Advanced Engineering Mathematics

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Lecture – 32
Application of Conformal Mappings to Potential Theory

Hello friends. Welcome to my lecture on application of conformal mappings to potential theory. Suppose that D is a domain in the z plane and that g is a function defined on the boundary C of D. Let us say we have a region D in the z plane, okay.

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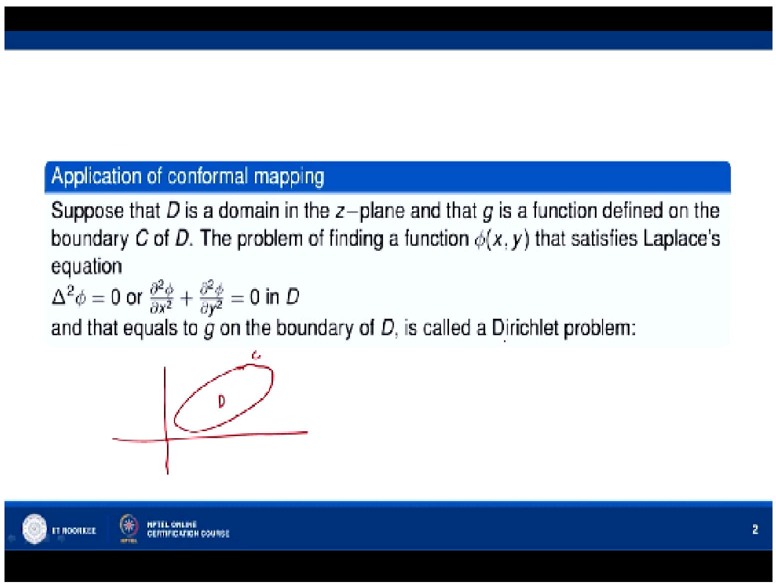

And C is its boundary, okay. So let us say C is its boundary. We have to find the function phi which satisfies the Laplace equation, del square phi=0 or phi xx+phi yy=0 in D and that equals to g on the boundary of D. Such a problem is known as Dirichlet problem. We will consider a Dirichlet problem and we shall solve the Dirichlet problem using the conformal mapping.

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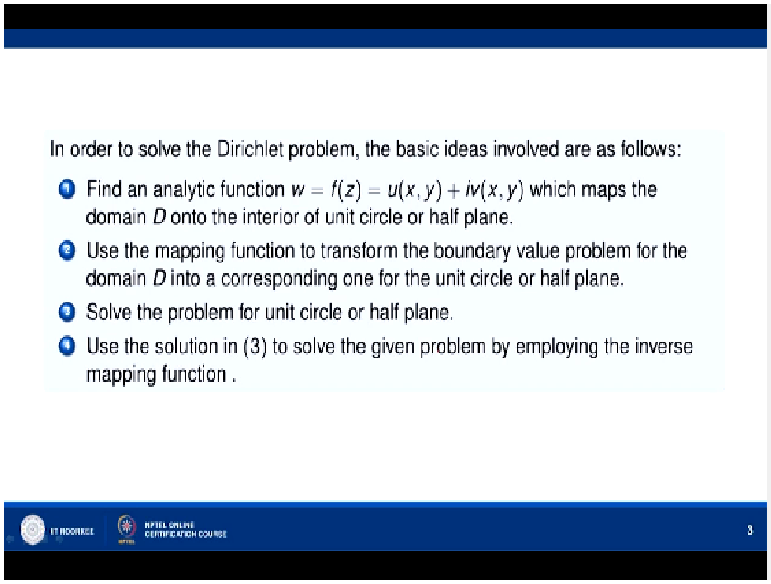

So in order to solve the Dirichlet problem, the basic ideas involved are, we have to find an analytic function w=fz=uxy+ivxy which maps the domain D on to the interior of unit circle or half plane. And then we use the mapping function to transform the boundary value problem, given boundary value problem for the domain D into a corresponding boundary value problem for the unit circle or half plane.

We shall solve the boundary value problem for unit circle or half plane and then use the solution to obtain the solution of the given problem by employing the inverse mapping function.

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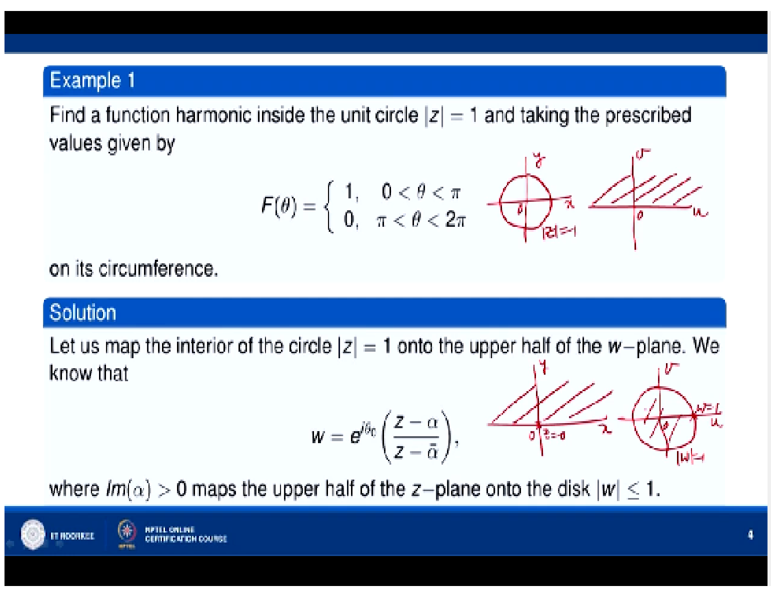

Now let us take example 1 where we have to find function which is harmonic inside the unit circle mod of z=1 and takes the prescribed values given by F theta=1 when theta lies between 0 and pi and 0 when theta lies between pi and 2pi on its circumference. So let us map the interior of the circle mod of z=1 on to the upper half of the w plane, okay. So we have, this is your circle mod of z=1 in the z plane and we have to map it to upper half of the w plane, okay.

Now so this is what we will do first. We know that w=e to the power i theta 0 z-alpha/z-alpha conjugate maps the upper half of the z plane on to the disc in the w plane. If this is your upper half plane, then it maps this upper half plane into mod of w less than or equal to 1, okay, if w is given by e to the power i theta 0*z-alpha/z-alpha conjugate, where imaginary part of alpha>0. So this maps upper half of z plane on to the disc mod of w less than or equal to 1.

This we have discussed earlier. So we are using this transformation. Now here we have circle in the z plane and upper half plane in the w plane. So this will interchange z and w here to get the required mapping. So let us first consider a particular bilinear transformation which maps the upper half of the z plane on to the disc mod of w is less than or equal to 1. Then we shall interchange z and w to arrive at the transformation which will map the region mod of z less than or equal to 1 on to the upper half of the w plane.

So let us first consider a particular value say alpha. Alpha is such that imaginary part of alpha>0.

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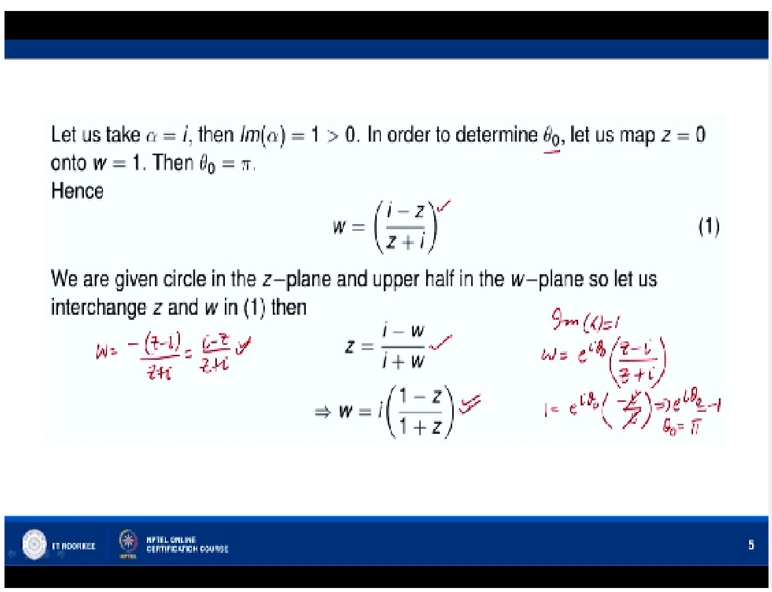

So I take alpha=i. Then imaginary part of alpha=1. So imaginary part of alpha=1>0. So then the transformation becomes w=e to the power i theta 0 z-i/z-i conjugate. i conjugate is -i, so z-i/z+i, okay. Now what we do? Let us map; now in order to determine this theta 0, we take a point on the boundary, the boundary is the real axis here. So let us take the point to be z=0 and we have to map it to a point on the boundary of the region here in this case, that is mod of w=1.

So that I take as w=1. So that point is here, w=1. So let us map z=0 to w=1, what we get? 1=e to the power i theta 0, then we have -i/i. So we get here e to the power i theta 0=-1, that means theta 0=pi, okay. So then what will happen? w will be equal to e to the power i theta 0 is -1, so -z-i/z+I and we get i-z/z+i. So this transformation maps the upper half of the z plane on to the disc mod of w less than or equal to 1.

Now as I said earlier, here we have the region circle in the z plane and upper half plane in the w plane, so we have to interchange z and w here. So let us interchange z and w in this mapping, okay. So then we get z=i-w/i+w and when we simplify this, for w, we get w=i*1-z/1+z. So this transformation now maps the disc mod of z less than or equal to 1 on to the plane v greater than or equal to 0.

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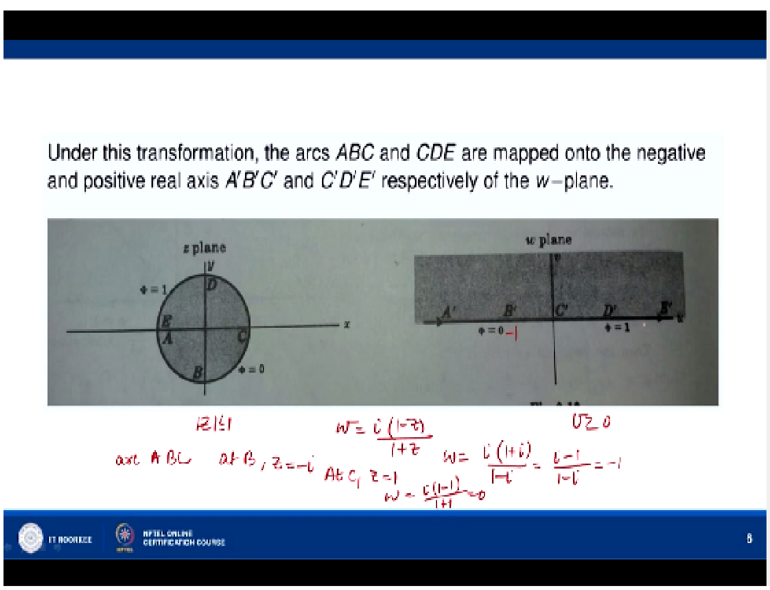

Now under this transformation, the arcs ABC and CDE, okay, so what we have now, let us see we have w=i*1-z/1+z, okay. So under this transformation, the mod of z less than or equal to 1 is mapped to v greater than or equal to 0, okay. Boundary mod of z=1 is mapped to the boundary here, that is v=0. Now let us see which part of the circle maps to which part of the boundary here, that is v=0.

So we notice that ABC, if you look at ABC, the arc ABC, the arc ABC at the point v we have z=-I, okay. Let us see where this v goes? So when you put z=-I here, what do you get? w=i*1-i/1+i, okay. We have z=-i, so this will be -I here. So this will be, okay and this will be -, okay. Now what we have? i-1/1-i which is equal to -1. So this v goes to v dash, okay. This v dash is -1, okay.

And at C, what you have? z=0. If you put z=0, what you get? No, at C, z=1, okay. At C, z=1, so what comes here? w=i*, okay, z=1 means 1-1/1+1. So this is equal to 0. So this is mapped with, the C is mapped to C dash, the origin of the w plane. This means that the lower arc ABC, the point A here. The point A goes to A dash that is infinity, okay. The point A goes to; at the point A, z=-1, okay, z goes to w=infinity, so A goes to A dash.

So ABC is mapped to A dash B dash C dash and CDE is mapped to C dash D dash E dash. We can easily check that D goes to D dash by putting the value of D, the complex number representing D point is z=i, so you put z=i here, you get this w=1 here. So this D is mapped to D dash. And E again goes to infinity, so E dash. So ABC is mapped to A dash B dash C dash. And CDE is mapped to C dash D dash E dash of the w plane.

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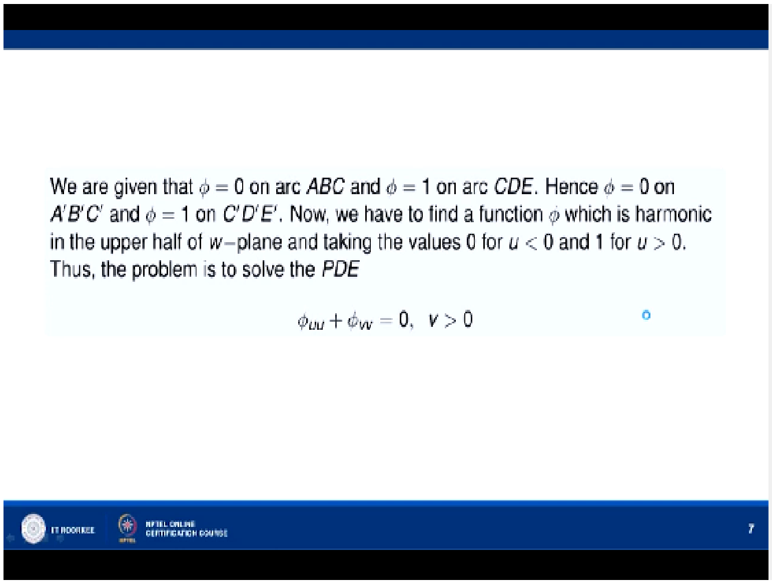

Now we are given that phi=0 on arc ABC. We are given that; we can see that. We are given that phi=0 when theta>0 but less than pi, that is on the upper arc and when theta varies from pi to 2pi, it is 0. So we have pi=0 on the arc ABC, okay, on this arc. This we have theta varies from pi to 2pi. So phi is 0 on the arc ABC. ABC is mapped to A dash B dash C dash. So phi is 0 here on the side negative real axis of the w plane and this one, CDE, okay.

On CDE, phi=1. So it is phi=1 on the positive u axis of the w plane, okay. So phi=0 on A dash B dash C dash and phi=1 on C dash D dash E dash. Now we have to find a function phi which is harmonic in the upper half of w plane and taking the value 0 for u<0, that is negative u axis and 1 for positive u axis. Thus the problem is to solve the Laplace equation phi uu+phi vv=0, where v>0. v>0 represents the upper half of the w plane.

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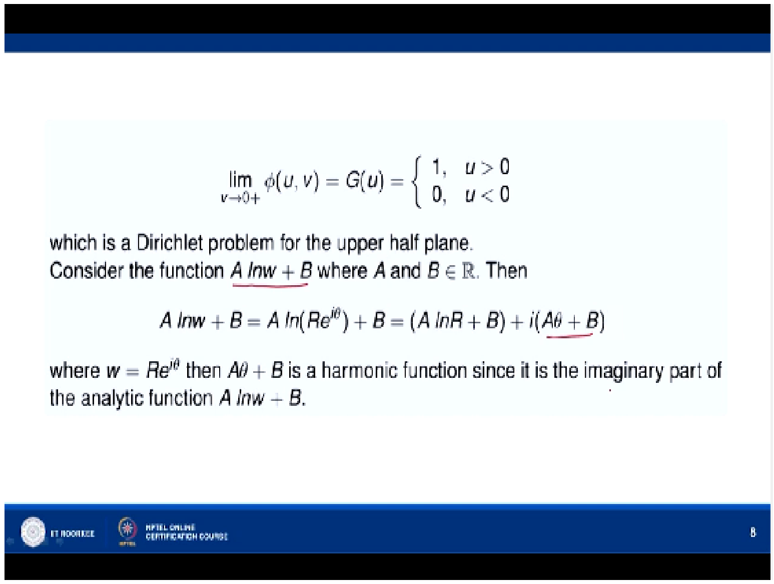

Now and the condition on phi uv is that when v goes to 0, phi uv=Gu and Gu is 1 when u>0 on the right half, positive u axis. Gu should take value 1 and Gu should be 0 when u<0. Now this is the Dirichlet problem for the upper half of the w plane. Now consider the function Alnw+B, okay. Consider this function Alnw+B where A, B belong to R. Then Alnw+B is A*lnRei theta. Let us put w=Rei theta.

So AlnRei theta+B which is AlnR+B+i*A theta+B. Now Alnw+B we know is an analytic function, okay. So its real and imaginary parts are harmonic functions. And therefore, A theta+B which is imaginary part of Alnw+B is a harmonic function. So A theta+B is a harmonic function as it is the imaginary part of the analytic function Alnw+B, okay.

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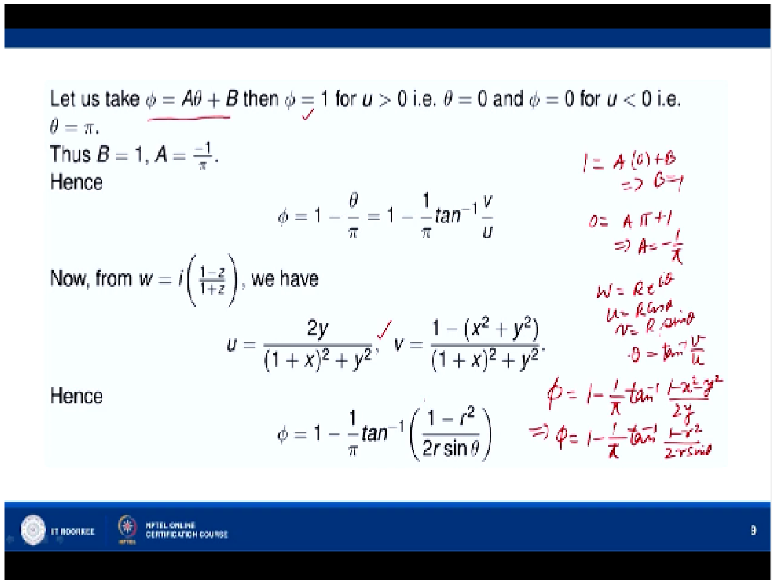

Now let us take phi=A theta+B. So as we said earlier A theta+B is harmonic because it is the imaginary part of Alnw+B. So let us take phi=A theta+B, then phi is harmonic and let us put the condition on phi that phi=1 for u>0 and u>0 means theta=0 because we are writing w as Rei theta, okay. So theta=0 gives you positive u axis. So that is theta=0 and phi=0 for u<0 meaning thereby theta=pi, negative u axis.

Now if you do that, then what you will get? Phi=1 when theta=0, okay. So A*0+B, so we get v=1, okay. So this condition gives you B=1 and then phi=0 when theta=pi. So A pi+1, okay, B is 1. So this gives you A=-1/pi, okay. So we get the values of A and B both. So phi=1-theta/pi from here, okay. And theta, because w=Re to the power i theta, we have put. So u=R cos theta and v=R sin theta.

So this gives you theta=tan inverse v/u, okay. Now we have the transformation w=i*1-z/1+z, this transformation. So from here, let us find the value of u and v, okay. So you put w=u+iB z=x+iy, then simplify this and equate the real and imaginary parts, we get u=2y/1+x whole square+y square and v=1-x square+y square/1+x whole square+y square.

So v/u will give you phi as 1-1/pi tan inverse, v=1-x square-y square, so 1-x square-y square/u, means 2y, okay. Now let us convert this to polar coordinates. So let us take x=r cos theta y=r sin theta. So this will give you phi=1-1/pi tan inverse 1-r square/2r sin theta, where x is r cos theta, y is r sin theta. So this is the solution of the given problem.

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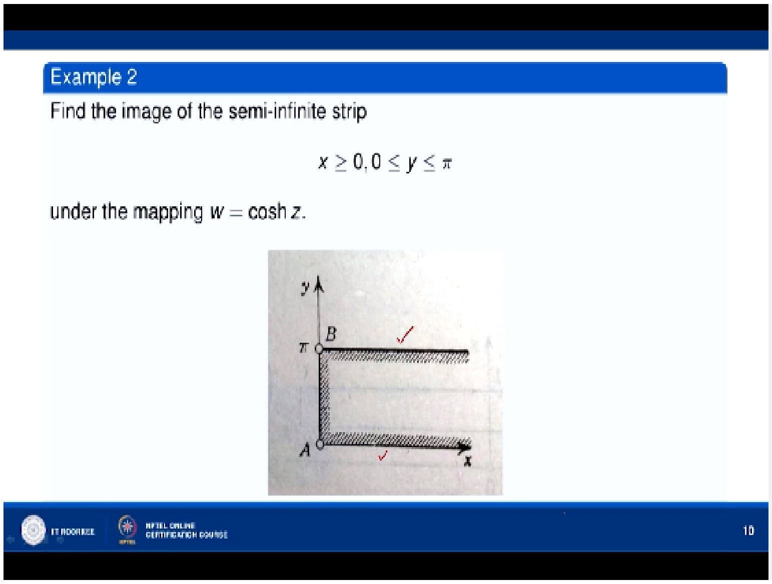

Let us take another example. Let us find the image of the semi-infinite strip. We are given the semi-infinite strip which is defined by the inequalities x greater than or equal to 0 and y varies from 0 to pi, okay under the mapping w=cos hyperbolic z.

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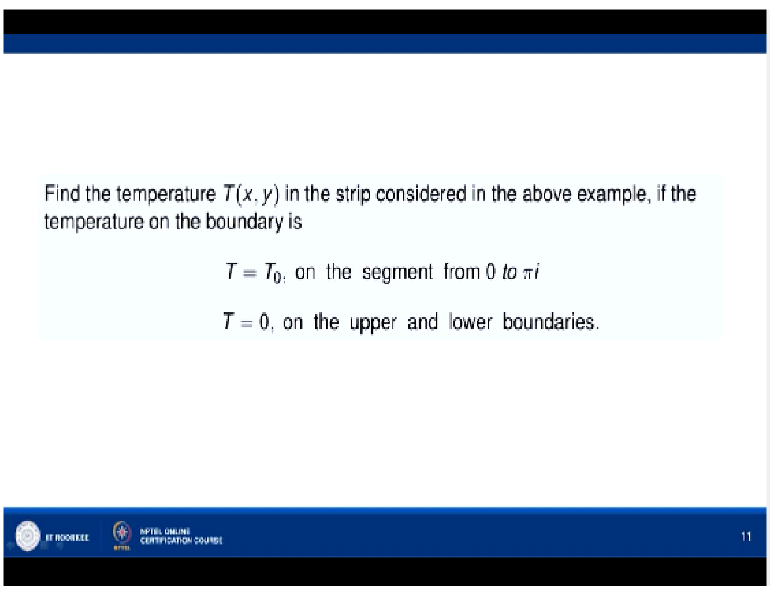

And the next, the second part of the problem is that we have to find the temperature Txy in the strip considered here, okay if the temperature on the boundary is T=T0 on the segment from 0 to pi i. So on this segment, okay, on the segment AB, temperature is given by T=T0 while temperature is 0 along this x axis and temperature 0 along the line this y-pi x greater than or equal to 0.

So this boundary and this boundary, okay. On this boundary and this boundary, temperature is 0 while here the temperature is T0, okay. So first let us see what is the image of this semi-infinite strip region bounded by the semi-infinite strip under the mapping w=cos hyperbolic z. Then we will take up the second part of the problem, okay.

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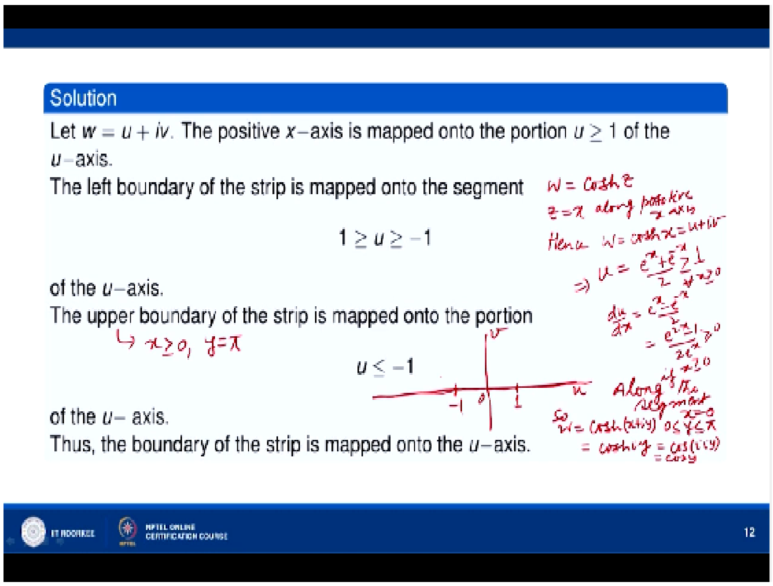

So let us say w=u+iv, okay. So first let us see where this positive x axis, this part, okay is mapped under the mapping w=cos hyperbolic x, okay. So w=cos hyperbolic z, okay, when we are looking at the image of the positive x axis, positive x axis means y=0, so z=2x, okay along positive x axis. And hence w will be equal to cos hyperbolic x, okay. Now x is greater than or equal to 0 along the positive x axis.

And so cos hyperbolic x is e to the power x+e to the power -x/2, okay. When x is greater than or equal to 0, it is greater than or equal to 1, okay. And moreover, cos hyperbolic x is a real quantity, so if you put w=u+iv, then u=cos hyperbolic x or you can say u=e to the power x+e to the power -x/2. So it is always greater than or equal to 1 for all x greater than or equal to 0, okay. And moreover, you can see du/dx, du/dx is e to the power x-e to the power -x/2, okay.

So this gives you e to the power 2x-1/2*e to the power x. So this is greater than or equal to 0 when x is greater than or equal to 0, okay. This means that the u monotonically increases, increases monotonically as x increases. So positive u axis is mapped on to the portion u greater than or equal to 1 in the w plane.

This is your u greater than or equal to 1. So on this part, positive axis is mapped on to u greater than or equal to 1. Now let us look at the left boundary of the strip, okay. Left boundary of the strip means this segment, okay, from 0 to pi, okay where y varies from 0 to pi. So let us look at this segment, okay.

So along the segment x=0 and 0 is less than or equal to y less than or equal to pi, okay. Now so w=cos hyperbolic z. So it is x+, cos hyperbolic z means cos hyperbolic x+iy, okay, x=0, so this is cos hyperbolic iy, okay. And cos hyperbolic iy is cos i of iy, okay. So cos -y which is cos y, okay. So w=cos y which means that u=cos y because y is real, so cos y is real. So we can write.

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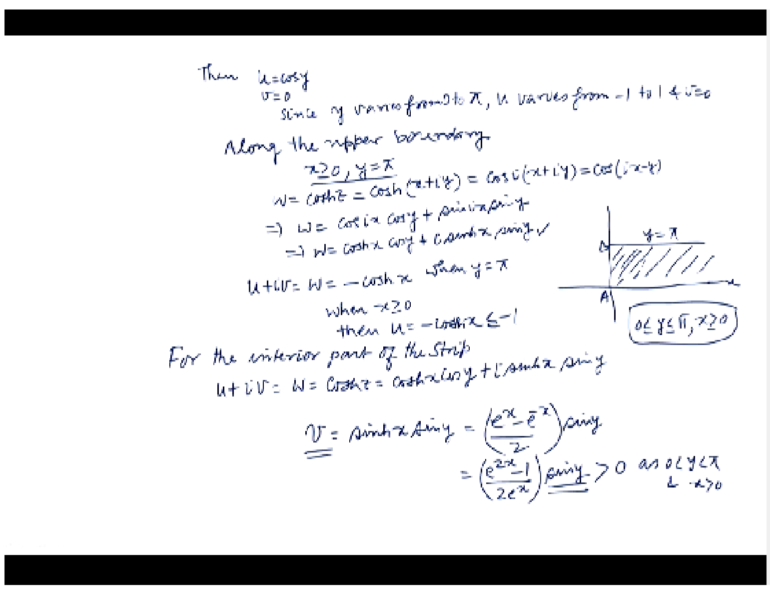

Then u=cos y, okay and v=0, okay. This now y varies from 0 to pi, okay. So u=cos y will vary from -1 to 1. So since y varies from 0 to pi, okay, u varies from -1 to 1, okay. And v=0. So that means, this part, the left boundary, okay, this left boundary, this boundary, okay, is mapped on to the segment -1 to 1, okay, on this segment, okay. When you reach the point pi i, pi i here, this pi i is mapped to this point -1, because that pi i, y=pi and when y=pi, cos pi gives you -1.

And this A point is mapped on to this 1. So we get 1 greater than or equal to u greater than or equal to -1. So this -1 to 1, this, the image of the left boundary of the strip. Now let us look at the upper boundary of the strip. Upper boundary of the strip is this one, this upper boundary. So along the upper boundary, we have y=pi and x is greater than or equal to 0, okay. So upper boundary means x is greater than or equal 0 and y=pi, okay.

Then let us see how we get u less than or equal to -1, okay. So along the upper boundary, x is greater than or equal 0, y=pi, okay, w=cos hyperbolic z, so we get cos hyperbolic x+iy. Now cos hyperbolic z=cos iz. So we get cos i*x+iy. So this is equal to cos ix-y, okay. And this gives you w=cos ix*cos y+sin ix*sin y. And which gives you w=cos x, cos ix is cos hyperbolic x. Cos hyperbolic x cos y+, sin ix is i sin hyperbolic x.

Now let us put the condition, this one, x greater than or equal to 0, y=pi. So when y=pi, what happens? Sin pi=0, okay. So w becomes-cos hyperbolic x we get when y=pi, okay. Now we are given that x is greater than or equal to 0. So when x is greater than or equal to 0, okay, cos hyperbolic x is greater than or equal to 1. So -cos hyperbolic x is less than or equal to -1. So then, now this is u+iv. u+iv=-cos hyperbolic x.

But -cos hyperbolic x is real quantity. So u=-cos hyperbolic x. So then u is equal to, less than or equal to -1, okay. So the upper end boundary of the strip is mapped on to this part, okay. So this means that the boundary of the strip is mapped on to the u axis in the w plane, okay.

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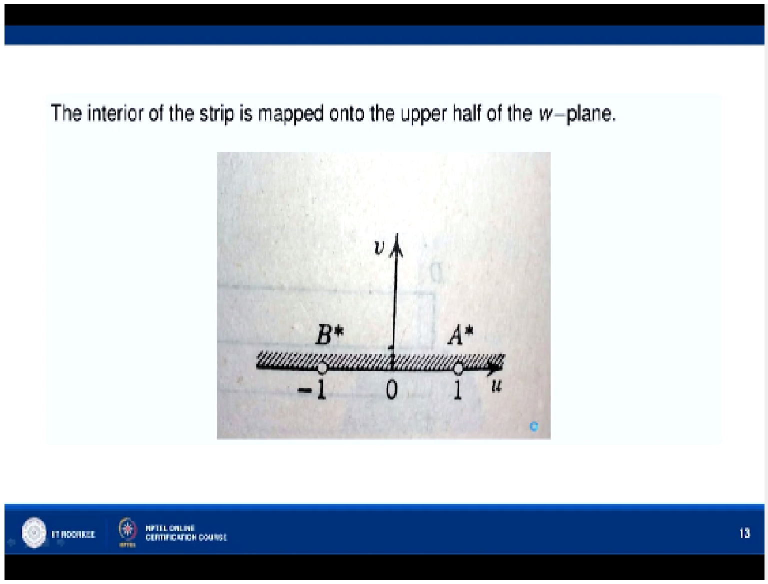

Now let us look at the interior of the strip, okay. The interior of the strip, this one, let us see where the interior of the strip goes, okay. So we want to show that the interior of the strip is mapped on to the upper half of the w plane, okay. So for the interior part, let us see again w=cos hyperbolic z. We have seen that cos hyperbolic z gives you cos hyperbolic x cos y and i sin hyperbolic x sin y, okay. w=u+iv.

So this gives you v=sin hyperbolic x*sin y, okay. Now sin hyperbolic x is e to the power x-e to the power -x/2*sin y. So this is e to the power 2x-1/2*e to the power x*, okay. Now let us see how the region bounded by this semi-infinite strip is defined, okay. This is your A B and this is your y=pi, okay. So 0 is less than or equal to y less than or equal to pi we have and x is greater than or equal to 0.

So this is how we are defining the region bounded by the strip, okay. So for interior region, 0<y<pi and x>0. So when y is lying between 0 and pi, sin y is positive and x>0 means sin hyperbolic x is positive. So this is greater than 0, okay. As 0<y<pi and x>0. So this portion, okay, the region bounded by the semi-infinite strip, that is the interior part of the strip goes to v>0. So the region bounded by the semi-infinite strip is mapped on to the upper half of the plane, w plane under the mapping w=cos hyperbolic z, okay.

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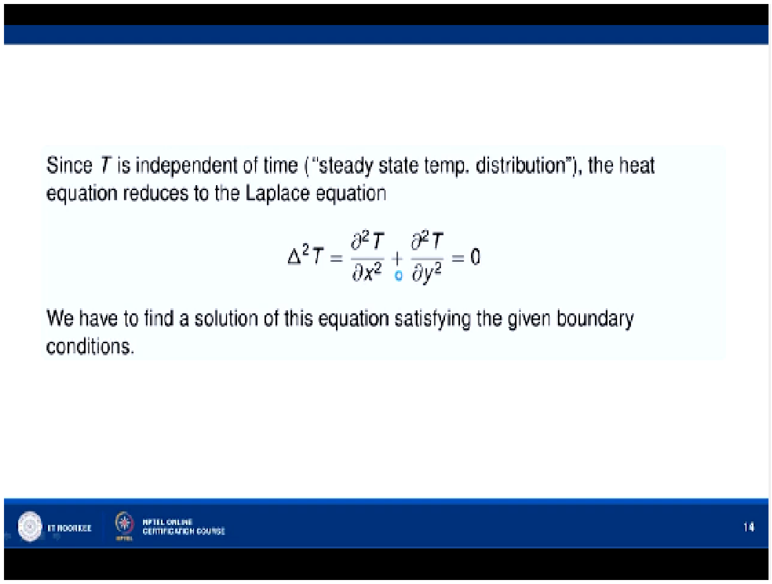

Now let us solve the second part of the problem, okay. The second part of the problem is that we have to find the temperature distribution in the semi-infinite strip which we have considered here, okay, the semi-infinite strip and the temperature on the boundary is given. It is T0 along the left part of the boundary and along the other 2 sides, upper and lower boundaries it is 0. So let us solve the Laplace equation del square T/del x square+del square T/del y square=0.

We have to find the solution of this equation satisfying the given boundary conditions. So the boundary conditions are T=T0 on the segment from now. Under the mapping w=cos hyperbolic z, we have seen this part, left end of the boundary, okay, left part of the boundary is mapped on to the segment -1 to 1. This is mapped on to u greater than or equal to 1. This is mapped on to u less than or equal to 1, okay.

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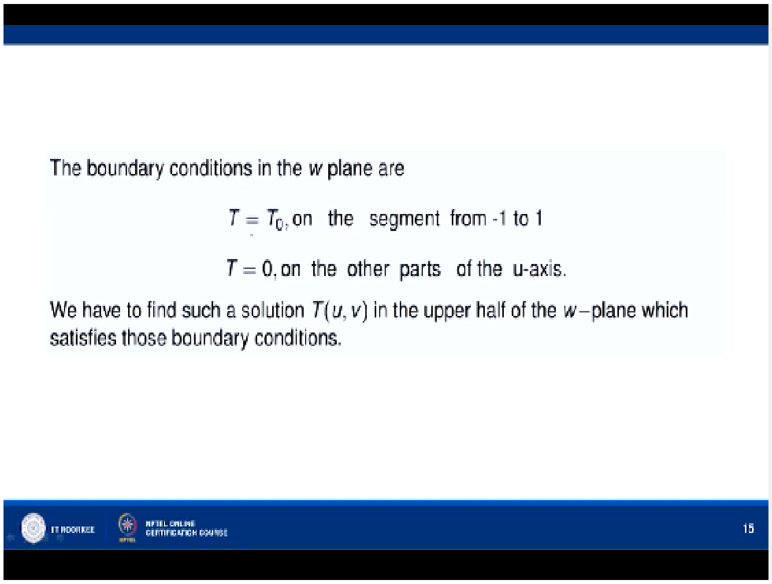

So making use of that, we can say that T=T0 on the segment from -1 to 1 and T=0 on the other parts of the u axis. So we have to find a solution Tuv in the upper half of the w plane which satisfies those boundary conditions. Let us consider the functions Ln w+1, this is principal value of the logarithm.

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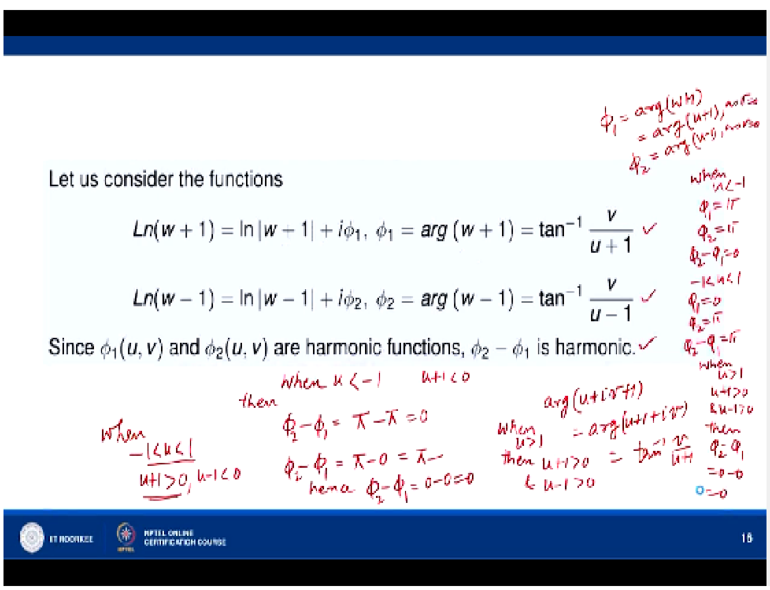

So Ln w+1=ln mod of w+1+iphi1 where phi1 is argument of w+1 and w=u+iv, so argument of u+iv+1=argument of u+1+iv which is equal to tan inverse v/u+1. So we get this argument. Now here Ln w-1 be considered which is ln mod of w-1+iphi2 in a similar manner as in the case of Ln w+1, we get phi2=argument of w-1=tan inverse v/u-1. Since phi1 and phi2 are imaginary parts of this analytic function Ln w+1 and Ln w-1, phi1uv and phi2uv are harmonic functions. And when phi1 and phi2 are harmonic, then phi2-phi1 is also harmonic function, okay.

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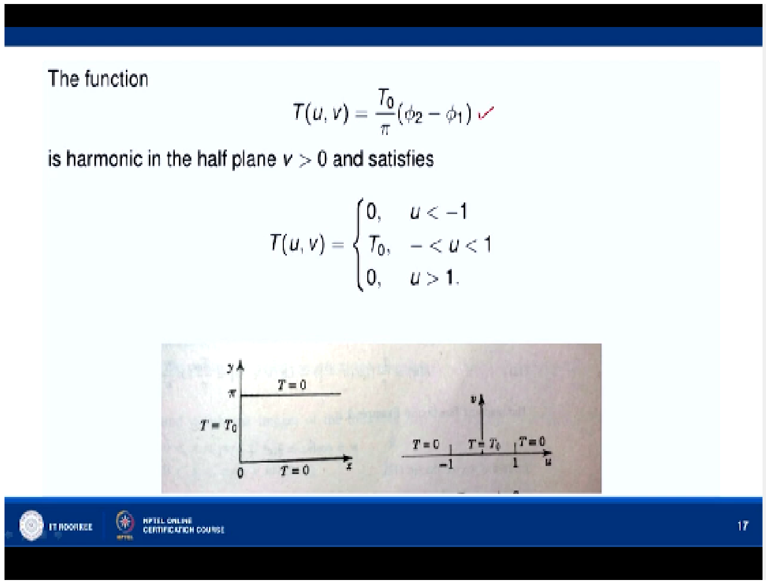

Now let us consider this function, Tuv=T0/pi phi2-phi1, okay. Since phi2-phi1 is harmonic, Tuv=T0/pi phi2-phi1 is also harmonic. Now let us see when u<-1, how Tuv=0, okay. So when u<-1, okay, then what happens to phi2-phi1, okay? So u<-1 means u+1<0, okay. So u+1<0, okay. This means that phi, the argument of phi2 is pi and argument of phi1 is also pi, okay. So u+1 is negative means phi2 is pi and this phi1 is also pi.

So we get pi-pi=0. Because when u<1, u+1<0 and u-1<0, so phi2-phi1=pi-pi=0. And when you take -1<u<1, okay. So when u is lying between -1 and 1, okay. When -1<u<1, okay, then what happens? u+1>0 while u-1<0, okay. So u+1>0 means the phi2 of u-1<0, u-1<0 means it is, okay. This is phi2, right. So phi2 is, okay. So u-1<0 means phi2 is pi, okay. So phi2 is pi and phi1=0. So phi2-phi1=pi-0=pi, okay.

So in this case, when u lies between -1 and 1, phi2-phi1==pi and when u>1, then what will happen? u+1>0, when u>1. Then u+1, okay, is greater than 0 and u-1 is also greater than 0, okay. Hence, phi2-phi1, okay, is equal to; phi2 means u-1>, so each one is equal to 0, 0-0=0, okay. Because u-1 is positive, u+1 is positive, so phi2-phi1 will be 0-0, that is equal to 0. This means that on the segment -1 to 1, we get pi.

On the other segments, on the other portions of the u axis, that is -1<u<1, u<-1 and u>1, okay. Actually what we are doing is, we are phi1, when you are looking at the conditions here, phi1=argument of w+1, okay. We are looking at these cases. What happens when we consider the part of this, when u<-1. u<-1 means, v=0, so this is argument of u+1, okay, as v=0. And phi2 similarly, is argument of u-1, okay, as v=0.

So along the real axis in the w plane, when u<-1, then u+1<0. u+1<0 means argument of u+1 is pi, okay. So phi1 is pi. And when u<-1, then u-1 is less than -2. So phi2 is also pi, okay. And so when u<-1, phi2-phi1=0, okay. And when u<1, then u+1>0. So phi1=0, okay. While u-1<0, so phi2=pi. So phi2-phi1=pi, okay. And when u>1, then u+1>0 and u-1 is also greater than 0. So argument of u+1 and argument of u-1, both are equal to 0.

So then phi2-phi1=0-0=0, okay. So this is how we look at this. So over -1 to 1, phi2-phi1 is pi. So that is Tuv becomes T0. So this is T0 and on other parts of u axis, phi2-phi1 becomes 0. So Tuv=0, okay. So Tuv function is harmonic in the half plane v>0 and satisfies these boundary conditions. You can see here T=0, here T=0, here T=T0, okay. And this T=0 corresponds to u<-1. It corresponds to, it is equal to T0 for the segment along the u axis, -1 to 1 and it is 0 when u>1. So this figure explains this.

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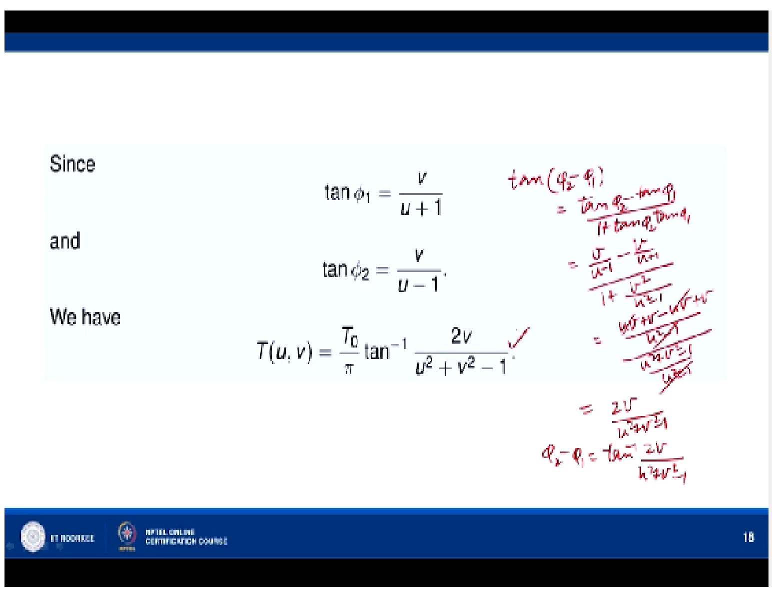

Now tan phi1=v/u+1, tan phi2=v/u-1, Tuv=t0/pi, we can write this. You see, T0/pi phi2-phi1 is that, this is the expression. So we can write tan phi2-phi1=tan phi2-tan phi1/1+tan phi2 tan phi1, okay. So here you put the value of tan phi2 is v/u-1 and this is v/u+1 and then we have 1+v square/u square-1. So this is uv+v-uv+v/u square-1/u square+v square-1/u square-1. So what you get is 2v/u square+v square-1. So phi2-phi1=tan inverse 2v/u square+v square-1. So we get this expression, okay.

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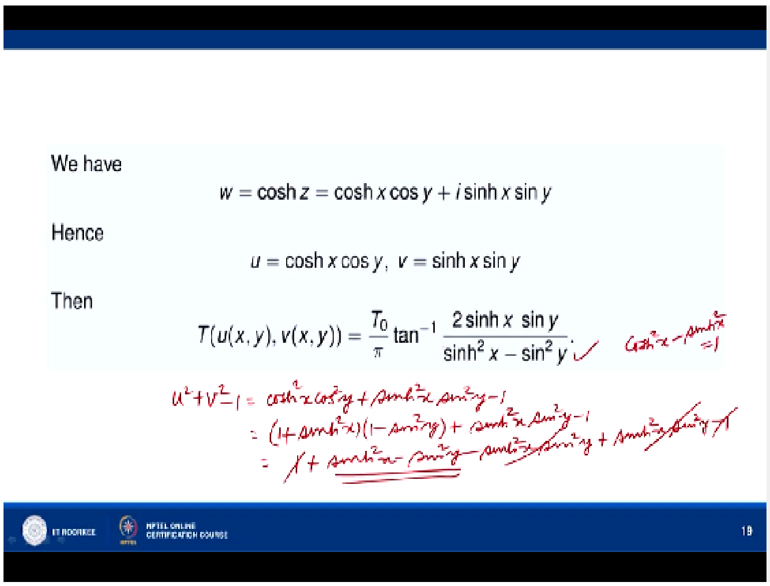

Now we have 2=cos hyperbolic z which is cos hyperbolic x*cos y+i sin hyperbolic x*sin y. So u=cos hyperbolic x cos y, v=sin hyperbolic x sin y. So let us put these values of u and v here. v=sin hyperbolic x sin y. So we put that value and here we see that u square+v square-1=cos hyperbolic square x cos square y+sin hyperbolic square x*sin square y-1. This gives you, we have to convert it to sin hyperbolic and sin y.

So let us use the relations cos hyperbolic square x-sin hyperbolic square x=1, okay. So we have this 1+sin hyperbolic square x and cos square y is 1-sin square y+sin hyperbolic square x sin square y-1. So we get this 1+, sin hyperbolic square x-sin square y-sin hyperbolic square x sin square y+sin hyperbolic square x*sin square y-1. So this cancels with this and this expression cancels with this and we get sin hyperbolic square x-sin hyperbolic square y. So we have put it here. So this is Tuv in terms of xy, Tuxyvxy.

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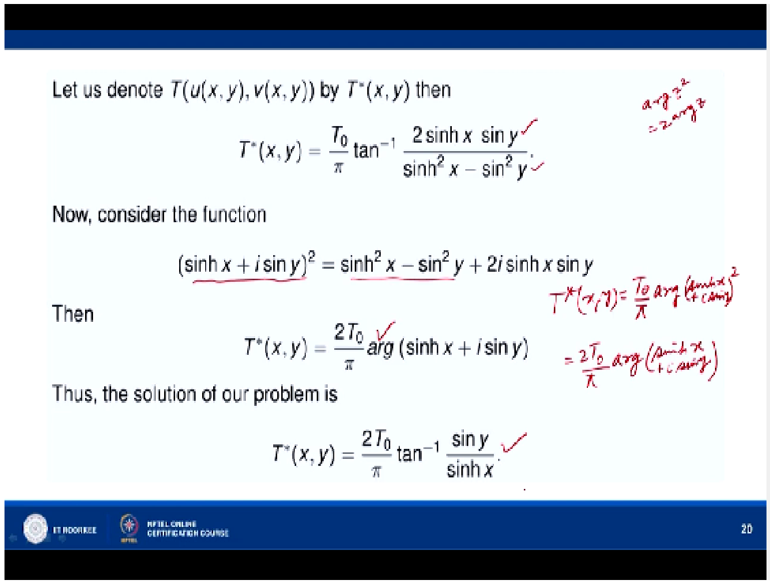

Let us now define Tuxyvxy by T*xy. Let us denote it by that. Then T*xy is T0/pi tan inverse 2sin hyperbolic x sin y/sin hyperbolic square x-sin square y, okay. Now if you consider this function, sin hyperbolic x+i sin y whole square, you square this, you get sin hyperbolic square x-sin square y+2i sin hyperbolic x sin y. And imaginary part here is 2sin hyperbolic x sin y which is here and real part is this one, okay which is here, okay.

So this means that tan inverse of imaginary part/the real part is nothing but argument of sin hyperbolic x+i sin y whole square, okay. So we can write T*xy=T0/pi argument of sin hyperbolic x+i sin y whole square, okay. And we know that argument of a complex number z square, argument of of z square is 2*argument of z. So this is equal to 2T0/pi argument of sin hyperbolic x+i sin y, okay.

So this is what we get. And then argument of sin hyperbolic x+i sin y, we can write as tan inverse sin y/sin hyperbolic x. So we get T*xy=2T0/pi tan inverse sin y/sin hyperbolic x. Now this is the solution to our problem. You can see that this T* also satisfies the boundary conditions. You see the boundary conditions are here y=0, okay. Here y=pi. So when y=0 and y=pi, we have temperature 0, okay.

And here temperature is T0. So let us see, let us show that these conditions are satisfied here. So when y is 0, sin y is 0, so this T*xy is 0. When y is pi, then again sin y=0, so this T*=0. And when x=0 because along the left part of the boundary, along this part, x=0. So x=0 means sin hyperbolic x=0. Sin hyperbolic x=0 means tan inverse infinity which is pi/2. So 2T0/pi and 2pi/2 will give you T0.

So T*xy satisfies the given boundary conditions and it is a harmonic function. So this is how we have to solve this Dirichlet problem. We use the function w=cos hyperbolic z to map the semi-infinite strip on to the upper half of the w plane. We solve it there, okay and then we use the inverse transformation to bring it, to get back the solution of the original problem.

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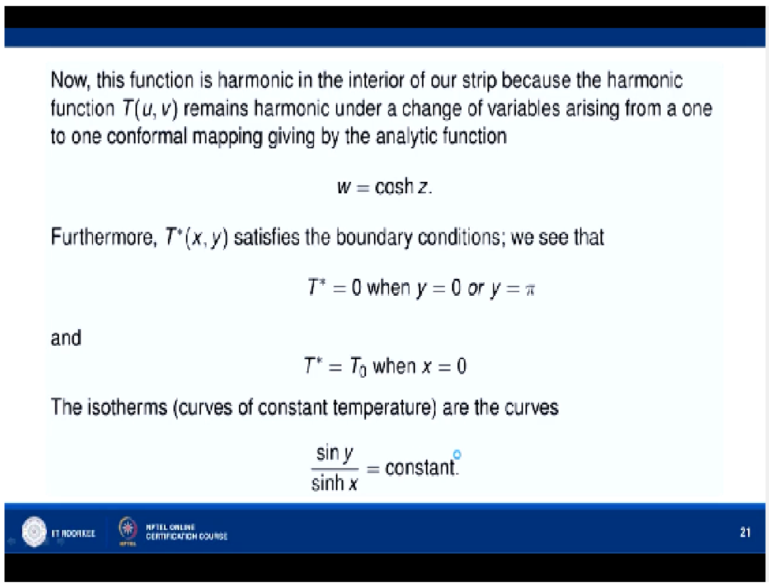

Now this function T*=2T0/pi tan inverse sin y/sin hyperbolic x is a harmonic function in the interior of our strip because the harmonic function Tuv remains harmonic under a change of variables as we have proved earlier arising from a one to one conformal mapping given by w=cos hyperbolic z. And T*xy satisfies the boundary conditions T*=0 when y=0 or y=pi, T*=T0 when x=0.

So this is the solution to our problem. Now the curves of constant temperatures are the curves given by sin y/sin hyperbolic x=constant. Because when T*xy=constant, then sin y/sin hyperbolic x=constant. So these are the isotherms. The isotherms are the curves given by sin y/sin hyperbolic x=constant. So with that I come to the end of this lecture. Thank you very much for your attention.