okay. In case mod of z is > mod of a, okay and then we know that z transform of n a to the power n = az divided by z – a whole square, when mod of z is > mod of a, okay.

So, here again we can easily show that z transform of; we have seen earlier, we have seen that z transform of the sequence yn, okay which is -2 raised to the power – n, okay is z over z -2, okay, when n is < = -1, this can be; here 2 can be replaced by a, so we can easily see that z transform of the – a raised to the power n is z over z – a, when mod of z is < mod of a okay, in a similar manner as we have done in the previous case, okay.

So, by following same steps, we can see that z transform of –a to the power n is z over z – a and n is < = -1, okay, so then z transform of n * a to the power n, this is -az divided by z – a, okay, so z transform of n a to the power n is again can be obtained by differentiating this and we get z transform of n a to the power, n =; this is z transform of a to the power n, z transform of a to the power n, we can write as – z over z – a.

And then z transform of n a to the power n = - az over z – a whole square, when n is < = - 1, it can be again found by the; using the recurrence relation as we have used earlier, we know the difference relation, if d over dz, yeah, if z ; -z times d over dz of yz =; suppose yz is the z transform of a to be power n, okay, then z transform of suppose, we know the z transform of yn; z transform of yn we know.

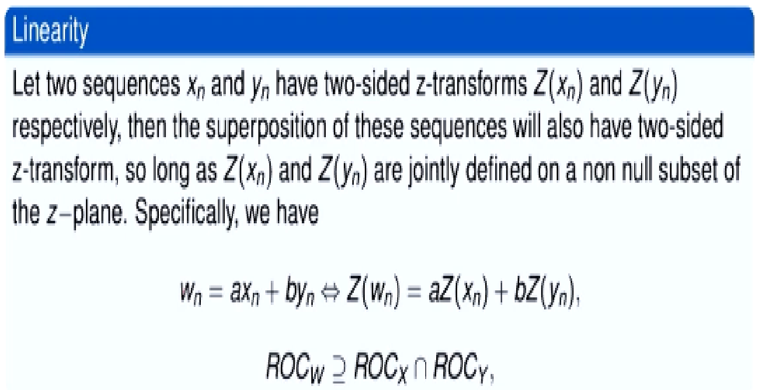
Suppose, z transform yn is sigma n = 0 to infinity yn z to the power – n, then z transform d over dz of; let us say this is yz, this is yz, then d over dz of yz = sigma n = 0 to infinity yn * - n z to the power –n – 1, okay, so we multiply by – z, then – z d over dz of yz = sigma n = 0 to infinity n yn * z to the power – n, so we get z transform of n yn, okay, so z transform of n yn; if we know the z transform of yn; z transform of n yn can be obtained by differentiating yz with respect to z.

And then multiplying by – z, so this formula holds for n > or = 0, this formula can also; will also be applicable, when n is < = -1, so here if you have z of a to the power n = -z over z – a, when you multiply a to the power n/n, we differentiate this with respect to z and then multiply by –z, let us see, d over dz of - z divided by z – a, this will give you derivative of numerator is – 1 * z – a, derivative denominator is 1.

So, minus; - z divided by z – a whole square and what we will get; - z + a - + z divided by z - a whole square and we will get a over z – a whole square, okay, so we multiply by – z; -z d over dz of –z divided by z - a and what we get; we get -az divided by z – a whole square, okay, so z, this is z transform of n a to the power n – az upon z – a whole square, so since n a to the power of the n u –n - 1 goes to –az/ z - a whole square, when mod of z < a.

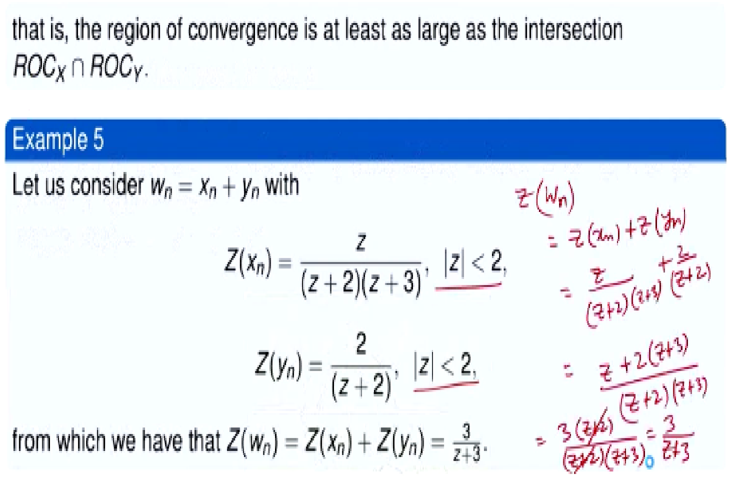
Thus we have yn =; so we can see yn from here, okay, we have z over z – 2 whole square, so replace a by 2 here, okay, so we have n 2 to the power n u - n -1 with minus divided by – 2, okay, so we will have n 2 to the power n u - n -1 divided by -; we can say -1 over 2, okay, this is nothing but the inverse z transform of z over z – 2 whole square, okay.

(Refer Slide Time: 20:50)



So, inverse z transform of this is given by this expression, okay now, let us say 2 sequences; xn yn have 2 sided Z transforms as Z xn and Z yn respectively then the superposition of these sequences will also have 2 sided Z transform, so long as Z xn, Z yn are jointly defined on a null subset of the z plane. If suppose, I write Wn sequence as axn + ybn then Z Wn = aZ Xn + bZ Yn, okay and ROCw will contain ROCx intersection ROCy that means ROCw, the region of convergence of the z transform of the sequence Wn will be at least as large as the intersection of ROCx and ROCy.

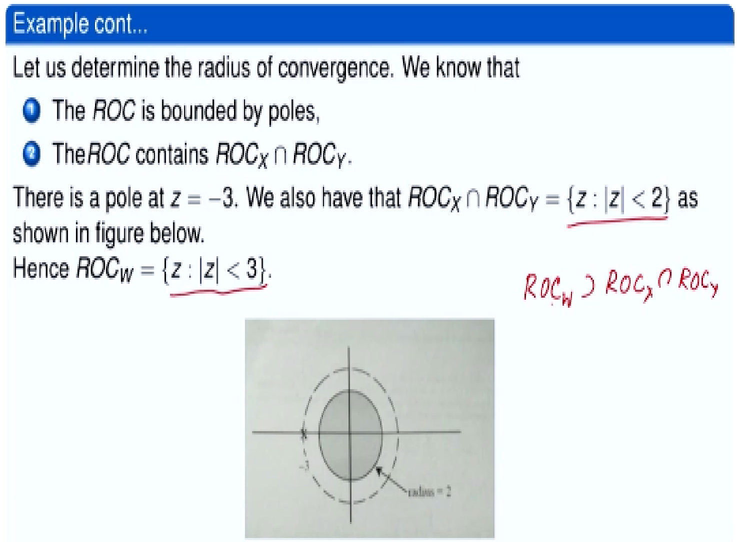
(Refer Slide Time: 21:32)



So, this is the theorem we can easily prove this theorem using the definition of z transform and here let us consider Wn = Xn + Yn, Z Xn suppose is Wz over z + 2z + 3, where mod of z is < 2, Z Yn is 2 over z + 2, where mod of z is < 2, okay, so then form this what do you notice; Wn = Xn + Yn, okay, so Wn =; Z transform of Wn = Z transform of Xn + Z transform of Yn and this will be = z + 2 over; z over z +2, z + 3 + 2 over z + 2, okay.

So, what do we get; z + 2 z + 3 and what we get here; z here and z + 2, 2 times z + 3, so what do you get; 2z + 6 that is 3z + 6, so 3 times z +2 divided by z + 2 z + 3, okay, so this will cancel and we get 3 over z +3, now you can see, the region of convergence here in mod z < 2, here mod z < 2, okay, what is the region of convergence here we will get; because it has a pole at z = -3, okay.

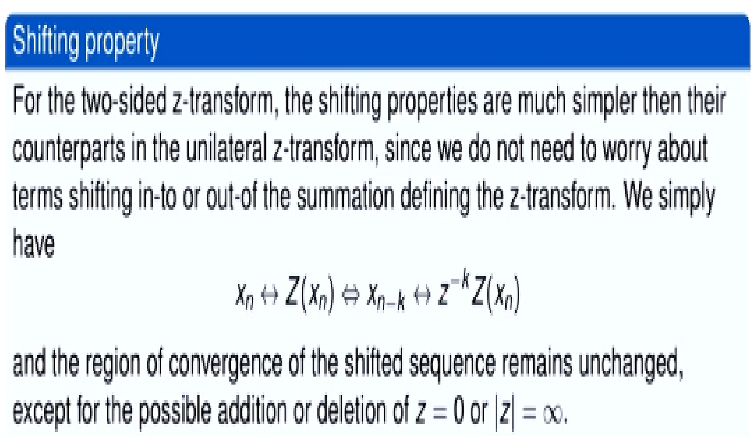
(Refer Slide Time: 23:19)



So, there is a pole at z = -3 ROCx intersection ROCy, okay ROCx intersection ROCy is the same, okay because they are same, if mod of z < 2, so this is z of; all z such that mod of z < 2 and ROCy is what; 3 over z + 3, so this is; because it has a pole, so ROCy will be the boundary will have a; bounded by this pole, z = -3, so this mod of z < 3, okay. So, mod of z < 3, you can see we know that ROCw contains ROCx intersection ROCy, okay.

So, it is mod of z < 3 and mod of z < 3 contains mod of z < 2, okay, so it is at least as big as the intersection of ROCx intersection ROCy.

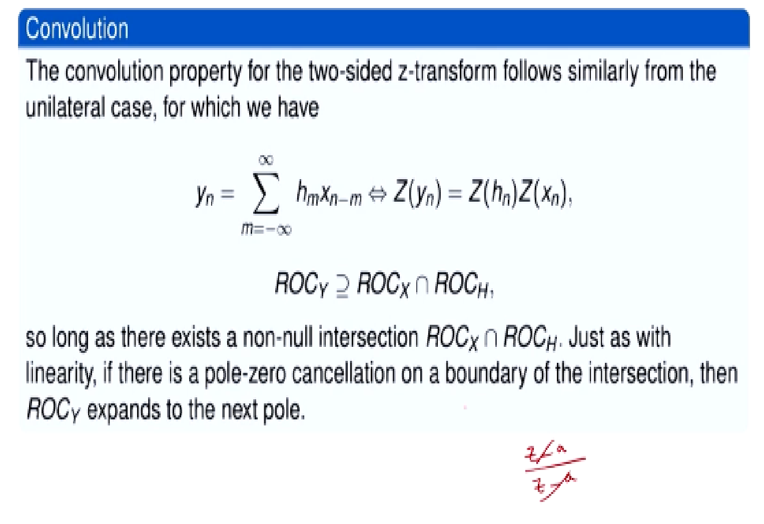
(Refer Slide Time: 24:26)



Now, for the two sided Z transform, the shifting properties are much simpler than their counterparts in the unilateral z transform, since we do not need to worry about the terms shifting into or out of the summation defining with the z transform. Let us say we simply have here Xn goes to Z Xn implied by Xn – k goes to z to the power –k Z Xn and the region of convergence of the shifted sequence remains unchanged except for the possible addition or deletion of z = 0 or mod of z = infinity.

So, if Xn goes to Z Xn, then the shifted sequence Xn – k will have Z transform z to the power -k Z Xn.

(Refer Slide Time: 25:15)

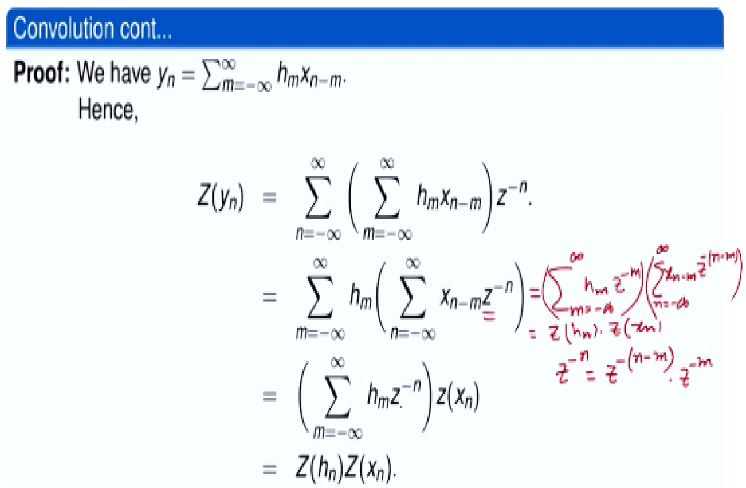


Now, the convolution property for the 2 sided Z transform follows in a similar manner as in the case of unilateral Z transform, we have; if we have 2 sequences; hn and xn then their convolution is defined as yn = sigma m = - infinity to infinity hm xn – m, then their Z transform; Z transform by n is product of the Z transform of hn and xn, okay and the region of convergence of y, okay, yz will contain at least as large as the region of intersection of; region of convergence of x intersection, region of convergence of h, okay.

So, long as there exist a non-null intersection, okay, ROCx intersection ROCh just as with linearity, if there is a pole zero cancellation or on a boundary of the intersection then the ROCy expands to the next pole, suppose it happened that the; we have a pole in the, we have a 0 in the numerator say at z = a, in the denominator also, we have a pole at z = a, so then they will cancel, okay just as in the; if there is a pole zero cancellation on a boundary of the intersection, then ROC expands to the next pole.

Just say, for example here, we have seen that z + 2 got cancelled and we move to the next pole, okay, so if there is a pole zero cancellation on a boundary of the intersection, then ROCy expands to the next pole, here we had at the intersection of ROCx, ROCy was the same said that mod of z < 2 and we cancelled this z + 2 factor here, okay. So, the such a situation if it occurs, okay then, if there is a pole zero cancellation on a boundary of the intersection, then ROCy expands to the next pole as in the previous case.

(Refer Slide Time: 27:17)

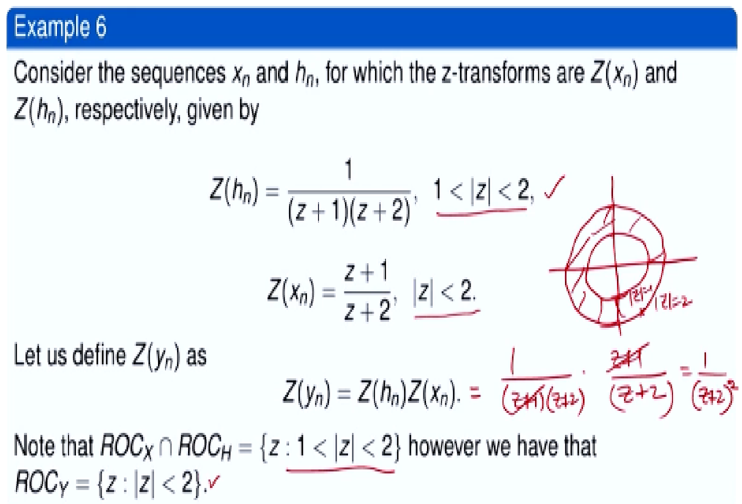


So, we have yn = sigma m = - infinity to infinity hm xn –m, then Z yn, okay, Z transform of Yn by definition of bilateral Z transform and sigma n = - infinity to infinity okay, this yn Z to the power - n and this can be then written as sigma m = - infinity to infinity hm, sigma n = - infinity to infinity Xm – m z – n, okay and then this is sigma m = - infinity to infinity hm z to the power, you can write it as z to the power - m it should be.

So, this is z to the power –n, can be written as z to the power –n - m and then z to the power – m, okay, so here we will have sigma m = minus; so this will be actually this will be = sigma m = - infinity to infinity hm z to the power - m and then sigma n = - infinity to infinity infinity Xn – m z – n – m, okay, so this becomes here Z Xn, okay and this becomes Z hn, okay. So, Z Hn =; this is = Z Hn * Z Xn.

So, this z to power - n will be written as z to the power –n – m * z to the power - m and we get the result, so this is the convolution theorem for the bilateral Z transform.

(Refer Slide Time: 29:07)



Now, consider the sequences xn and hn for which the z transforms are Z xn, Z hn respectively, Z hn is 1 over z + 1 z +2 and the region of convergence is 1 < mod z < 2, Z xn is z + 1 over z + 2 region of convergence is mod z < 2, let us define yn; Z yn = Z hn * Z xn. Now, ROCx * intersection ROCh, okay is you can see here, this is the annular, okay, this is mod z = 1, this is mod z = 2, okay. Then, here we have annular region, okay, this is annular region this one and here we have mod z < 2.

That means complete disc, so intersection when you take, we will get the annular region, so the ROCx intersection ROCh = this which is non null set and ROCy is what; ROCy is Z hn * Z Xn; Z Hn * Z Xn is how much; 1 over z + 1 * z +2 and we get z + 1 here over z + 2, okay. Now, you can see z = -1 is a 0 of Z yn, okay and z = -1 is also a; is a pole of Z yn and so in the case of the cancellation of zero and pole, okay, we move to the next pole.

So, here we get 1 over z +2 whole square, so we now have a pole of order 2 at z = -2, so we move to this mod of z = 2 and therefore, the region of convergence is z such that mod of z < 2, okay, so this is what had said here, if there is a pole zero cancellation on a boundary of the intersection, okay then ROCy expands to the next pole, okay, so that is what we have to say in this lecture, thank you very much for your attention.