Advanced Engineering Mathematics
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Lecture – 33
Review of Z-transforms - I

Hello friends. Welcome to my lecture on Review of Z-transforms. In this lecture, we shall discuss Z-transforms and their properties.

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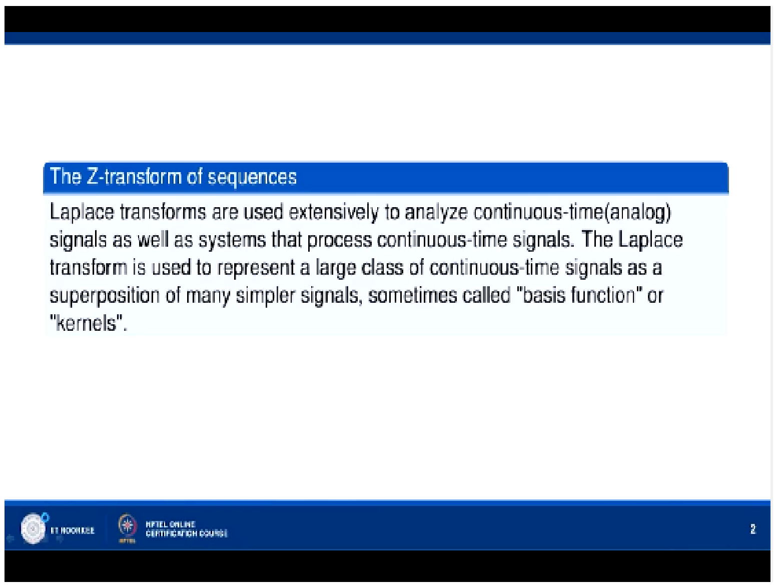

Let us first discuss the Laplace transform because Z-transform is a discrete analog of Laplace transforms. Laplace transforms are used extensively to analyse continuous time analog signals as well as systems that process continuous time signals. The Laplace transform is used to represent a large class of continuous time signals as a superposition of many simpler signals, sometimes called basis function or kernels.

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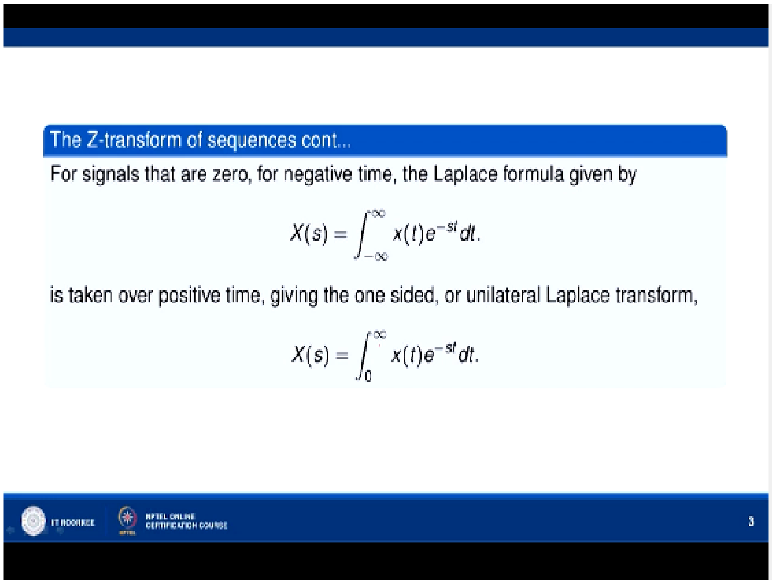

For signals that are 0, for negative time, the Laplace formula given by Xs=integral over -infinity to infinity xte to the power -st dt is taken over positive time, giving the one sided or unilateral Laplace transform Xs=integral 0 to infinity ste to the power -st dt.

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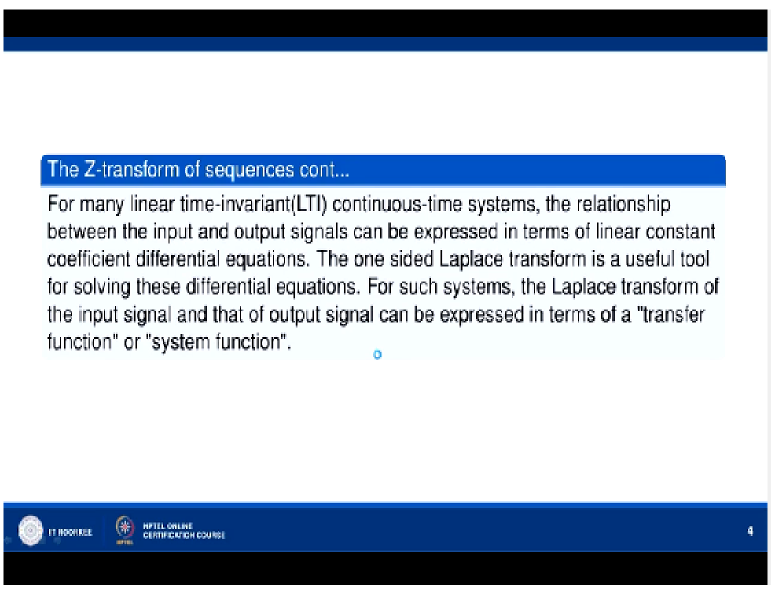

For many linear time invariant continuous time systems, the relationship between the input and output signals can be expressed in terms of linear constant coefficient differential equations. The one sided Laplace transform us a useful tool for solving these differential equations. For such systems, the Laplace transform of the input signal and that of output signal can be expressed in terms of a transfer function or system function.

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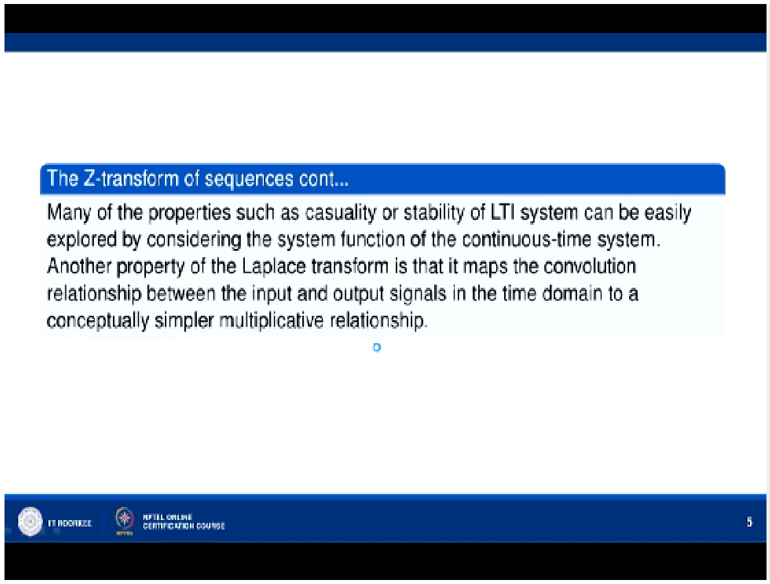

Many of the properties such as casuality or stability of LTI system that is a linear time invariant system can be easily explored by considering the system function of the continuous time system. Another property of the Laplace transform is that it maps the convolution relationship between the input and output signals in the time domain to a conceptually simpler multiplicative relationship.

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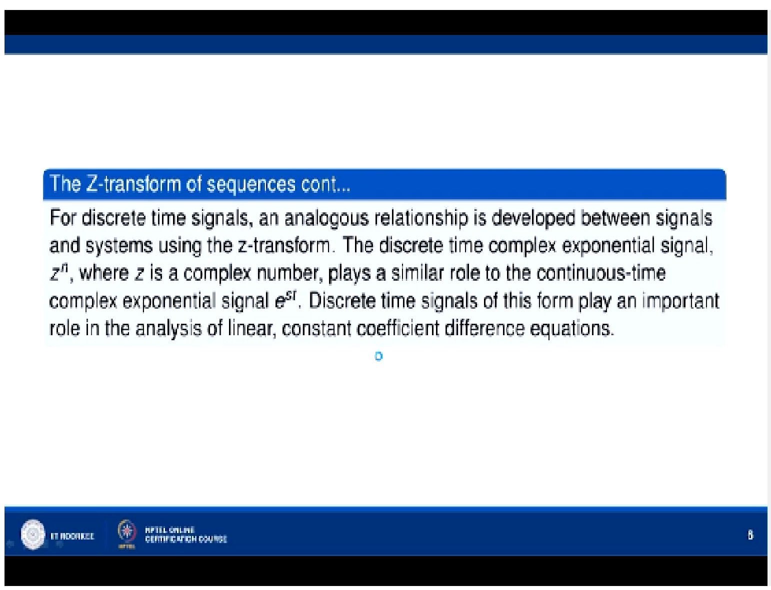

For discrete time signals, an analogous relationship is developed between signals and systems using the z-transform. the discrete time complex exponential signal, z to the power n, where z is a complex number, plays a similar role to the continuous time complex exponential signal e to the power st. Discrete time signals of this form play an important role in the analysis of linear, constant coefficient difference equations.

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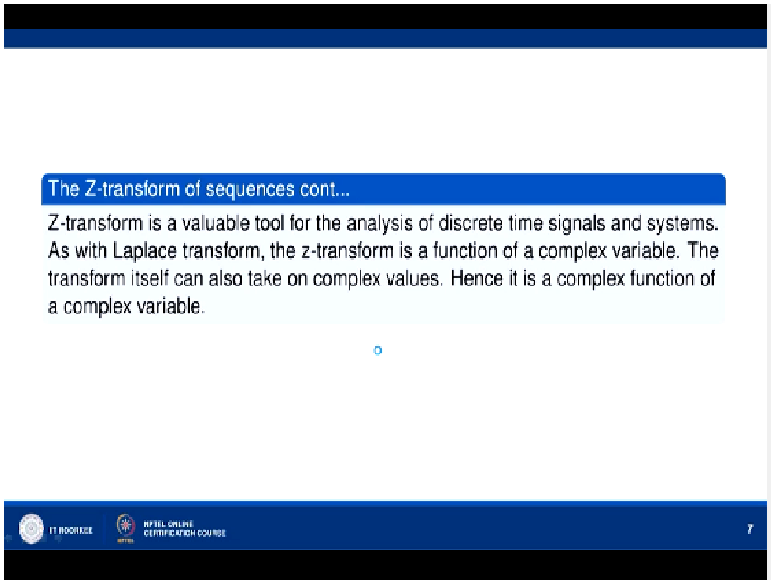

Z-transform is a valuable tool for the analysis of discrete time signals and systems as with the case of Laplace transform, the Z-transform is a function of a complex variable. The transform itself can also take on complex values. hence it is a complex function of a complex variable.

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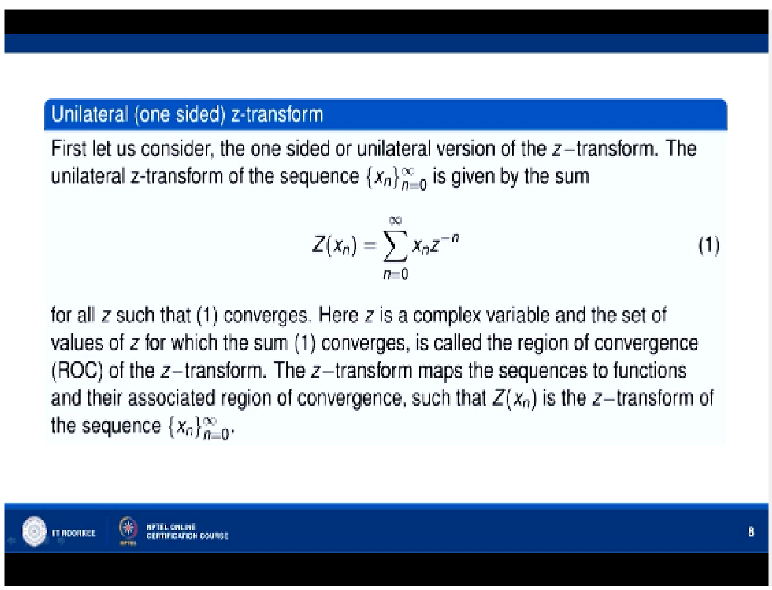

Let us first define a unilateral Z-transform that is one sided Z-transform. So the one sided or unilateral version of the Z-transform is defined as follows. Suppose we have a sequence xn where n runs from 0 to infinity through the positive integers. So n=0 1 2 3 and so on. So when we are given a sequence xn.

The unilateral Z-transform of the sequence xn is defined as Zxn=sigma n=0 to infinity xn*z to the power -n for all z such that 1, the summation, the infinite series in 1 converges. Here z is a complex variable and the set of values of z for which the sum converges, is called the region of convergence of the Z-transform. The Z-transform maps the sequence to functions and their associated region of convergence, such that Zxn is the aztr- of the sequence xn.

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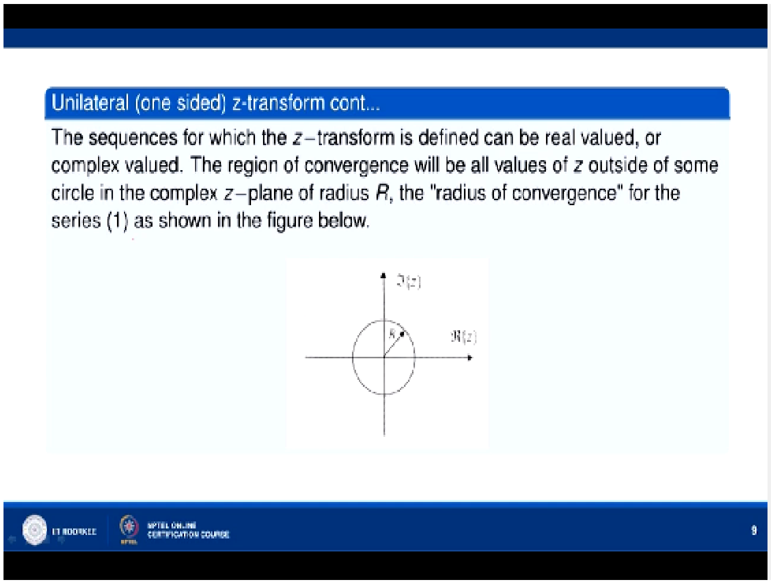

Now let us consider the sequences for which the Z-transform is defined, the sequences for which the Z-transform is defined can be real valued or complex valued. The region of convergence will be all values of z outside of some circle in the complex z plane of radius R, the radius of convergence for the series 1 as shown in the figure below. So the radius of convergence will be the some region outside the circle of convergence of radius R with center at the origin where R is given as the radius of convergence.

So here for example the region of convergence is the shaded portion and it is outside the circle with center at origin and radius R in the complex z plane. So this region can be represented by mod of z>R.

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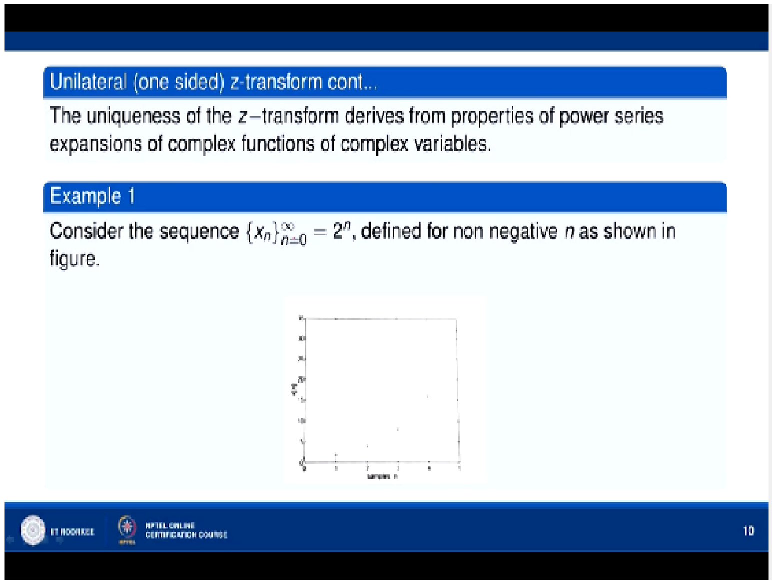

Now the uniqueness of the Z-transform derives from properties of power series expansions of complex functions of complex variables. Let us consider the sequence xn=2 to the power n where n begins from 0 to infinity. So then consider this sequence which is defined for non-negative integers n=0 1 2 3 and so on. So here you can see in this figure, these are the values of n. When n=1, x1=2, so we have this point 2.

Then at n=0, we have 2 to the power 0, so 1 and we have for n=1, we have 2. n=2, we have 2 square that is 4. n=3, we have 2 cube that is 8 and n=4, that is, here, 2 to the power 4 which is 16. n=0, you can see it is 1 here. Then let us find the Z-transform of this sequence.

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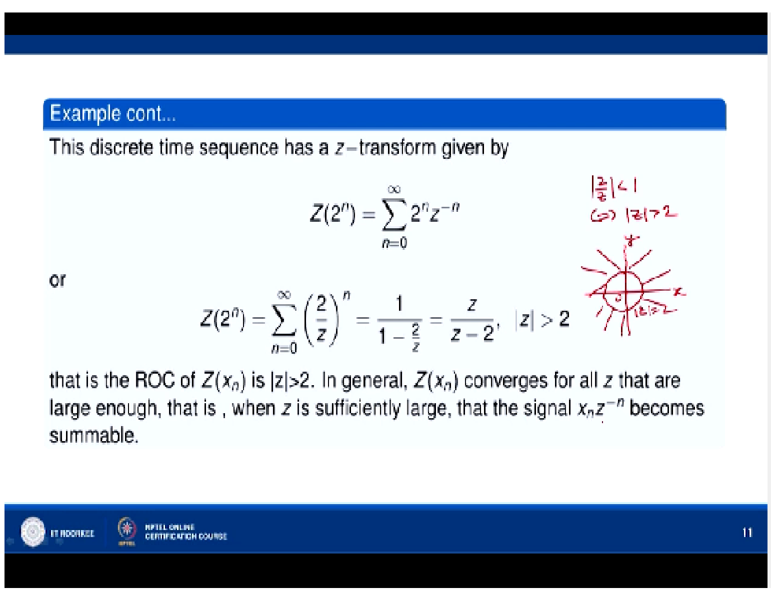

So the discrete time sequence has a Z-transform given by Z of 2 to the power n=sigma n=0 to infinity 2 to the power n z to the power -n. We can also write this as Z of 2 to the power n=sigma n= 0 to infinity 2/z to the power n. Now this is a geometric series and so its sum will be 1/1-2/z provided mod of 2/z<1. So mod of 2/z<1 gives you mod of z>2. So the sum of this series is z/z-2 provided mod of z>2.

So mod of z>2 is the region of convergence of this series and you can draw this region of convergence. So this is your circle with center at origin and radius 2. So this is mod of z=2. Then the region of convergence is this. So for this series, region of convergence is mod of z>2. Now in general, Zxn is sigma n= 0 to infinity xn z to the power -n. So in general, Zxn converges for all z that are large enough. That is when z is sufficiently large that the signal xn z to the power -n becomes summable.

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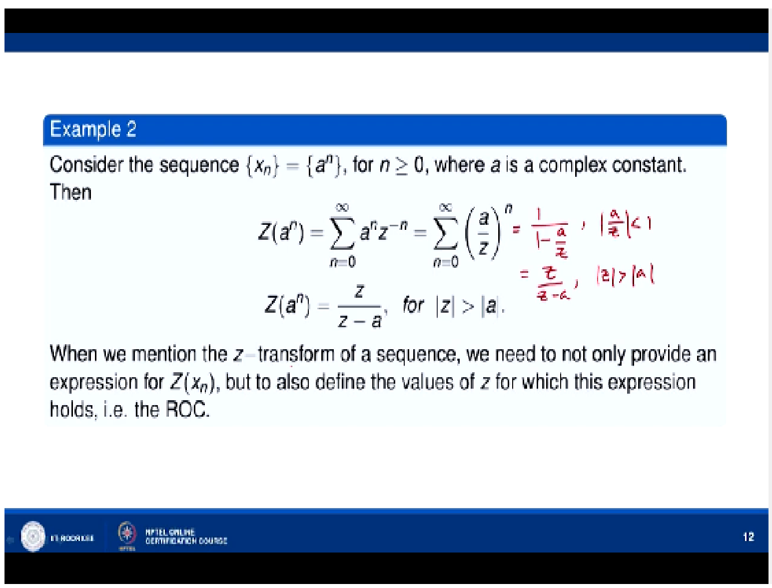

Now consider the sequence xn=n to the power n where n greater than or equal to 0 and is an integer. So then this will be Za to the power n, a is any complex constant here. So Z of a to the power n will be sigma n=0 to infinity anz to the power -n and this can be written as sigma n=0 to infinity a/z raise to the power n. Now again this is a geometric series. So its sum is 1/1-a/z provided mod of a/z<1.

So this can be written as z/z-a provided mod of z>mod of a. So the sequence an has Z-transform given by z/z-a where mod of z>mod of a. Now we mentioned that the Z-transform of a sequence whenever we mentioned the Z-transform of a sequence, we should also write the region of convergence of the series. So not only the expression of Zxn is to be given, its region of convergence must also be specified. Now let us show that Z-transform is linear, okay.

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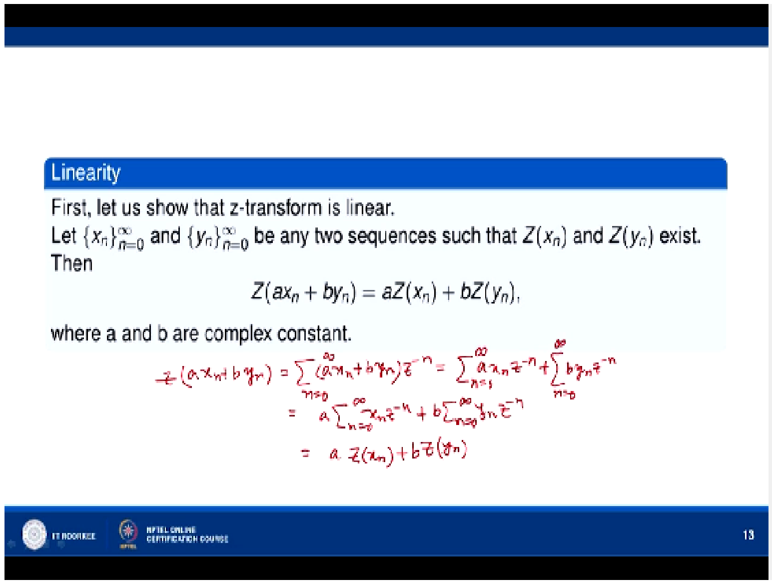

So let xn and yn be any 2 sequences such that their Z-transforms exist, then z of axn+byn is given by, by definition, let us write it as sigma n=0 to infinity axn+byn*Z to the power -n and this can be written as then sigma n=0 to infinity axnZ to the power –n + summation n=0 to infinity bynZ to the power -n. Now sigma n=0 to infinity xnZ to the power -n is convergent. So this is equal to a*sigma n=0 to infinity xn Z to the power -n and we have b* similarly n=0 to infinity ynZ to the power -n.

So since sigma n=0 to infinity xnZ to the power -n and sigma n=0 to infinity ynZ to the power -n are convergent, okay, we can write this sum, infinite sum as sum of these 2 series and then we can bring this colour a and b outside. So then this is a*Z of xn+b*Z of yn. So we have 2 sequences, xn and yn whose Z-transforms are known and a and b are any complex constant, then Z-transform of axn+byn will be a*Zxn+b*Zyn. So that we can obtain. It shows that Z-transform is a linear operation.

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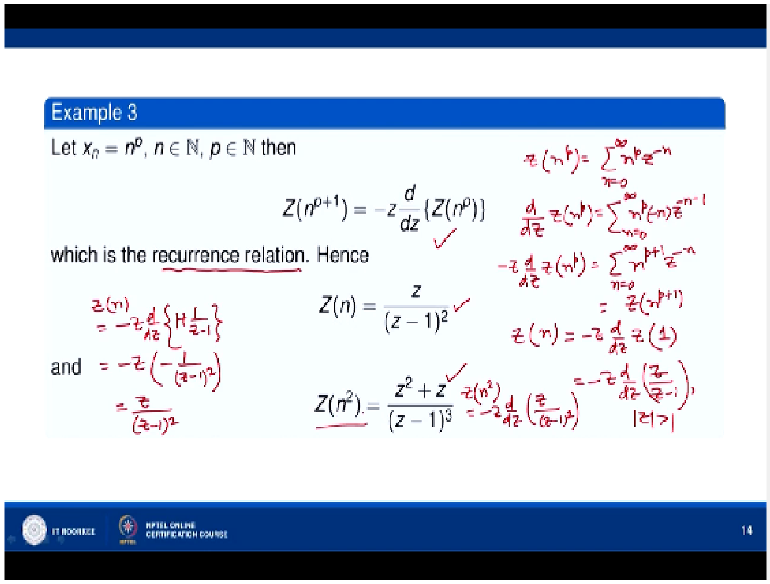

Now let us consider a sequence xn=n to the power p where n is there in N and p belongs to N. So Z of n to the power p=sigma n=0 to infinity n to the power p z to the power -n, okay. Now let us differentiate z of n to the power p. So then what we get? Summation n=0 to infinity n to the power p*-n z to the power -n-1. Let us multiply both sides by -z, then -zd/dz of z n to the power p=sigma n=0 to infinity n to the power p+1*z to the power -n, okay.

Now right side is Z-transform of n to the power p+1. So Z-transform of n to the power p+1 is given by -z*d/dz of Z-transform of n to the power p. And this relation is called as the recurrence relation, okay. If you know the Z-transform of n to the power p, you can find Z-transform of n to the power p+1. By differentiating z of n to the power p with respect to z and then multiplying by mod z.

For example, let us find the Z-transform of n, okay. Z-transform of n, you can take p=, if you take p, we have taken p as 1, okay. So Z-transform of n if we want to know, we Z-transform of n square we can find here. If we take p=0, even for p=0 it is true because Z-transform of n we can find here. So if you take p=0 here, we get z of -zd/dz of Z-transform of, p is 0, so we get 1, okay. Now Z-transform of a to the power n we have found.

Z-transform of a to the power n where a is any complex number. We can take a as 1, okay. Then 1 to the power n will be 1 for all n. So we will have the sequence xn as 1 for all n, okay. And that sequence we can consider. So then that gives you Z-transform of 1=z/z-1, where mod of z>1. So this is equal to -zd/dz of z/z-1 provided mod of z>1. So then we can differentiate this. We can write it as Zn=-zd/dz of, we can write it as 1+1/z-1.

So then z/z-1 can be written as 1+1/z-1, we can differentiate this. This is -z and then we get -1/z-1 whole square. So what we get is z/z-1 whole square, okay. So we get Z-transform of n. Z-transform of n is z/z-1 whole square. Now we know the Z-transform of n. We can find the Z-transform of n square. So Z-transform of n square will be equal to -z*d/dz of z/z-1 whole square. So when you differentiate z/z-1 whole square with respect to z and multiply by -z, we get the Z-transform of n square which is z square+z/z-1 whole cube.

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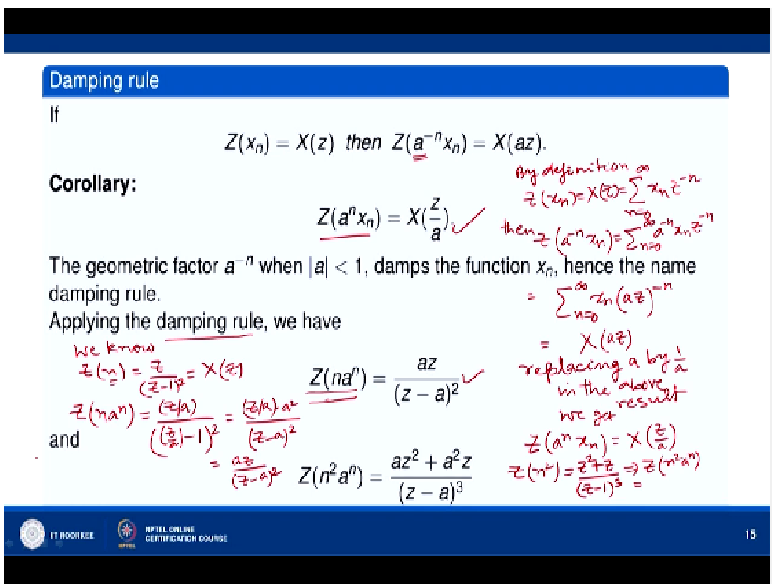

Now let us discuss a property of Z-transform, another property. if Z of xn=Xz, then Z of a to the power -n xn=Xaz. And a corollary to this result is Z of a to the power n*xn is X of z/a. So let us first show that if Z of xn is Xz, then Z of a to the power -n*xn is Xaz. So by definition, Z of xn=Xz=sigma n=0 to infinity xnz to the power -n. Now let us multiply xn by a to the power -n. So then Z-transform of a to the power -n*xn will be equal to limit sigma n=0 to infinity a to the power -nxn*z to the power -n.

Now this right hand side can be written as sigma n=0 to infinity xn az to the power -n. And therefore, by our definition, this is X of az, okay. So Z-transform of a to the power -n xn=X of az, okay. Now here if you replace 1/a by a, then you get the corollary, okay. So replacing a by 1/a in the above result, we get Z-transform of a to the power n xn=X of z/a. So we get this corollary, okay.

Now the geometric factor a to the power -n, when mod of a<1, damps the function xn and that is why we call it as a damping rule. So now if you use the damping rule, we can determine the Z-transform of the sequence n*an, okay. Because we know the Z-transform of n. So recall that Z-transform of n is given by z/z-1 whole square, okay. We know Z-transform of n=z/z-1 whole square. So this is your Xz, okay. Sequence Xn is n, okay. Now we want Z-transform of n*a to the power n. So Z-transform of n*a to the power n. We can apply the corollary, okay.

So Z-transform of Xn*a to the power n is Xz/a. So replace z by z/a in the expression for Xz. So we get z/a/z/a-1 whole square, okay. So this is z/a/z-a whole square/a square. So this a square comes here and what we get? az/z-a whole square, okay. So z of n a to the power n is az/z-a whole square. Similarly, Z-transform of n square a to the power n can be determined because we now know the Z-transform of n a to the power n, okay.

So we know the Z-transform or we can apply, we know the Z-transform of n square, okay. We have to use that. Z-transform of n square is z square+z/z-1 whole cube. Now we want the Z-transform of n square*a to the power n. So we can apply the damping rule. So let us recall that Z of n square=z square+z/z-1 whole cube. So this gives you Z-transform of n square a to the power n=, replace z by z/a, okay.

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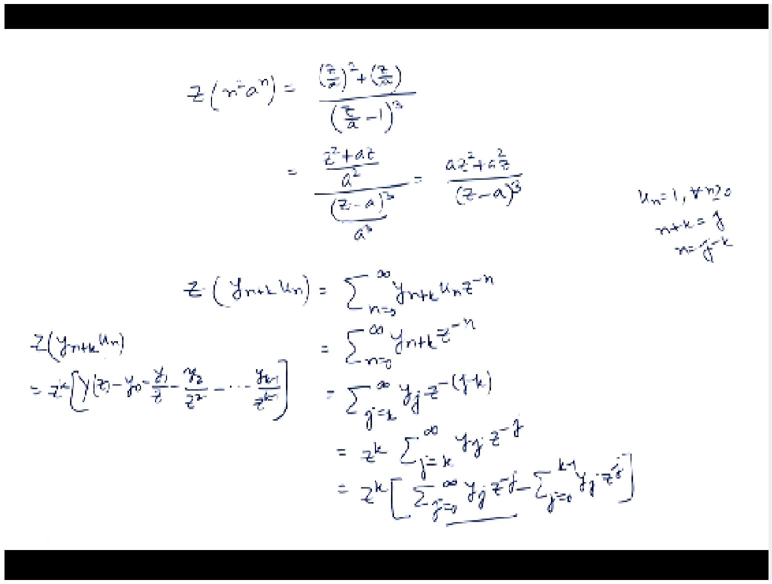

So we can write that Z-transform of n square a to the power n=z/a whole square+z/a/z/a-1 whole cube, okay. So this will give you z square+az/z squre/z-a whole cube. This is z square+az/a square and here we get z-a whole cube/a cube. So we get az square+a square z/z-a whole cube. So Z-transform of n square a to the power n is az square+a square z/z-a whole cube.

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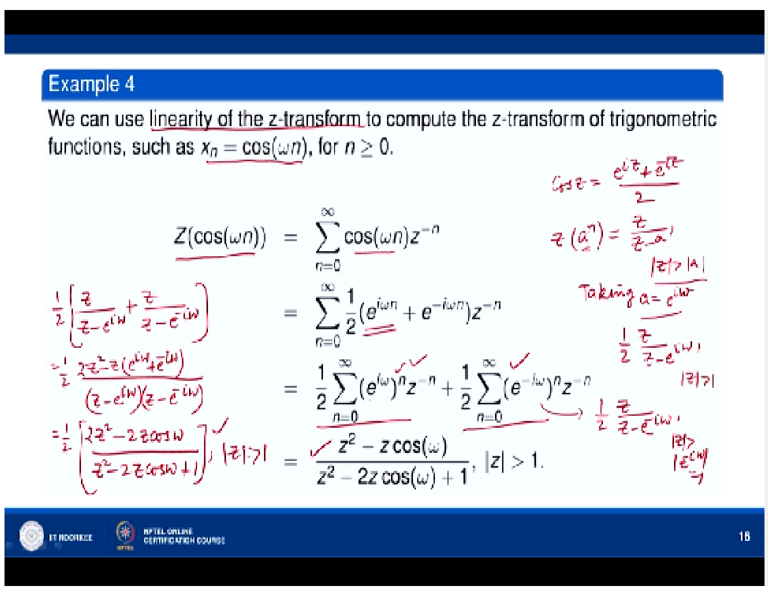

Now we can find the Z-transform of sequence which is of trigonometric functions. Let us say xn=cos omega n where n greater than or equal to 0. So for this, we can use the linearity of the Z-transform, okay. Linearity of the Z-transform can be used to determine the Z-transform of xn=cos omega n. So Z of cos omega n=, by definition, sigma n=0 to infinity cos omega n z to the power -n.

Now we know that cos z=e to the power iz+e to the power -iz/2, okay. So we have sigma n=0 to infinity 1/2 e to the power i omega n, okay, +e to the power -i omega n z to the power -n. We put the value of cos omega n here. So e to the power i omega n+e to the power -i omega n/2. Now e to the power i omega n can be written as e to the power i omega whole to the power n*z to the power -n and we are using linearity here, okay.

So using linearity, this half comes out and we write sigma n=0 to infinity e to the power i omega n*z to the power -n. e to the power i omega n is further=e to the power i omega raise to the power n and then we have 1/2 sigma n=0 to infinity e to the power -i omega raise to the power n*z to the power -n. Now this omega is a real quantity, okay. This omega is real quantity. So what we have?

Let us recall that Z-transform of a to the power n, okay. Z-transform of a to the power n is z/z-a where mod of z>mod of a, okay. So taking a=e to the power i omega, okay. Taking a=e to the power i omega because a here could be a complex number. So we get the Z-transform of the first summation, Z-transform of this will be 1/2z/z-e to the power i omega, where mod of z>mod of e to the power i omega which is 1, okay.

So mod of z>1. And second term will be? 1/2, similarly we now take e to the power -i omega and mod of e to the power -i omega when omega is real, is equal to 1. So again when mod of z>1, Z-transform here will be 1/2. This will be 1/2*z/z-e to the power -i omega, okay. So what we get is this. 1/2z/z-e to the power i omega+z/z-e to the power -i omega. And this is 1/2*z-e to the power i omega*z-e to the power -i omega.

Here we have z square*-z*e to the power i omega+e to the power -i omega, okay. z square-, and then we have, this is equal to 1/2 2z square, this will be 2z square, so 2z square-2z cos omega e to the power i omega+e to the power -i omega is 2cos omega/; here in the denominator, we will have z square-z*e to the power i omega+e to the power -i omega. So -2z cos omega and then we have +e to the power i omega*e to the power -i omega which is e to the power 0, so that is 1.

So this 2 will cancel with this 2 and we will have z square-z cos omega/z square-2z cos omega+1. Now this series converges when mod of z>mod of e to the power i omega. So mod of z>1. And this series converges when mod of z>mod of e to the power -i omega, which is also 1, okay. So here also mod of z>mod of e to the power -i omega which is equal to 1. So for both the series, okay, the region of convergence is the same, mod of z>1, okay. So this is equal to Z-transform of cos omega n=this where mod of z>1, okay.

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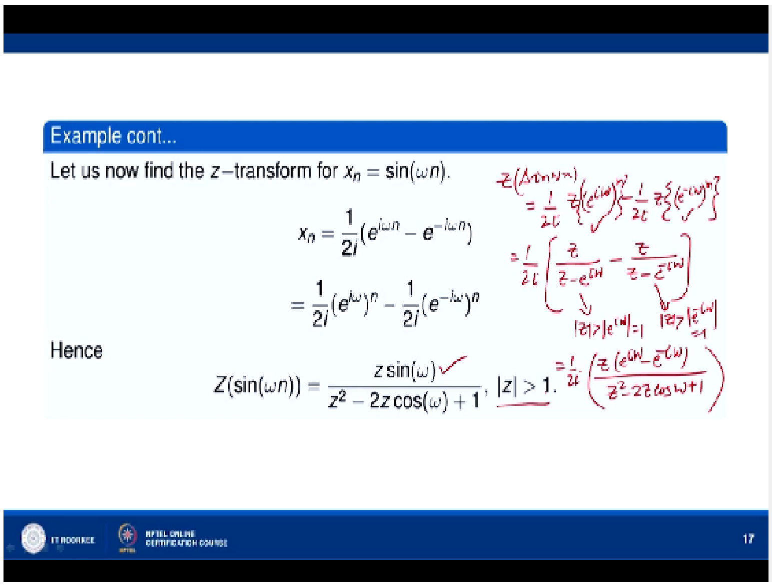

Now let us similarly, we can find the Z-transform of the sequence sin omega n when xn=sin omega n. So xn is 1/2i e to the power i omega n-e to the power -i omega n. And we can write as in the case of the cos omega n, we can write it as 1/2i e to the power i omega raise to the power n and -1/2i e to the power -i omega raise to the power n.

Then Z of sin omega n will be equal to 1/2i, using linearity, Z-transform of e to the power i omega-1/2i omega e raise to the power n and Z-transform of e to the power -i omega raise to the power n, okay. So this is 1/2i z/z-e to the power i omega-z/z-e to the power -i omega. Again, we are applying here the formula z of a to the power n. So this series, the Z-transform of a to the power n is z/z-a when mod of z>mod of a.

So mod of z for the convergence of this, we have mod of z>mod of e to the power i omega which is equal to 1. And for the convergence of the series here, okay, the region of convergence is mod of z>mod of e to the power -i omega. So this is equal to also 1, okay. So region of convergence is again same for both the series, the series that occurs here and the series that occurs here.

So then this is 1/2i, we can write z-ei omega*z-e-i omega, that is z square-2z cos omega as in the pre-case of cos omega n we have seen, +1 and here we will get z square and here -z square, that will cancel. So here we will get -ze to the power i omega. Here +ze to the power i omega. So z*e to the power i omega-e to the power -i omega, okay. Now z of e to the power i omega, z*e to the power i omega-e to the power -i omega/2i is sin omega, okay. So we get z sin omega/z square-2z+1, okay. And this region of convergence is mod z>1.

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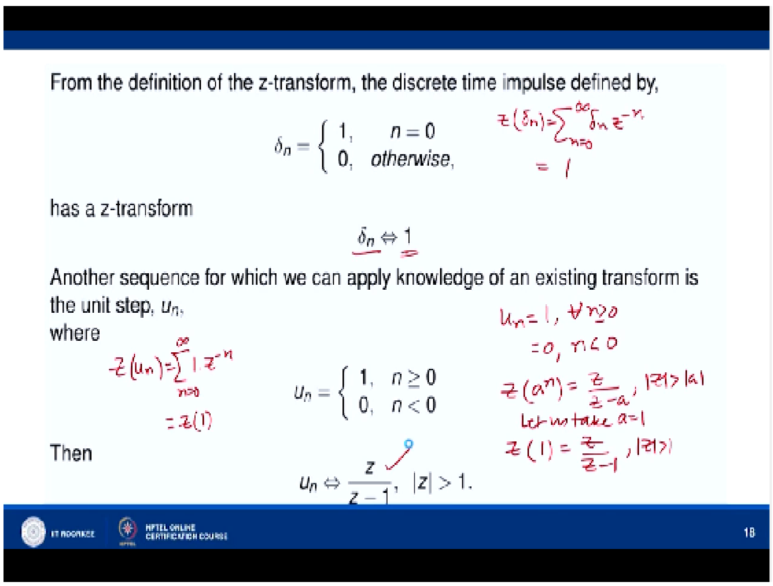

Now let us take the discrete time impulse delta n, okay. Delta n is defined as 1 when n=0 and 0 otherwise. So unilateral Z-transform here will be what? Z-transform of delta n, okay will be equal to sigma n=0 to infinity delta n z to the power -n. Now at n=0, delta n=1. So when you put n=0 here, we get delta 0 which is 1 z to the power 0 is also 1, so we get 1. And delta n=0 for all values of n other than 0.

So delta n=0 makes other terms 0. And therefore, Z-transform of delta n=1. So delta n goes to 1 and 1 goes to delta n, the inverse Z-transform of 1 will be equal to delta n, okay. So Z-transform of delta n is 1 and inverse Z-transform of 1 is delta n. Now another sequence for which we can apply the knowledge of an existing transform is the unit step function. Z-transform of un. un=1 for all n greater than or equal to 0, okay and 0 when n<0.

So we can apply the formula that we have already found for a to the power n sequence. We have seen that Z-transform of a to the power n=z/z-a when mod of z>mod of a, okay. Here a is any complex constant. So let us take a=1, okay. So Z-transform of 1 to the power n. 1 to the power n is 1, so 1=z/z-1, okay, when mod of z>1, okay. So here the sequence un=a to the power n where a=1 for all n greater than or equal to 0.

Because other terms of the sequence are all 0. So sigma, if you find the Z-transform of un, it is equal to sigma n=0 to infinity 1xn is 1 or you can say un is 1*z to the power -n, okay. So this is same as Z-transform of the sequence 1. So Z-transform of the unit step function is z/z-1 where mod of z>1.

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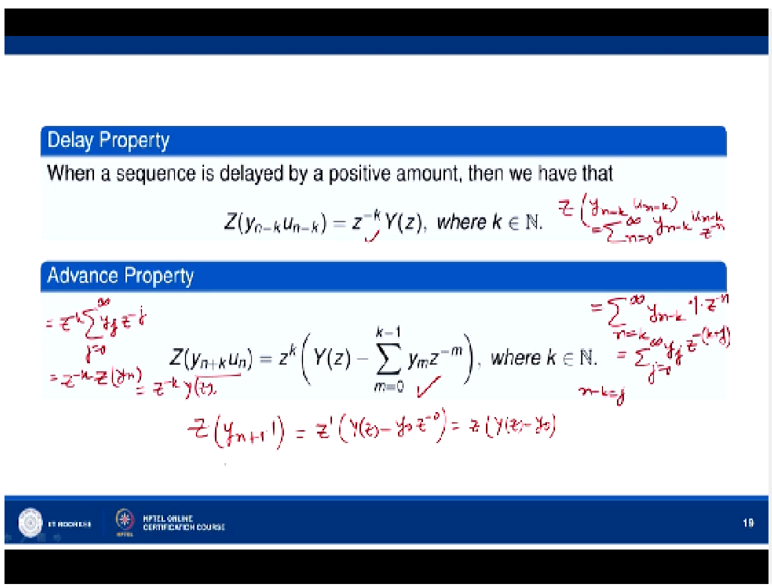

Now let us consider the delay property. When a sequence is delayed by a positive amount, then we have Z of yn-k un-k=z to the power -k Yz. Here k is any natural number. Now this we can easily see Z-transform of, where we have yn-k sequence, yn-k un-k, this equal to sigma n=0 to infinity yn-k un-k, okay. When n=0 1 2 and so on, then un-k is 0 whenever n is negative, okay. un is we have seen, un is equal to 0, okay when n is negative.

So here, this summation will be reduced to sigma n=k to infinity because if n<k, n-k<0, so un-k is 0. So yn-k un-k. And un-k now is equal to 1. So we get here, and we also have z to the power -n here. So 1*z to the power -n we have. Now replace n-k by equal to j, okay. So when you do that, this is sigma j=0 to infinity, okay, yj, and n=k+j, so z to the power -k+j. So we can write this as z to the power -k we can write outside the summation and sigma j=0 to infinity and we have yj z to the power -j.

So this is z to the power -k*Z-transform of yn sequence, okay. So which is Yz. So z to the power -k*Yz, okay. So we get this formula. This is the delay property of the Z-transform. Now advance property, okay. When yn is advanced by an amount k. So Z of yn+k un we want to find, okay. So Z-transform of yn+k*un, we want to find this, okay. So this is sigma n=0 to infinity yn+k*unz to the power -n, okay.

Replacing this we can write. So this is when n takes values from 0, this is what? un=1, okay. un=1 for all n greater than or equal to 0, okay. So we get here sigma n=0 to infinity yn+k z to the power -n, okay. Now n+k if I write as j, then what will happen? j will begin with n=0, okay. So j will begin with k and go up to infinity and here we shall have yjz to the power n=k-j, so -k-j, okay.

So we shall write it as z to the power -k*sigma j=k to infinity or it is j-k, okay. n=j-k, z to the power k sigma j=k to infinity yjz to the power -j. Now I can write it as z to the power k*sigma j=0 to infinity yjz to the power -j, when I subtract the terms from j=0 to k-1, yjz to the power j, okay. So this means that Z-transform of yn+kun=z to the power k and I replace this, this is Z-transform of yj sequence, okay.

Z-transform of yn sequence it is, so i write it as yz, okay. Then we have here yz-y0-y1/z-y2/z square and so on, -yk-1/z to the power k-1. So this is the advanced property. Z-transform of yn+kun is z to the power k*yz-y0-y1/z-y2/z square and so on, yk-1 z to the power k-1. So this is what we get, okay.

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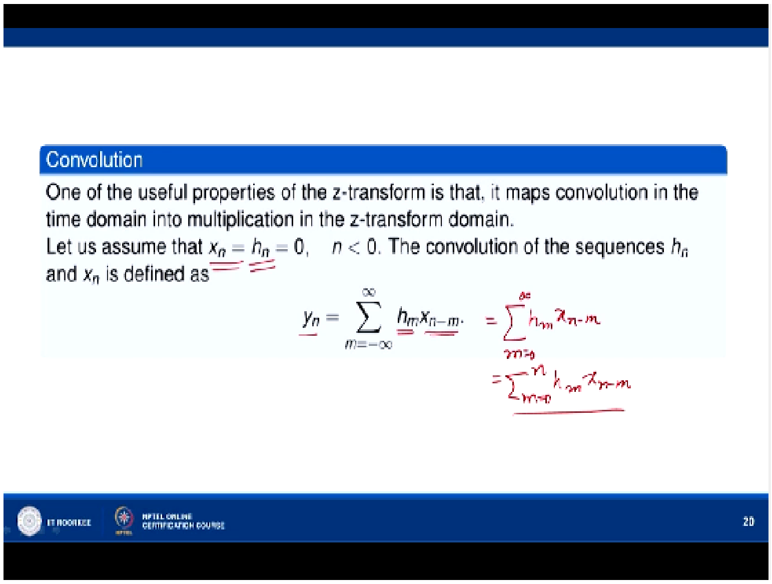

Now let us go to convolution property. So one of the useful properties of the Z-transform is that, it maps convolution in the time domain into multiplication in the time domain. If you have 2 sequences xn and hn which are defined as 0, when n<0, then the convolution of the sequences hn and xn is defined as yn=.

Convolution of hn and yn gives another sequence which is given by yn, yn=sigma m=-infinity to infinity hmxn-m. Now in this case xn=0 when n is negative and hn=0 when n is negative. So here hm will be 0 as long as m is negative. So this m will vary from 0 to infinity. hmxn-m, because of hm, okay, sum will be 0, I mean terms will be 0 here so long as m is taking negative values.

Now xn-m, okay, will be 0 whenever n<m, okay. So this means that m>n, okay. So this means that this is equal to sigma m-0 to n hmxn-m, okay. So if you assume that xn and hn are 0's for n<0, negative integer values of n, then yn=sigma m=0 to n hmxn-m because if m is more than n, okay, then n-m is negative, so xn-m is 0, okay.

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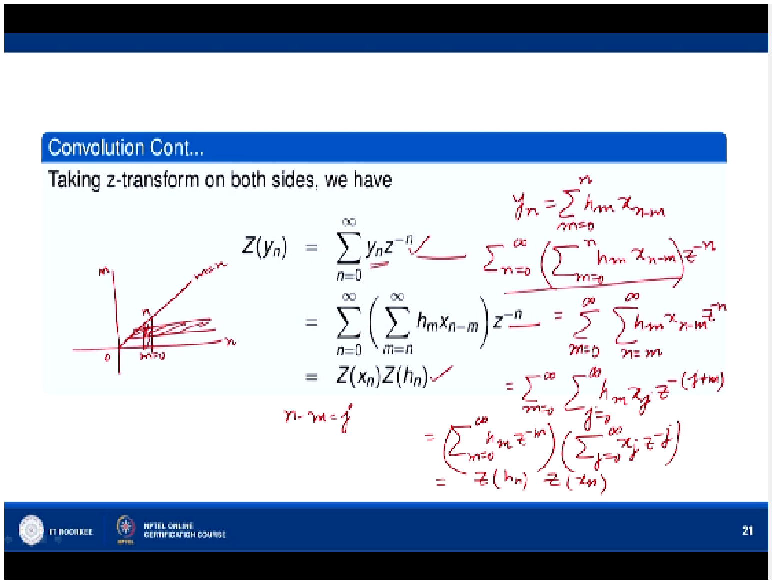

So now let us find the Z-transform of yn. We are going to show that Z-transform of yn is the product of the Z-transforms of the 2 sequences xn and hn, okay. So Z-transform of yn is sigma n-0 to infinity ynz to the power -n and let us now put the value of yn, okay. So yn if you put, yn will be equal to, sigma n=0 to n hmxn-m, okay. So this will be yn=sigma m=0 to n hmxn-m, alright.

So we have this, this gives you sigma n=0 to infinity sigma m=0 to n, okay, hmxn-mz to the power -n, okay. So this will be equal to, now let us see m depends on n, right. So this is your m=n, okay. m varies from, n is independent variable, n is here and m is here, okay. So m varies from 0 to n, okay. So we are taking this like this and then I change the order of integration. So I put it like this.

So if you do this, then we shall change the order of integration and we will get n varies now from m to infinity, so this is, can we put as n varies from m to infinity, okay. And m varies from 0 to infinity, okay. And we have hmxn-mz to the power -n, okay. This interchanging the order of the summation, we get this. Now put n-m=j, okay. So what we will have? Sigma m=0 to infinity and here we shall have n-m=j, so j varies from 0 and goes up to infinity, okay.

And we have hm, xn-m will be xj and z to the power -j+m, right. So this will be equal to sigma m=0 to infinity we can now separate it, we have hmz to the power -m and then we have summation j=0 to infinity xjz to the power -j, okay. Now this is Z-transform of hn, this is Z-transform of xn, okay. So we get Z-transform of xn*Z-transform of hn here, okay. So this is the proof.

So Z-transform, if we know the Z-transforms of 2 sequences xn and hn, okay, then the Z-transform of their convolution can be found. It is the product of their Z-transforms. Now this property of Z-transforms is used in the solution of difference equations. Also we use this property in the probability theory. Now let us discuss the initial value problem for casual signal.

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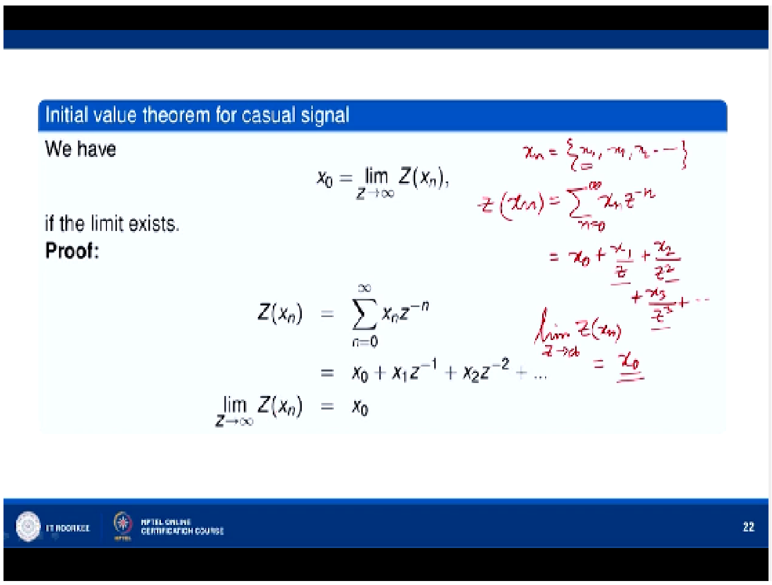

So let us say we want to prove that the initial value, the sequence we know is xn, the first term of the sequence is x0, okay, x0, x1, x2 and so on, okay. So we want to determine the initial value or initial term of the sequence that is x0. So the initial value theorem says that the initial value of the sequence x0 can be found if we take n tends to infinity and Z of xn, okay. So Z of xn is what? Z of xn is sigma n=0 to infinity, xnz to the power -n.

So this is x0+x1/z+x2/z square+, okay. So limit z tends to infinity Zxn, okay, =x0 because when z goes to infinity, 1/z goes to 0. So these terms all will be 0, will tend to 0. So x1/z, x2/z square, x3/z cube, all will tend to 0. So in the Z-transform of xn, when we take z to go to infinity, we get x0. So if the limit of Z of xn as z tends to infinity exist, then it will give us the value of x0, the initial value of the sequence. So this is the proof.

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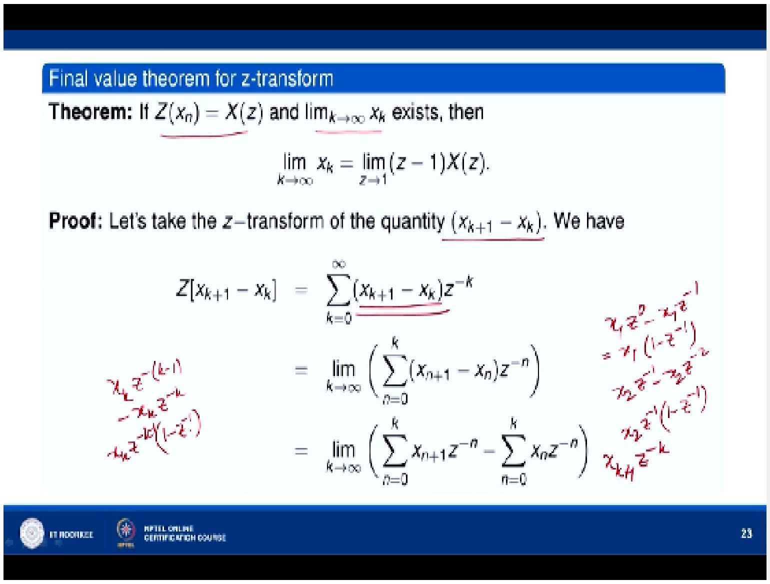

Let us go to next theorem which is final value theorem for Z-transform. If Z of xn=Xz and limit of xk as k goes to infinity, the final term of the sequence is X infinity. So that is why we call it as final value theorem. So final value X infinity can be found or limit of xk as k tends to infinity can be found form this formula, limit z tends to 1, z-1*Xz. Now let us, to prove this result, let us take the Z-transform of the expression xk+1-xk.

So Z-transform of xk+1-xk is sigma k=0 to infinity xk+1-xk*z to the power -k and this can be written as, now this xk+1-xk we can write in this form, limit k tends to infinity n=0 to k xn+1-xn*z to the power -n. Now here what is happening; we can now write this finite sum, these are finite sum, so we can write it as sigma n=0 to k xn+1z to the power -n -sigma n=0 to k xnz to the power -n. So then what we will do?

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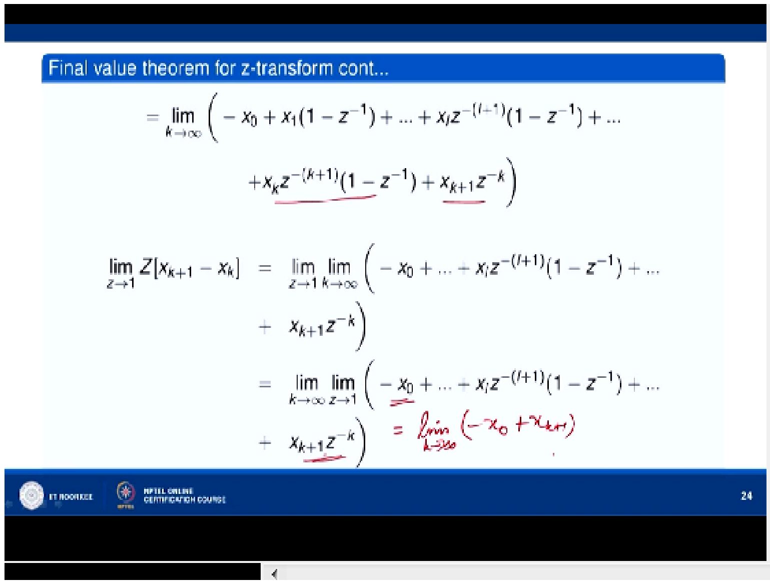

So we can write down this further as limit k goes to infinity, okay, and you put the values of n and first term will be -x0, okay. Here you will get when you put n=0, you get -x0. When you put n=0 here, what you get? x1z to the power -n and here when you put n=1, what you get? -x1. Here when you put n=0, you get x1z to the power 0, okay. When you put n=0 here, we get x1z to the power 0.

Here you get -x1z to the power -1. So this is equal to x1*1-z to the power -1. Similarly, we can put n=1 here. You get x2z to the power -1 and you will get here n=2, if you put, you get x2z to the power -2, okay. So this way we will get these terms. So at -x0, x1 1-z to the power -1, then the next term will be x2z to the power -1 1-z to the power -1, okay like this. So if we go like this, here what will happen?

xl*z to the power -l-1, okay. So like we had here, x2z to the power -1, okay. So when you put here l, okay, xlz to the power, we will have -l-1. So we will get this one, xlz to the power -l-1 when -z to the power -1 and then the term, last term is this one. Last term is xn+1. So last term will be xk+1z to the power -k and here what we will get? xkz to the power -k, you get xk+1z to the power -k and here you get -xk, just a minute, xk+1z to the power -k, that is the last term, okay.

And the term before that is, when you put n=k-1, so xkz to the power -k-1, okay and here what we will get? -xkz to the power -k. So xkz to the power -k*1-z to the power -1 we get. Here when we put here n=k-1, so we got this, okay. And n=k here gives you this, okay. So z to the power -k-1 we write, okay. So xkz to the power -k-1*1-z to the power -1, this is the term here, okay. And here we have xk+1z to the power -k.

So then limit z tends to 1 Z of xk+1-xk will give you limit z tends to 1 limit k tends to infinity-x0 x1z to the power -l+1 1-z to the power -1 and so on xn+1z to the power xk+1 z to the power -k. And this will give you, interchanging limits, limit k goes to infinity z goes to 1 limit z goes to 1 -x0+xlz to the power -l+1 1-z to the power -1 and so on xk+1z to the power -k. And then when z goes to 1, these terms will all go to 0, okay.

Those who contain 1-z to the power -1 as a factor, only these 2 terms we will get. This term and this term, okay. So what we will get? Limit k goes to infinity, this will become -x0 and then we have xk+1, okay.

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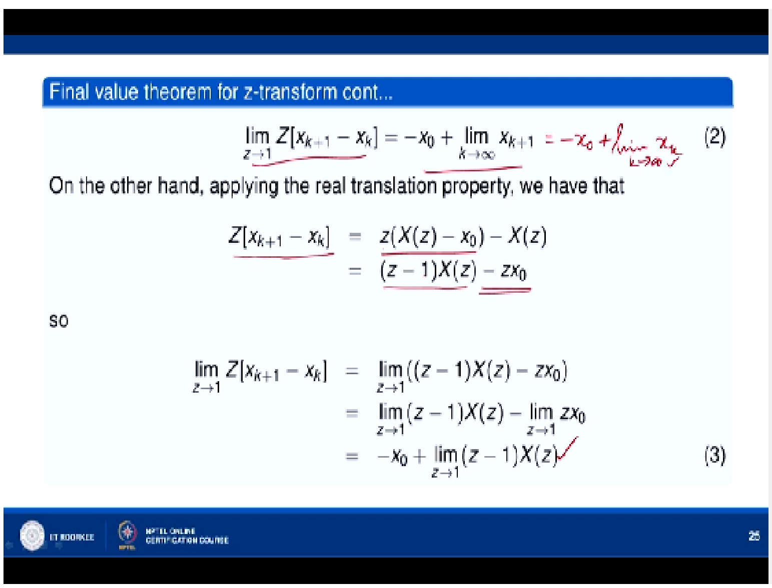

So what we get is this. -x0 limit k goes to infinity xk+1, this is the value of limit of zk+1 at Z of xk+1-xk as z goes to 1. Now on the other hand, if you find Z of xk+1-xk. By the advanced property, let us apply this advanced property, this one, okay. Z of yn+k we have found. The Z of yn+k is zk*yz-sigma m=0 to k-1. So here if you want Z of yn+1, okay, take k=1, so Z of yn+1, when n is greater than or equal to 0, un is 1.

So we write un as 1, then this is z to the power 1yz-, when k is 1, so m is 0, so we get y0z to the power 0, okay. So we get z*yz-y0. Now in our case, it is the sequence xn+1, xk+1, Z of xk+1 will be z*Xz-x0. So we get this here, okay. Z-transform of xk+1 is z*Xz-x0, okay by the advanced property. And Z of xk is Xz. So this is z-1*Xz-z*x0. This is z*x0. Now what we have, okay?

So limit z tends to 1 Z of xk+1-xk, okay, this will be equal to this value, when your z goes to 1, this value will tend to -zx0, okay because this will tend to 0. So limit z tends to 1, Zxk+1-xk is limit z tends to 1 z-1Xz-zx0 which is limit z tends to 1 of this expression-limit z tends to 1 of this expression, so we get -x0+limit z tends to 1, okay, this will not go to 0 because Xz may contain z-1 in the denominator. So this is limit z tends to 1 z-1 Xz. So what we will get? This value now can be put as -x0+limit k goes to infinity xk+1, okay.

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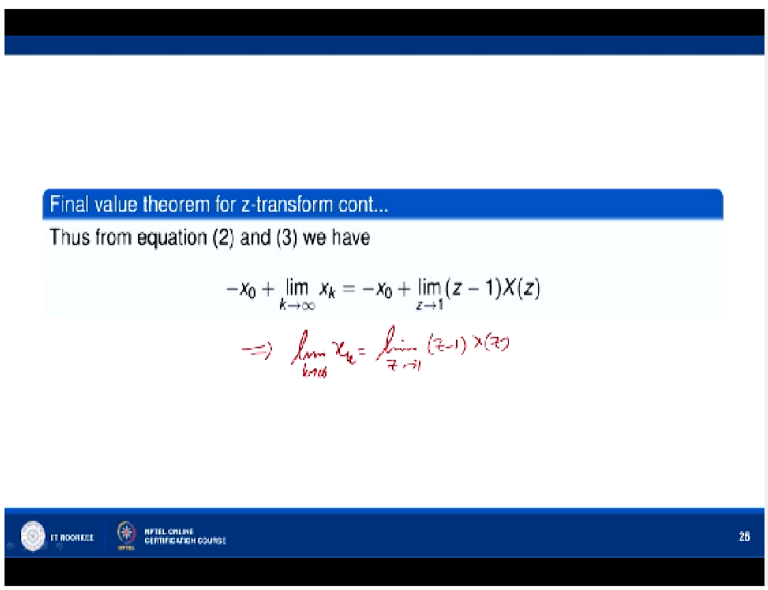

So what we will have is? and limit of xk+1 as k goes to infinity is same as limit of xk, so I can also write it as -x0+limit k goes to infinity xk, okay. So this value is equal to this value and therefore, this is equal to this. So we get this result here. -x0+limit k goes to infinity xk=-this and this implies that limit k goes to infinity xk=limit z tends to 1z-1Xz, okay. So this is the final value theorem.

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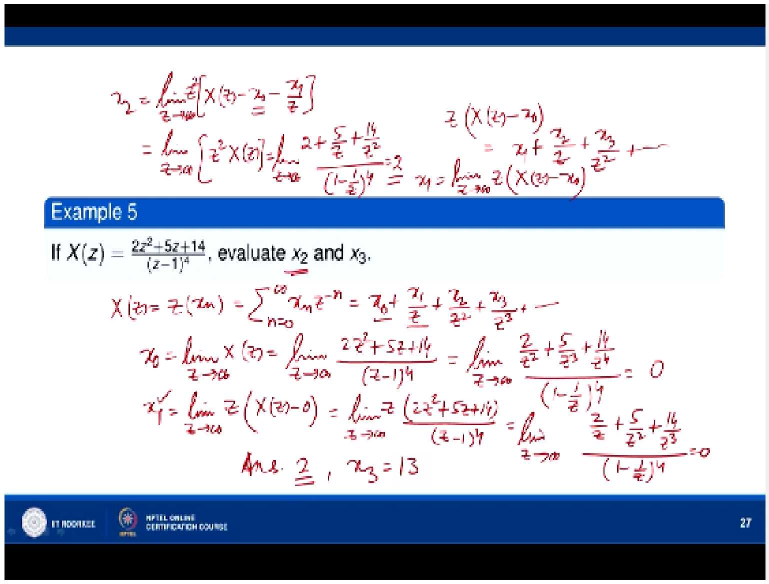

Now let me take an example on this initial and final value theorem. Here we are given the Z-transform of a sequence Xz=2z square+5z+14/z-1 to the power 4. We want to determine x2, okay. So x2 is, let us look at the result. limit k goes to infinity xk, okay. So limit, in order to find x2, we shall have to determine x0 first, okay, the initial, this one we will have to determine first, okay.

So we have to go there. See Xz is what? Xz is Z-transform of xn, okay. So this is sigma n=0 to infinity xnz to the power -n. So this is x0+x1/z+x2/z square and so on, x3/z cube and so on, okay. Let me find x0 first. x0=limit z tends to infinity, okay, yes. We can do like this. See x0=limit z tends to infinity Xz, okay. So this will be now limit z tends to infinity 2Xz square+5z+14/z-1 to the power 4, okay.

Now in the denominator we have power of z 4, in the numerator we have power of z as 2. So dividing by z to the power 4 in the numerator and denominator, we write limit z tends to infinity 2/z square+5/z cube+14/z to the power 4 and denominator we have 1-1/z to the power 4, okay. Now when z goes to infinity, 1/z goes to 0. So we get 0, okay. And then we find x1. Limit z tends to infinity.

Now what we will do? From Xz, we subtract x0 and then multiply by z. So we get Xz-x0*z, then what we get? x1+x2/z x3/z square and so on. So we can determine x1, if we take the limit of this expression. So x1=limit z tends to infinity z*Xz-x0, okay. So x1=limit z tends to infinity z*Xz-, x0 is 0. So limit z tends to infinity z*, again we put the value of Xz, 2z square+5z+14/z-1 to the power 4.

So again we divide by z to the power 4. In the numerator, the power of z, highest power of z is 3, okay. So we will have limit z tends to infinity 2/z 5/z square and then we have 14/z cube/1-1/z to the power 4. And again we get the limit 0, okay. So x1 is also 0. Now we can find x2. x2 can be found if from Xz we subtract x0, then x1/z and multiply by z square and take the limit as z tends to infinity.

So x2=limit z tends to infinity Xz-x0-x1/z, okay. So we subtract x0, we subtract x1/z and then multiply by z square and take the limit as z tends to infinity, so we will get x2. So this is now limit z tends to infinity. Now x0 is 0 here, okay. x1 is 0 here. So we get limit z tends to infinity Xz and you can see here Xz, multiply by z square here, okay. So we have to multiply by z square, okay.

So limit z tends to infinity. When you multiply this expression by z square, the power of z4 becomes 2. In the denominator, power of z4 is 1, okay. So we get, when you divide by z to the power 4 in the numerator and denominator and take the limit as z tends to infinity, we will get the limit as 2. So this is z square after you multiply and divide by z to the power 4, you get 2+5/z, okay, and then you have 14/z square/1-1/z to the power 4.

And then you take the limit as z goes to infinity, you get the limit as 2, okay. So answer is 2 here, okay. x2 is 2. And now we have to determine x3. So again what we will do? We subtract from Xz, x0, x1/z, x2/z square and then multiply by z cube, okay.

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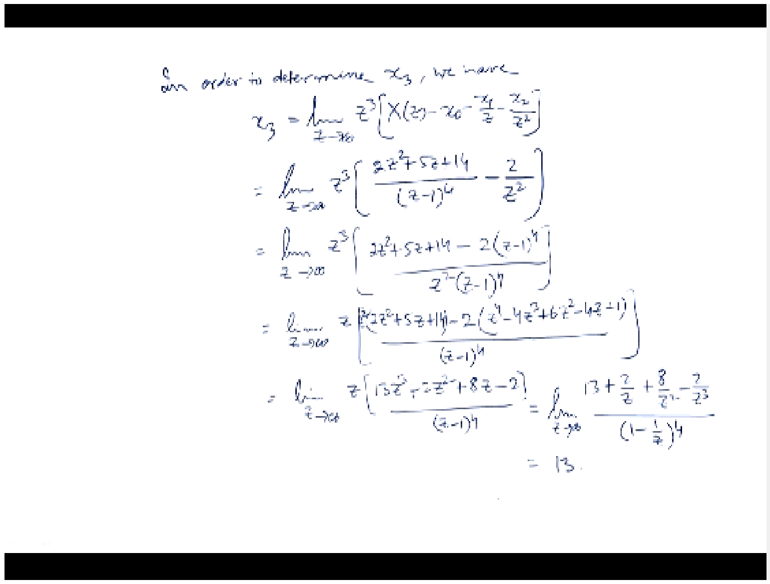

So in order to determine x3, we have x3=limit z tends to infinity z cube Xz-x0-x1/z-x2/z square, okay. So this is limit z tends to infinity z cube, Xz is this, 2z square+5z+14, z-1 to the power 4. x0 is 0, x1 is 0, x2 is 2. So -2/z square, okay. x2 is 2. So this gives you limit z tends to infinity z cube*2z square+5z+14-2*z-1 to the power 4, okay which is equal to limit z tends to infinity z times 2z square+5z+14-2*z to the power 4-4z cube+6z square-4z+1.

And here we will multiply by z square. Let me multiply by z square. So we have this is equal to limit z tends to infinity z*, 2z to the power 4 will cancel with 2z to the power 4, we have 5z cube here and here we have 8z cube, so 13z cube. And we have 14z square-12z square, so 2z square. 2z square+8z-2/z-1 to the power 4, okay. So this is equal to limit z tends to infinity. When we divide by z to the power 4, we get 13+, 2 z cube (()) (01:03:18) to the power 4, so 2/z+8/z square-2/z cube/1-1/z to the power 4.

So when z goes to infinity, the limit comes out to be 13. And therefore, the value of x3 is equal to 13. So x2 is 2, x3 is 13. So this is the example on the initial and final value theorem. With this, I would like to end my lecture. Thank you very much for your attention.