Advanced Engineering Mathematics
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Lecture – 35
Review of Z-Transforms-III

Hello friends, welcome to my third and last lecture on review of Z transforms, in this lecture, we shall discuss the inverse Z transform of some complicated rational functions and also we shall see how we can solve the difference equation where we will have to take the; while taking the inverse Z transform, some complications might arise.

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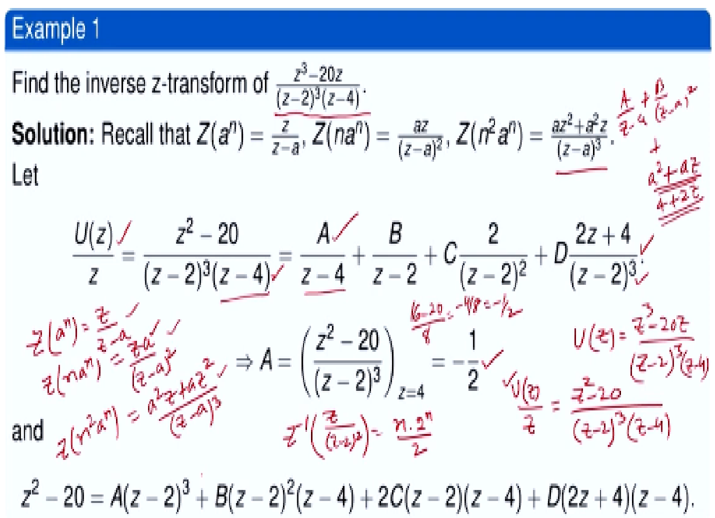

So, let us first discuss the inverse Z transform of the rational function, z cube – 20 z over z – 2 whole cube * z – 4, you can see that this rational function, so this we call as Uz, okay so Uz = z cube – 2z divided by z – 2 raised to the power 3 * z – 4, now you can see that this Uz has got a pole of order 3 at z = 2 and a pole of order 1 at z= 4, okay. So, what we will do while taking the inverse Z transform, the usual technique is that we first consider Uz over z, okay, Uz over z.

And when you consider Uz over z, what is left is z square - 20 divided by z -2 raised to the power 3* z - 4 now, why we do this; because if you straight away you break Uz into partial fraction, then the partial fractions will not come in the standard form. We know that z of a to the power n, okay, z of a to the power n is z over z – a, okay, z of n times a to the power n is z over z

– a whole square * az, z of n a to the power n is az over z – whole square, okay z n square a to the power n.

This is = a square z + az square divided by z – a whole cube, so there are some standard forms of the z transforms of the sequences, a to the power n, n a to the power n, n square a to the power n, so if you straight away you factorise Uz * partial fractions and the partial fractions are of the form say a over z – a + b over z – a whole square and so on, then when we take the inverse Z transform, it will not come in standard form.

We will like to consider the fractions of Uz in such a way that they come in the standard form, so we can apply the standard formulae and you can see that in the z transform of a to the power n or na to the power n, n square a to the power n, z is common in the numerator, z here; here also we have az, here we have, we can take out z, so we will have a square + az, okay, so what we will do; w will consider Uz over z and whatever remaining rational function we have that we bracket into partial fractions.

So, Uz over z is z square – 20 over z - 2 whole cube * z – 4, so this is how we write Uz over z, z square – 20 over z – 2 whole cube z – 4 and this is so corresponding to z – 4 we will write the partial fractions a over z - 4 then z -2 raised to the power 3, we will write as z – 2, one will be the fraction z – 2, the other will be z -2 whole square. Now, here we could write c over z – 2 whole square.

But when we will take the inverse z transform, then this z, okay this z we will multiply on the right side and here we will get z over z – 4 * a, z over z – 2 * b and here we will get c times z over z -2 whole square. When we take the inverse transform of Uz, z over z - 4 will be 4 to the power n, inverse z transform of z over z – 4 will be 4 to the power n, inverse z transform of z over z – 2 will be 2 to the power n.

Here, inverse z transform of z over z – 2 whole square, we cannot write directly; directly if you want to write then z of n a to the power n is az over z – a whole square, so and here we have only z over z – 2 whole square, so we will have to divide na to the power n/ a, okay, inverse z transform of z over z – 2 whole square, this will be na to the power n that means, n2 to the power n divided by 2.

What we do is; in the beginning itself, this is scalar a which is 2 here, we take along with the constant c, okay, so c * 2 over 2 – z whole square; 2 over z – 2 whole square, so when you will multiply this; by this z right side by z, then it will be 2z over z – 2 whole square and so we will be able to write the inverse z transform of 2z over z - 2 whole square directly as n * 2 to the power n.

Now, here corresponding to this z – 2 whole cube, z – 2 whole cube if you compare with this expression, okay or with this expression, there what we have; this z we have already taken, okay one z we have already taken from here, so what is left is az + a square z + z , a square z, one z we have already taken, so that means we have a square + az, okay, a square + az we need here, okay, a square + az we need here, so a = 2.

That means we have 2 to the power, 2 means 4 + 2z, okay, so we need in the numerator 2z + 4, so what we do is; along with the constant D which we need to determine, okay from the identity, we multiply D by 2z + 4 over z – 2 whole, okay why we shall we write Uz, okay, right side will be A times z over z – 4, B times z over z – 2, C times 2z over z – 2 whole square, D times 2z square + 4z divided by z – 2 whole cube.

And we will be able to write inverse of 2z square + 4z over z -2 whole cube directly as n square 2 to the power n, okay, so that is the technique to factorise this Uz, okay, Uz; to factorise Uz, we first consider u z over z and then the partial fractions are written in such a way that we while taking the inverse z transform, we can directly put the values, we can use this standard formulae, okay.

Now, corresponding to this z – 4, okay z – 4 factor, the value of A can be determined directly, you need to consider only z square – 20 divided by z – 2 whole cube, you do not have to consider z -4 in the remaining rational function, you put z = 4, you get the value of A, so z square - 20 divided by z – 2 whole cube, when you put z = 4., gives you 16 – 20; 16 – 20 divided by 4 – 2 that is 2 whole cube, so 8.

So, this is = -4/8 and this is =1/2; -1/2 okay, so this is -1/2, so we get the value of A and for the values of this B, C and D, we write this equation, okay, so we have write the identity, z square – 20, then you take the LCM, A will be multiplied by z – 2 whole cube, B will be multiplied by z – 2 whole square * z – 4, 2C will be multiplied by z – 2 z - 4 and D will be multiplied by 2z + 4 * z – 4. Now, this is an identity, it is valid for all values of z, so to get the values of A, B, C, D, we equate the coefficients of like powers of z on both sides.

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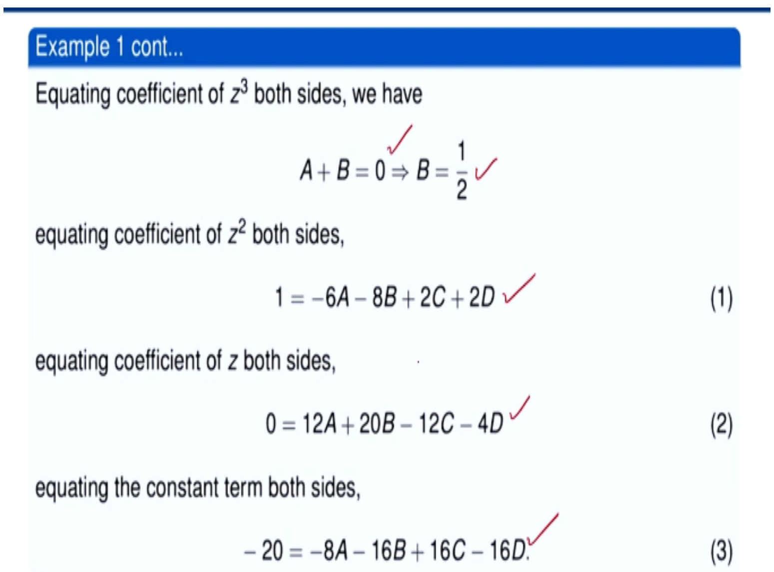

So, first we equate the coefficient of z cube on both sides, so when you see here the coefficient of z cube is A and there is no z cube term here, there is no other z cube term except here, okay the coefficient of z cube is A here, this is; ah, we have coefficient of z cube here also, so coefficient of z cube is A and here we have B, okay, so A + B = 0, okay, so one equation is A + B = 0.

Now, we have A = -1/2, okay A = -1/2, so B = 1/2, now let us equate the coefficient of z square both sides, then when you cut the coefficient of z square both sides, here the coefficient of z square is 1, on the right side, you collect the coefficient of z square, you get -6A - 8B + 2C + D and so we have this equation, if 1 = -6 A – 8B + 2C + D, now next we equate the coefficient of z both sides, there I no z term here, okay, so the coefficient of z is 0.

Here, on the right side, when you collect the coefficient of z, you get t this; 12A + 20 B – 12 C – 4D, so let us write there is no term in z, so we have 0 = 12A + 20B – 12C – 4D and then lastly we equate the constant both sides, on the left side the constant is -20, okay, the constant is -20 and on the right side, the constant is here it is -8A, here it is -16B, here we have 8 * 2; 16C and here we have -16D, okay.

So, we have this equation, -20 * - 8A – 16B + 16C – 16D, now we have the values of A and B and there are 3 equations, we need to determine only 2 constants C and D, so we can determine from them.

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Example cont...

Putting the values of A and B in (1), we get

$$C + D = 1. \quad (4)$$

Putting the values of A and B in (2), we get

$$3C + D = 1. \quad (5)$$

Solving (4) and (5), we get

$$C = 0 \text{ and } D = 1.$$

Hence

$$\frac{U(z)}{z} = -\frac{1}{2(z-4)} + \frac{1}{2(z-2)} + \frac{2z+4}{(z-2)^3}$$

We will put in the values of A and B in equation 1, in this equation, when you put the value of A and B in this equation, okay, what you get is; so putting the value of A and B in this equation 1, this one, okay, we get C + D = 1, similarly let us put the values of A and B in equation 2, in this equation, the relation between C and D as 3C + D = 1. Now, this equation is 3C + D = 1, this is 3C + D = 1, so from these 2 equations, it turns out that C = 0 and D = 1.

And thus we get the value of Uz/z; Uz/z, you put the values of A, B, C, D, okay, so Uz/z becomes this, - 1over 2 * z – 4 1 over 2 * z – 2 + 2z + 4 divided by z – 2 whole cube. Now, you can see; if you just multiply by z this equation, you will get the expression of Uz in the extended form where we can directly use the formula for the inverse z transform. So, Uz = -z over 2 times z – 4 + z over 2 times z – 2 2z square + 4z/ z - 2 whole cube.

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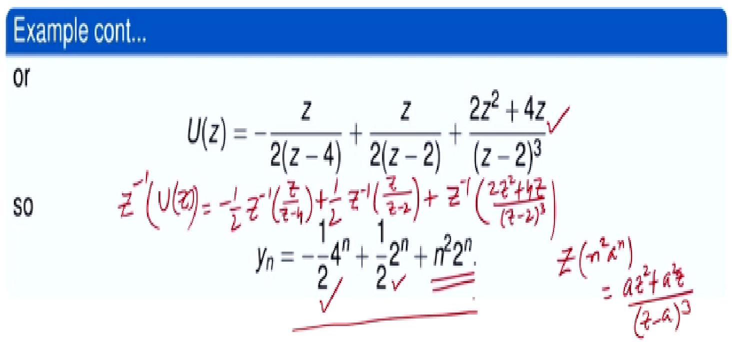

So, we get this, okay and now, let us take the inverse z transform, so taking inverse z transform, z inverse Uz, okay, z inverse Uz will be = -1/2 z inverse z over z – 4 + 1/2 z inverse z over z – 2 + z inverse 2z square + 4z divided by z - 2 whole cube, okay, z inverse of z over z – 4 is 4 to the power n, so we have -4, -1/2 * 4 to the power n, then z inverse of z over z -2 is 2 to the power n, so we have 1/2 * 2 to the power n here we have z inverse of 2z square + 4z divided by z – 2 whole cube is n square 2 to the power n.

Because let us again recall z transform of n square a to the power n is az square + a square z divided by z - a whole cube, okay so in this you put a = 2, you get inverse z transform of 2z square + 4z divided by z – 2 whole cube n square 2 to the power n, so this is how we can find the inverse z transform of Uz.

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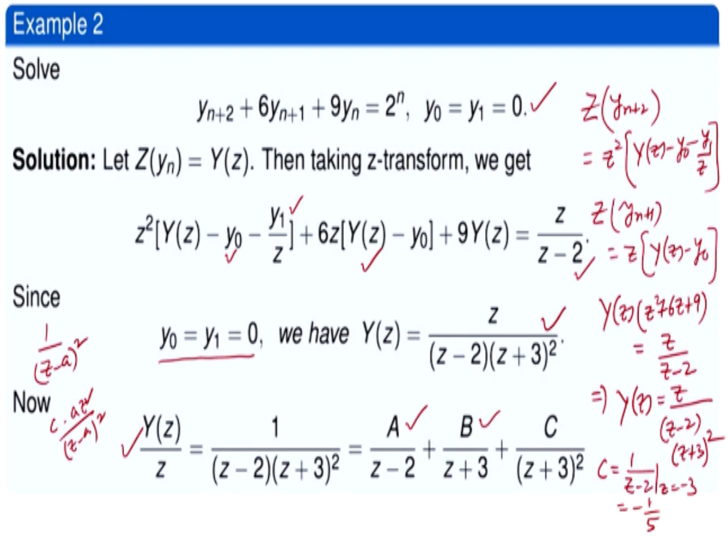

Now, let us solve this difference equation; yn + 2 + 6yn + 1 + 9yn = 2 to the power n where we are given that y0 and y1 are = 0, let us take z transform of yn sequence as yz, then taking the z transform of the given difference equation, we get z square, z of yn + 2, let us recall the advance property, z of yn + 2, y advance property is z square * z of yn, okay which is yz – y0 –y1 over z, okay, so we get z square yz – y0 – y1 over yz then yn + 1 z transform of yn +1, by advance property, z times yz – y0, so we get this okay.

Then z transform of yn is; so yn is 9 yz, z transform of 2 to the power n is z over z – 2, okay, now substituting the value of y0 and yn, y0 is 0, yn is 0, so we get yz times z square + 6z + 9, okay, we get yz times z square + 6z + 9 = z over z – 2, z square + 6z + 9 is z + 3 whole square, so we get yz, okay, yz = z over z – 2 * z + 3 whole square, okay that is this okay. Now, as usual we have to correct; we have to consider yz over z, okay.

So, let us consider yz over z then we have the rational function 1 over z – 2 z + 3 whole square, this we have to write in the standard from, so A over z – 2 B over z + 3 and C over z + 3 whole square, here in the last example on inverse z transform, corresponding to 1 over z – A whole square, okay, we have taken the fraction as C times Az divided by z – A whole square, this z we have already taken in yz over z.

So, we have to consider C times A, okay, what we can do here that this is A over z – 2 + B over z + 3 + C over z + 3 whole square, you might consider here C times -3 because A is -3 here and if you do not take -3 here along with C, then you will have to divide while taking the inverse z transform of z over z + 3 whole square/ -3. So, here we have not written -3 separately, so that I will do in the end while taking the inverse z transform, okay.

So, now let us find the values of A, B, C here, so the value of A can be found directly, A because z corresponding to z – 2 okay, the power of z – 2 is 1, so when you have simple pole at z = 2, the value of A can be found directly, 1 over z + 3 whole square, in that you put z = 2, so when you put z = 2 in 1 over z + 3 whole square, you get 1 over 25, so A is 1 over 25, okay and the value of C also be found directly.

Because the power of z + 3 is 2 here and here also the power of z + 3 is 2, so in the; remove z + 3 whole square from this expression in the remaining rational function, you put z = -3, so 1 over z – 2, okay, C = 1 over z – 2, when you put z = -3 gives you -1 over 5, okay.

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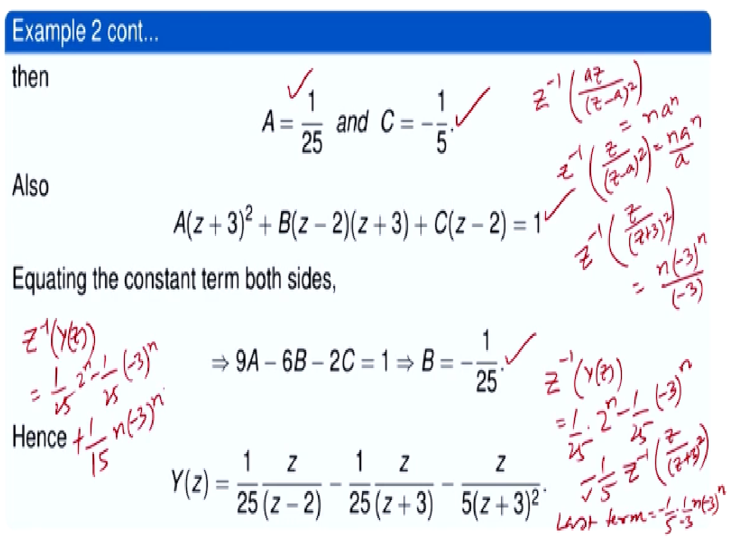

So, we get the value of C as -1 over 5 but we still need the value of B, for that we will have to write the identity, you take the LCM on the right side, okay and then equate both left and right side, so 1 will be = times z + 3 whole square + B times z – 2 z + 3 + C times z – 2, okay, so this equation we get and from this, we can easily find the value of B because we know already A and C, so you can equate the constant term both sides.

The constant term on the left side is 9A – 6B – 2C, on the right side it is 1, so 9A – 6B – 2C, when you put the values of A and C as 1 over 25 and -1 over 5 gives B = -1 over 25 and hence you can get the values of yz, okay, yz is now A times z over z – 2, so you have 1 over 25 z over z – 2, here B is now -1 over 25, so we get -1 over 25 z over z + 3 and C is now -1 over 5, so we get C that -1 over 5 times z over z + 3 whole square.

Now, let us take the inverse z transform here, okay, so inverse z transform of yz will give us the yn sequence, so z inverse of yz will be = 1 over 25, z inverse of z over z – 2 that is 2 to the power n – 1 over 25 z inverse of z over z + 3, so -3 to the power n, okay, now let us find -1 over 5 z inverse of z over z + 3 whole square, okay. Now, we know that z inverse of Az over z – A whole square, this is = n times A to the power n, okay.

If you take A = - 3, okay then if you take, if you have -3z upon z + 3 whole square then you can write directly n times - 3 to the power n but here we have z only, so z inverse of; you can write z over z – A whole square from here as n A to the power n divided by A, okay, so if you take A = -3, what we will get; z inverse of z over z + 3 whole square will give us n times -3 to the power n divided by -3, okay.

So, this will be last term; last term will be -1 over 5 okay, this -1 over 5 * 1 over -3 and then n times -3 to the power n, okay, so this is 1 over 15, okay, so we can now write z inverse of yz, we can write, so this is = 1 over 25 2 to the power n – 1 over 25 – 3 to the power n and then = 1 over 15 n times -3 to the power n, okay.

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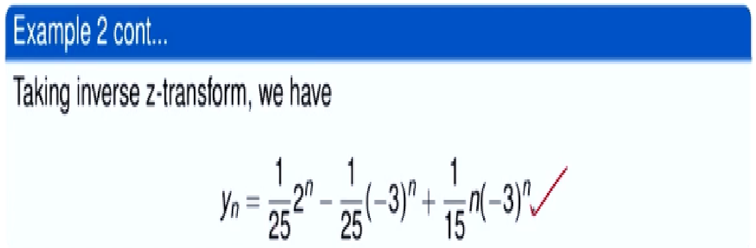

So, this is what we have here, okay, yn = z inverse of yz, this is yn okay, this = yn, so this equals yn is given by 1 over 25 2 to the power n – 1 over 20 1 over 25 -3 to the power n + 1 over 15 n times -3 to the power n.

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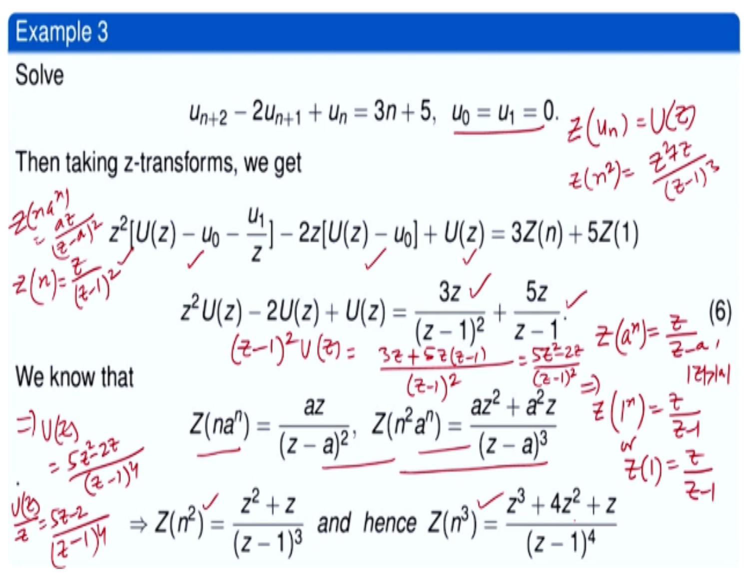

Now, let us take one more question, here we have a difference equation, un + 2 – 2 un + 1 + un = 3n + 5, okay, so here z transform of un be denote by uz, we are again given u0 u1 = 0, so taking z transform, z of un + 2 by advance property will give us z square –uz - u0 – u1/ z and then -2 times z of un + 1 will be z times uz – u0 + z of un will be uz and z of n okay, 3 times z of n + 5 times z of 1, we know that z transform is a linear operation, okay, it satisfies linearity property.

So, z transform of 3n + 5 is 3 times z of n + 5 times z of 1, okay and we know that z of a to the power n = z over z- a, then mod of z is > mod of a, so put a = 1 here, so then this implies z of 1 to the power n = z over z – 1, okay or z transform of 1 = z over z – 1 okay and z transform of n, okay and z transform of n we can write, z transform of n is z over z – 1 whole square, I got it from z transform of n a to the power n.

In that I put a as 1, okay but it can found directly, okay, z transform of n as z over z – 1 whole square, so this is = this, okay, 3 times z over z – 1 whole square + 5 times z over z – 1, okay, so I have made use of these formulas, okay. Now, we will need the formula for z of n square and z of n cube later on, okay, when we simplify this. Now, this left hand side is z square – 2z * uz + uz, so this is z – 1 whole square.

So, left side is z – 1 whole square * uz, okay and right side you can see, this is z – 1 whole square and we have 3z * 3z + 5z * z – 1, okay, so we get 3z; so we get 5z square – 2z divided by z – 1 whole square, okay, so this z – 1 whole square we divide here on the right side and get uz = 5z square – 2z divide by z – 1 to the power, okay, so uz comes out to be 5z square – 2z divided by z – 1 to the power 4 okay.

What we will do again; we will write uz yz okay, we will write uz yz and what we will have left with; uz yz will be = 5z – 2 divided by z -1 to the power 4, this we will break into partial fractions but we will write it in the partial fractions such a way that we can easily get the inverse z transform, so z – 1 to the power 4, okay.

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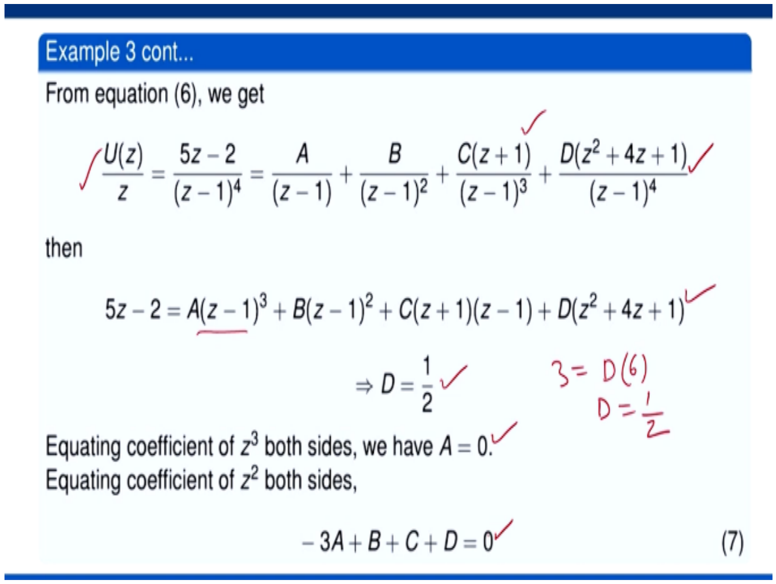

Now, let us see how we do this, so uz / z is 5z 2 divided by z – 1 to the power 4, now there will be 4 fractions corresponding to z – 1 to the power 4, there will be z - 1 1 over z – 1 1 over z – 1 square, okay, z – 1, z- 1 square, z- 1 cube, z – 1 to the power 4, now for z – 1, you need only z in the numerator, okay that z is here, so we will multiply in the end and will have A times z over z – 1, we can easily write the inverse z transform.

For z – 1 whole square, you need to have az, okay, since a is 1, you need to have z and z we will get from here, so B times z over z – 1 whole square will can also be easily inverted, now here we have z – 1 whole cube, for z – 1 whole cube, you use this formula, z transform of n square a to the power n and this gives you z transform of n square as A = 1, okay so z square + z divided by z -1 to the power 3.

Now, one z from the numerator we have already taken and in writing; to write uz over z, so what we need here is that z + 1 we need in the numerator, so we write C times z + 1 over z – 1 whole to the power 3 and corresponding to z – 1 to the power 4, you need this formula, z of n cube. Now, z of n cube over; the value of z of n cube can be found easily from the recurrence relation for z of n to the power p.

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$$-z\frac{d}{dz}Z(n^p) = Z(n^{p+1})$$

$$Z(n^3) = -z\frac{d}{dz}Z(n^2)$$

$$= -z\frac{d}{dz}\left(\frac{z^2+z}{(z-1)^3}\right)$$

$$= -z\left[\frac{(2z+1)(z-1)^3 - 3(z-1)^2(z^2+z)}{(z-1)^6}\right]$$

$$= -z\left[\frac{(2z+1)(z-1) - 3(z^2+z)}{(z-1)^4}\right]$$

$$= -z\left\{\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right\}$$

$$= \frac{-z(-z^2-4z-1)}{(z-1)^4} = \frac{z^3+4z^2+z}{(z-1)^4}$$

If you recall z of n to the power p, we had proved this one, p is a positive integer, we have shown that –z d over dz of z of n to the power p is z of n to the power of p + 1, okay, so if you know the value of z of n, you can find the value of z of n square, if you know the value of z of n square, you can find the value of z of n cube, so from here you can see, z of n cube will be = -z d over dz of z of n square, okay.

Z of n square we know, so z of n square is this one, okay, z z square + z over z – 1 whole cube, so d over dz of z square + z divided by z – 1 whole cube, okay. So, you can differentiate this okay, this will give you –z times 2z + 1 * z – 1 whole cube – 3 times z – 1 whole square * z square + z divided by z – 1 to the power 6, we can cancel z – 1 whole square to have 2z + 1 * z – 1 – 3 times z square + z divide by z – 1 to the power 4, okay.

We have cancel z – 1 whole square from the numerator and denominator, so we get – z times 2z square + z – 2z – 1 – 3z square – 3z divided by z – 1 to the power 4, okay, what we get; 2z square – 3z square, so we get –z square, okay and then we get here z – 2z is –z -3z is – 4z, okay and we get -1 divided by z -1 to the power 4 and this is z cube + 4z square + z divided by z – 1 to the power 4 okay, this is what we have taken here, okay.

So, what we will do; now one z we have already selected from the numerator to write uz /z, so we need on the right side, z square + 4z + 1 divided by z – 1 to the power 4, so this coefficient of z we have taken, coefficient of D we have taken, z square + 4z + 1 over z – 1 to the power 4, okay. Now, we can see that the; we have z – 1 to the power 4, okay, so here what we do; we can get the value of D directly, okay.

The value of D can be found directly, you put z = 1, this is identity equation, okay, this identity, okay it is true for all values of z, so in this you put z = 1, okay, what you get; -3 = this; this will be 0, this will be 0, this will be 0, so we will get D times 6, so we get D =; no, this is +3, okay so we get D = 1/2 okay 5 -2 is 3, now equating coefficient of z cube both sides, let us equate the coefficient of z cube both sides, this is the term which has term in z cube, here we do not have a term in z cube, here also and here also we have no term in z cube.

So, the coefficient of z cube is A here and here the coefficient of z cube is 0, so we get A =0, so we get the value of D and the value of A, okay, now let us equate the coefficients of z square both sides, so here there is no term in z square on the left side, so we get 0 and there the right side, the coefficient of z square if you collect, you get -3A + B + C + D, so you get this equation, -3A = B + C + D = 0, this is one equation.

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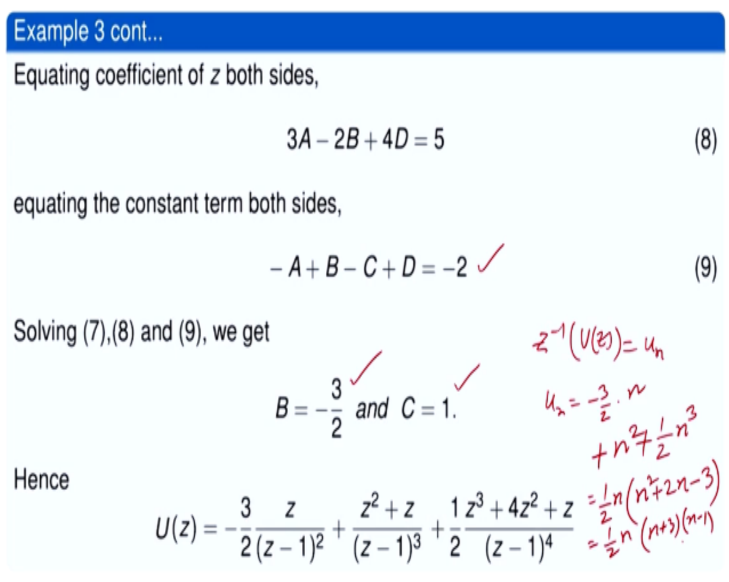

And when you equate the coefficient of both sides, here the coefficient of z is 5 on the right side, the coefficient of z is 3A - 2B + 4D, so we get 3A – 2B + 4D = 5, okay, now equating the constant term both sides, here the constant term is -2, here the constant term will be –A + B –C +

D, okay so we have –A + B –C + D = -2, okay. Now, solving 7, 8 and 9, we know the values of A and D, we can put them, we get the value of B and C.

B = -3/2, C = 1, so we get uz, okay uz = -3/2; -3/2 is the value of B, okay, A = 0, okay, so this term become 0, so -3/2 z – 1 whole square, z over z – 1 whole square + C = 1, okay, so we get z square + z divided by z – 1 whole cube and then we have D; D = 1/2 okay, so we get 1/2 z cube + 4z square + z divided by z – 1 to the power 4, D = 1/2 okay. So, this is +1/2, here we have -3/2 z over z – 1 whole square + z square + z over z – 1 whole cube + 1/2 z cube + 4z square + z over z – 1 to the power 4 and now take the inverse z transform, okay.

So, inverse z transform of uz, z inverse of uz you have to find, okay, this is = u1, okay, so u1 will be = -3/2 okay, this is z transform of n okay, z transform of -3/2 * n, okay + this is z transform of n square, okay, z transform of n square is z square + z over z =- 1 whole cube, so you get n square here, okay and this is z transform of n cube, so -1/2 n cube, so this is +1/2 n cube, okay. So, what we can do; this is `1/2 n times, we will have n square + 2n + 3, -3, okay.

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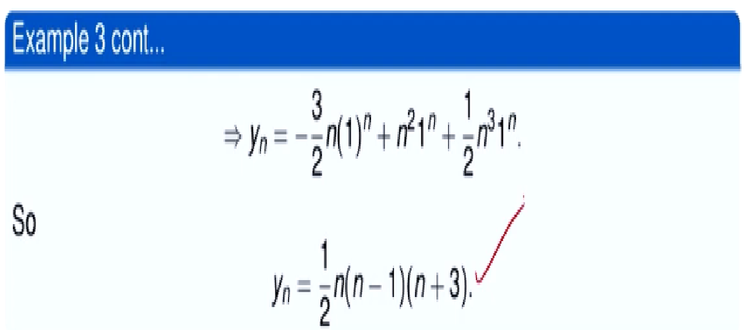

So, this will be having factor 1/2n n +3 * n -1, okay, so un will be = 1/2n * n + 3 * n -1, okay, this, all right, so this is how we solve this differential; difference equation, we have seen that there was some complicated rational functions and we have to use some; we have to write the partial fraction in a particular form, so that we can easily take the inverse z transform. I think now this; how we have to solve the given difference equation.

And how we have to find the inverse z transform there, this thing must be; must have been come clear to you. With this, I would like to end my lecture, thank you very much for your attention.