Advanced Engineering Mathematics
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Lecture - 31
Conformal Mappings from Disk to Disk and Angular Region to Disk

Hello friends. Welcome to my lecture on conformal mappings from disk to disk and angular region to disk. Here we shall also consider the conformal mapping w=exponential z. These conformal mappings are required by us when we solve problems arising in potential theory. So let us first consider the general bilinear transformation which transforms the disk mod of z<=rho onto the disk mod of w<=rho dash.

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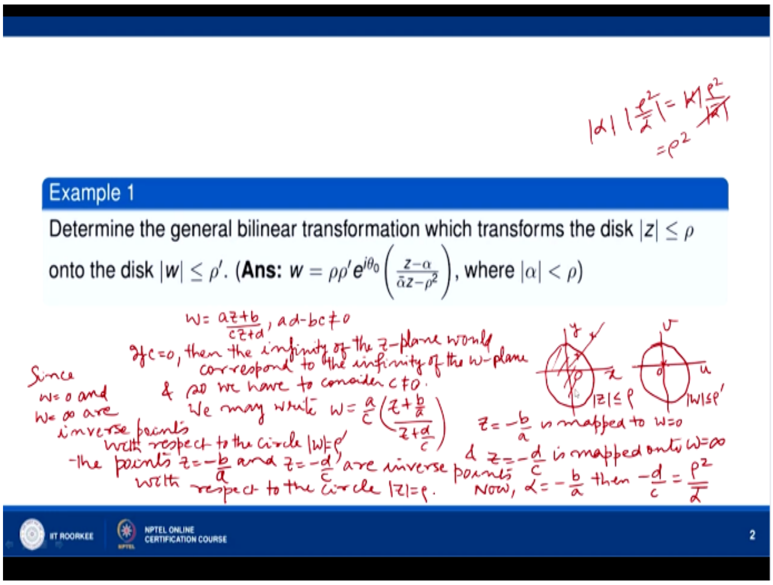

So we have the disk in z-plane which bounds the region mod of z<=rho and then we have the disk in the w-plane given by mod of w<=rho dash. So let us consider the bilinear transformation w=az+b/cz+d where ad-bc is not equal to 0 okay. Now here again we start with the condition on c. If c is=0, then the infinity of the z-plane will correspond to the infinity of the w-plane.

Then, the infinity of the z-plane would correspond to the infinity of the w-plane and so we have to consider c not equal to 0 okay and when c is not equal to 0 we may write w as a/c*z+b/a/z+d/c okay. Now from here it is clear that z=-b/a is mapped to w=0 and z=-d/c is mapped onto w=infinity. Now we know that w=0 and w=infinity are inverse points with respect to circle mod of w=rho dash. So since w=0 and w=infinity are inverse points with respect to the circle mod of w=rho dash.

And under bilinear transformation, inverse points with respect to a circle are mapped into inverse points with respect to its image. So these points 0 and infinity should correspond to the inverse points –b/a and –d/c with respect to the circle mod of z=rho. So therefore so the points z=-b/a and z=-d/c okay are inverse points with respect to the circle mod of z=rho okay. Now let us say let alpha be=-b/a.

Then, -d/c will be=rho square/alpha conjugate because –b/a and –d/c are inverse points with respect to the circle mod of z=rho. So alpha and its inverse point rho square/alpha conjugate they must be on the same side of the center and they should be collinear with the center okay, so one point if it is here, the other point will be here. So they are collinear with the center and on the same side of it.

And moreover product of the distances with the center is=radius square. So here you can see if argument of alpha is theta then argument of rho square/alpha conjugate is also theta and moreover modulus of alpha*modulus of rho square/alpha conjugate is=modulus of alpha. If it is alpha the other –d/c is=rho square*alpha conjugate. So what we have, the product of the two must be equal to rho square, the product of the moduli of the two.

So if modulus of alpha*modulus of rho square/alpha conjugate okay is=modulus of alpha*rho square/modulus of alpha conjugate. So these two cancel and we get rho square okay. So if –b/a is=alpha then –d/c must be equal to rho square/alpha conjugate. Now we notice that alpha=-b/a is mapped to the interior of mod of w=rho dash that is it is mapped to w=0, so –b/a okay must be such that modulus of alpha should be<rho, it should be an interior point of mod of z=rho okay.

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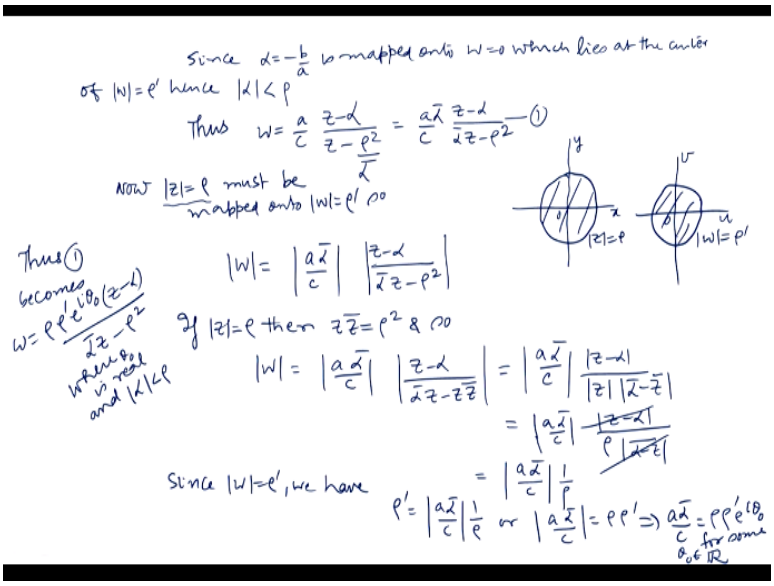

Since alpha=-b/a is mapped onto w=0 okay which lies at the center of mod of w=rho dash. Hence, mod of alpha must be<rho, it must be an interior point of the circle mod of z=rho okay. Now what do we get, we have w=a/c z+b/a/z+d/c and b/a is –alpha, d/c is=-rho square/alpha conjugate, so let us put these values. So we get thus w=a/c z+b/a z-alpha and z+d/c is z-rho square/alpha conjugate okay.

So this we can write as a/c*z-alpha/alpha conjugate z-rho square and then alpha conjugate here okay. So we can put it like this. So now we want to map the boundary of the disk in the w-plane to the boundary of the disk in the w-plane okay. Modulus of z=rho should be mapped to modulus of w=rho dash. So let us take a point on the boundary. Let us take a point. So what we get?

Okay now mod of z=rho must be mapped onto mod of w=rho dash, so mod of w we can take here this equation let we call it as 1. So mod of w=mod of a alpha conjugate/c*mod of z-alpha/alpha conjugate z-rho square okay. So if z lies on the boundary mod z=rho if mod of z=rho then zz conjugate is=rho square and so mod of w will be=mod of a alpha conjugate/c*mod of z-alpha/alpha conjugate z-alpha alpha conjugate okay.

And I can write it as mod of a alpha conjugate/c*mod of z-alpha/this is alpha conjugate z, this is zz conjugate okay. So we will write it as z times alpha conjugate-z conjugate okay. Mod of z=rho so mod of a alpha conjugate/c okay*mod of z-alpha/rho times mod of now alpha conjugate-z conjugate is alpha-z conjugate okay and mod of z-alpha is same as mod of alpha-z conjugate.

So these two cancel okay and we get mod of a alpha conjugate/c*1/rho=mod of w, mod of w=rho dash okay. So since mod of w=rho dash we have rho dash=mod of a alpha conjugate/c*1/rho or mod of a alpha conjugate/c=rho rho dash okay. So thus transformation 1 becomes thus 1 becomes w=rho rho dash*e to the power i theta 0*z-alpha/alpha conjugate z-rho square okay where theta 0 is real and mod of alpha is<rho okay.

So this implies that a alpha conjugate/c=rho rho dash e to the power i theta 0 for some theta 0 which is real belonging to R okay. So under the transformation w=rho rho dash*e to power i theta 0*z-alpha/alpha conjugate z-rho square where theta 0 is real and mod of alpha is<rho. The disk mod of z<=rho in the z-plane will be mapped onto the disk mod of w<=rho dash in the w-plane, so this is the transformation.

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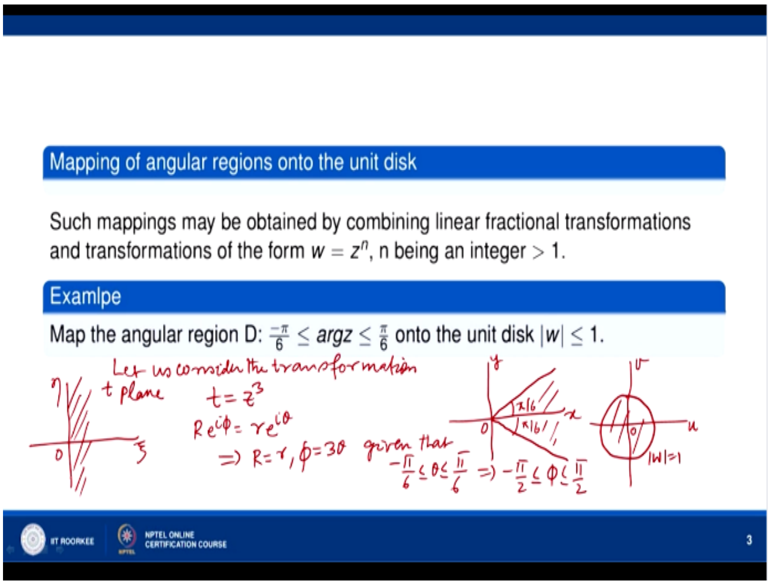

Now let us go to another case where we have to map an angular region onto the unit disk. Such mappings can be obtained by combining the linear fractional transformations and transformation of the form zw=z to the power n where n is an integer>1. Let us say for example you want to map the angular region -pi/6<=argument of z<=pi/6 onto the unit disk mod of w<=1.

Then, let us draw the figure first. So this is the angular region in the z-plane okay. This is also pi/6, so this is the angular region in the z-plane and we want to map it to the unit disk in the w-plane mod of w=1 okay. This is where we want to map okay. So what we do is first we will map this angular region to the right half plane okay. Let us consider the transformation t=z cube, so we will go from the z-plane to the t-plane okay.

Under this transformation if you say that t is R e i phi and z=r e i theta then what we get R=r and phi=3 theta okay. So here we are given that theta lies between –pi/6 to pi/6, so 3 theta will lie in the region –pi/2 to pi/2. So this implies that –pi/2 is<=phi<=pi/2 okay. So in the t-plane okay this angular region will be mapped into the right half plane. This is t-plane okay. So let us say the real axis is zeta and the imaginary axis is eta okay.

So in the t-plane okay, the angular region –pi/6<=theta<=pi/6 is mapped onto right half plane. Now from right half plane we will go to the unit disk. So let us recall how we can do that.

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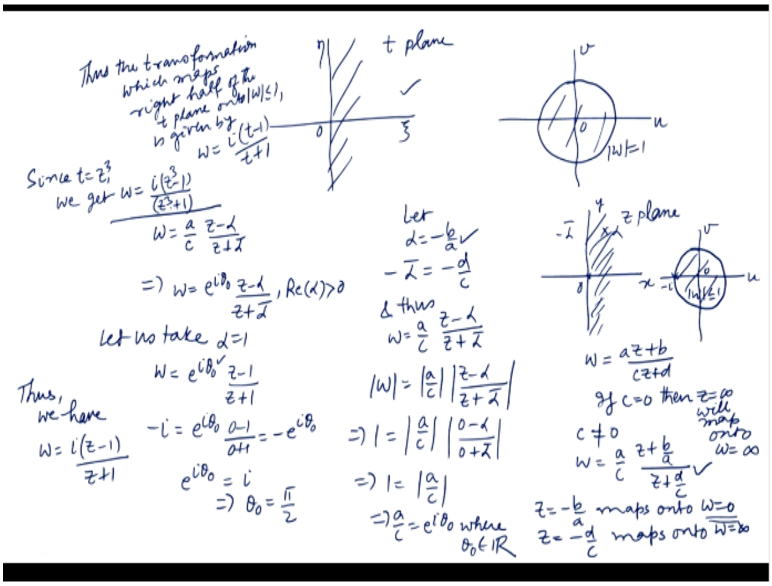

So our problem is now to map the right half plane to the disk mod of zw<=1 okay. Now we can easily do this. Suppose we have let us recall the transformation instead of t-plane we are considering w-plane. So let us say we are mapping from w-plane we are getting this w-plane okay, so this is 0, x, y, this is z-plane and we want to map z-plane to w-plane. Later, we shall replace the z-plane by t-plane.

For convenience, we can do this okay. So let us say we want to map this right half plane into the disk mod of w<=1 okay. So we can see that the bilinear transformation w=az+b/cz+d okay. Now here if c is=0 then z=infinity will map onto w=infinity but what we want, we want to map unbounded region into the bounded region, so infinity of the z-plane will go to some point in the disk mod of w<=1, so we cannot allow this z=infinity to go to w to infinity.

And therefore c is not equal to 0 okay, so c=0 is not possible, so c is not equal to 0 we have to take. Then, we can write w=a/c z+b/a/z+d/c okay. Now z=-b/a goes to maps onto w=0 and z=-d/c maps onto w=infinity, w=0 and w=infinity are inverse points with respect to mod of w=1, so -b/a, -d/c must be inverse points with respect to the imaginary axis okay. So let us say let alpha be=–b/a, then –d/c will be=-alpha conjugate.

Because if we take any complex number here say alpha okay, then its complex conjugate is alpha conjugate but its image in the imaginary axis is –alpha conjugate okay. So –alpha conjugate will be=-d/c and thus w=a/c*z-alpha/z+alpha conjugate we have okay. This becomes this okay. Now we want to map a point of the right half of the z-plane into the interior of mod of w=1.

So let us take okay we want to map the boundary of this region that is the imaginary axis onto mod of w=1, so let us take z=0 here, z=0 must go to some point on the boundary of the disk that is mod of w=1. So mod of w=mod of a/c*mod of z-alpha will give you mod of w=1, so 1=mod of a/c*z is 0, 0-alpha/0+alpha conjugate okay. So this implies 1=mod of a/c because mod of alpha and mod of alpha conjugate is same okay.

So this means that a/c=e raised to the power i theta 0 where theta 0 is real okay. So this we can say that w=a/c*z-alpha/z+alpha conjugate this becomes w=e to the power i theta 0*z-alpha/z+alpha conjugate. So this is the most general bilinear transformation which maps the right half of the z-plane onto the disk mod of w<=1. Now we can get a specific transformation from here.

Here the condition on alpha is that alpha lies in the right half plane, that means real part of alpha must be>0 okay. So that condition we have to have on alpha okay because this alpha=-b/a goes to w=0. So alpha must be a point of the right half plane. Right half plane is mapped into mod of w<1 and the boundary that is the y-axis is mapped onto mod of w=1, so this condition we will have on alpha.

Now we want a specific transformation here. So let us take alpha=1. This is most general bilinear transformation. We can take a particular case here, particular transformation we can find. So let us take alpha=1 then the real part of alpha is 1 okay. So it is >0 and w becomes e to the power i theta 0 z-1/z+1 okay because alpha is real it is=1 okay. Now what we do? Let us map a point in order to determine this theta 0, we have to map a point on the y-axis that is the boundary of the z-plane onto some point particular point of the mod of w=1.

So let us say suppose I map z=0 to this –i okay. We can take any point, so –i=e to the power i theta 0*0-1/0+1. So this is=-e to the power i theta 0 okay. So e to the power i theta 0=i which implies that theta 0=pi/2 okay. So here I have mapped w=0 onto this point –i okay in the w-plane okay. So thus we get the transformation, thus we have w=i times z-1/z+1. This transformation maps the right half of the z-plane to the disk mod of w<=1.

Now I am finding this transformation for the t-plane, the right half of the t-plane to map to mod of w<=1. So instead of z, I will write now t okay. So thus the transformation which maps right half of the t-plane onto mod of w<=1 is given by w=i+z/t so i times t-1/t+1 okay. Now we have t=z cube okay, t=z cube so let us put that value of t here. So since t is=z cube, we get w=i times z cube-1/z cube+1 okay.

So this is the desired transformation. The transformation w=i times z cube-1/z cube+1 maps the angular region-pi/6<=argument of z<=pi/6 onto the disk mod of w<=1.

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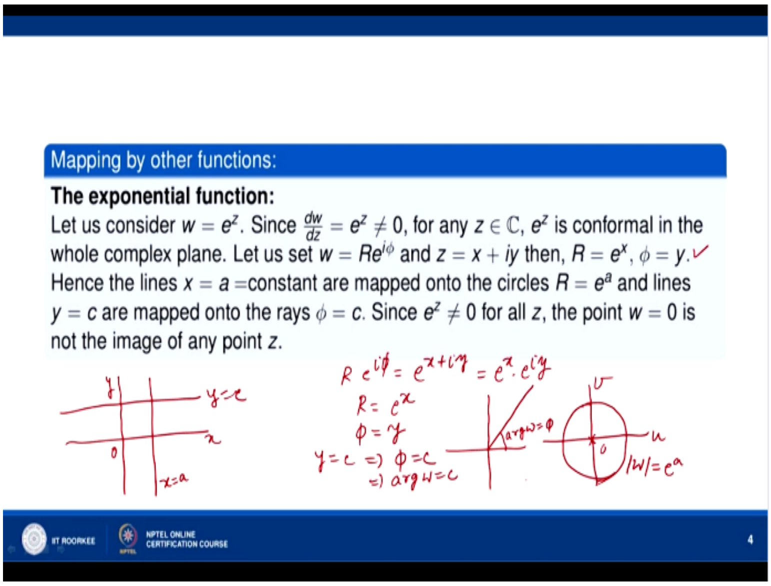

Now let us go to some other mappings which are important when we discuss the applications of conformal mappings. So the mapping that is w=e to the power z, the exponential function is one such mapping which is important. So the exponential function w=e to the power z we know is a conformal mapping for all values of z belonging to the complex plane because dw/dz is=e to the power z and e to power z does not vanish for any value of z.

So e to the power z is conformal in the whole complex plane and if you take w=Re i phi and z=x+iy then what we get, Re i phi=e to the power x+iy which we can write as e to the power x*e to the power iy and so equating the absolute values, we get R=e to the power x and phi=y okay. So R=e to power x and phi=y hence the lines x=a. If you take the lines in the z-plane parallel to y–axis say x=a in the z-plane, then the line x=a will be mapped onto the circle given by R=e to the power a.

R is the radius; it is the radius of the actually R=constant gives us the circle in the w-plane. So this is R=this is the circle which is having radius e to the power a okay. So the equation of the circle is mod of w=a e to the power a sorry e to the power a okay. So x=a is mapped into mod of w=e to power a, so the lines are mapped the straight lines which are parallel to y-axis are mapped onto to circles R=e to power a.

And the lines y=c if you take a line parallel to x-axis y=c then you can see y=c gives you phi=c and the phi is the argument of w. So this implies argument of w=constant. Argument of w=constant means the lines parallel to the real axis of the z-plane they are mapped onto the ray okay because for the ray argument of w is constant okay. So this is the argument of w that is we are taking as phi okay.

This is equal to phi. So this phi is constant, so y=c is mapped onto the ray phi=c. Now e to the power z is not 0 for any value of z that means w=0, this w=0 is not the image of any point z in the z-plane okay.

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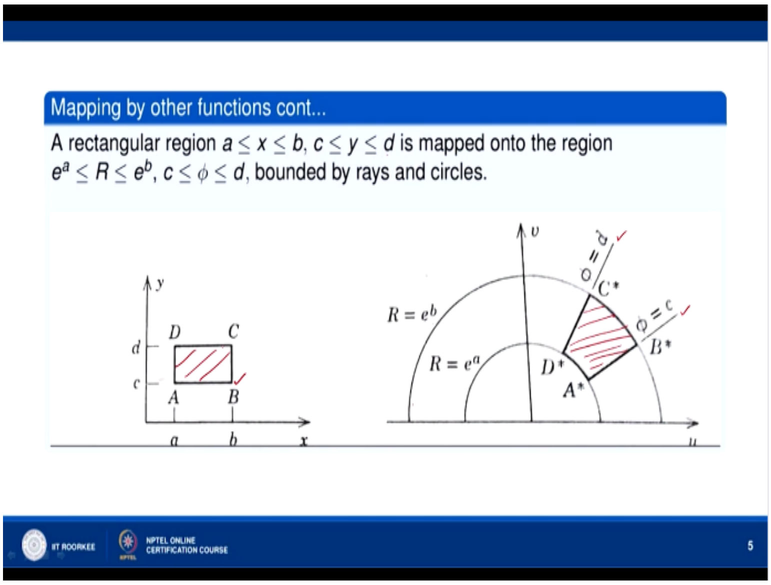

Now let us look at the mapping of a rectangular region in the z-plane. So let us take a rectangular region in the z-plane. It is given by a<=x<=b, c<=y<=d okay. So this rectangular region now this rectangular region is bounded by x=a, x=b, so x=a go to R=e to power a, x=b goes to R=e to power b, therefore the region which is bounded by a<=x<=b goes to e to power a<=R<=e to power b.

And c, y=c goes to phi=c, y=d goes to phi=d, so c<=y<=d goes to c<=phi<=d. So this rectangular region okay goes to the region bounded by the rays and circles, rays are this, this x=a goes to phi=c, x=a goes to this R=e to the power a this circle okay. This is the image of x=a and this is the image of x=b, the ray R=the circle R=e to the power b and the line y=c this y=c goes to phi=c and y=d goes to phi=d okay.

So a, b goes to this A star, B star and c, d goes to this one, d becomes D star here okay because a, d goes to this R=e to the power a okay. So this is C star, this is D star, so this is the region okay which is the image of this region okay under this w=e to power z.

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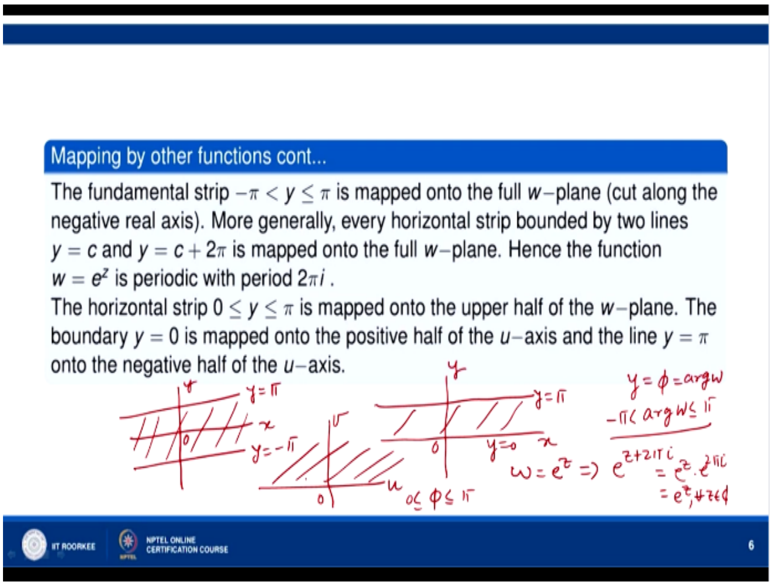

Now let us take the fundamental strip –pi<y<=pi okay. So this fundamental strip let us consider okay this goes to yeah y=phi, y=argument of w okay. So if y lies it in –pi and pi, argument of w lies from –pi to pi okay and this means that this fundamental strip is mapped onto the full w-plane and there is a cut along the negative real axis. More generally every horizontal strip bounded by two lines y=c and y=c+2pi will be mapped onto full w-plane.

Because y is argument of w, so argument of w will vary from c to c+2pi okay, that will cover the whole w-plane. So hence the function w=e to the power z is periodic with period 2 pi i. This we can see directly also w=e to the power z gives you e to the power z+2 pi i=e to the power z*e to the power 2pi i okay and e to power 2 pi i is cos 2 pi+i sin 2 pi which is 1. So e to the power z+2 pi i is=e to power z for all z belonging to the complex plane c okay for all complex number z.

So e to power z is periodic with period 2 pi i. The horizontal strip 0<=y<=pi let us consider. So the horizontal strip this is y=0 and this is y=pi. So the horizontal strip 0<=y<=pi is mapped onto the upper half of the w-plane because then the argument will vary from argument of w will vary from 0 to pi. So argument of w will vary from 0 to pi that means phi varies from 0 to pi okay.

So it is mapped over here okay. The boundary y=0 okay y=0 boundary of the strip is mapped onto positive x-axis okay, positive u-axis this is because we are here in the w-plane, so positive u-axis and the line y=pi is mapped onto negative half of the u-axis.

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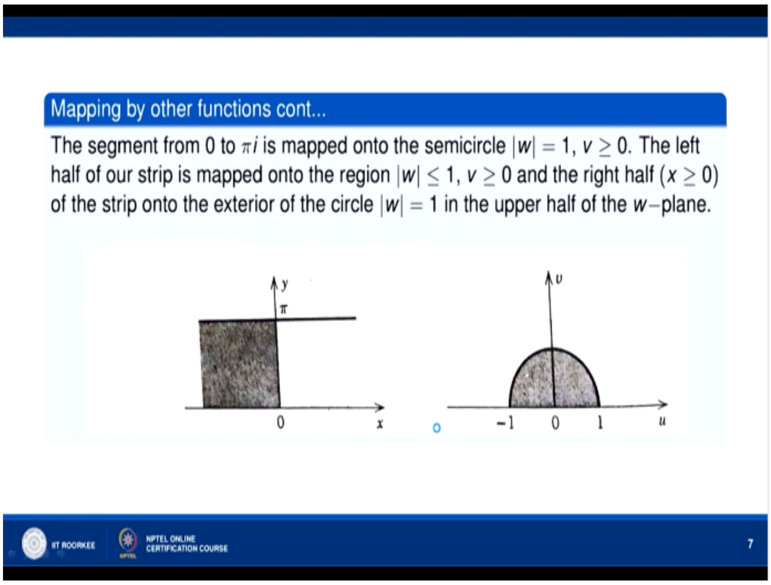

The segment from 0 to pi i let us look at the segment from 0 to pi i this segment okay. The segment from 0 to pi i, so this segment from 0 to pi i is mapped onto the semicircle mod of w=1, let us see how we get this because we have looked at the boundary only okay. We have seen that y=0 is mapped onto to positive u-axis, y=pi maps onto negative u-axis okay. Let us look at where the interior of this strip goes.

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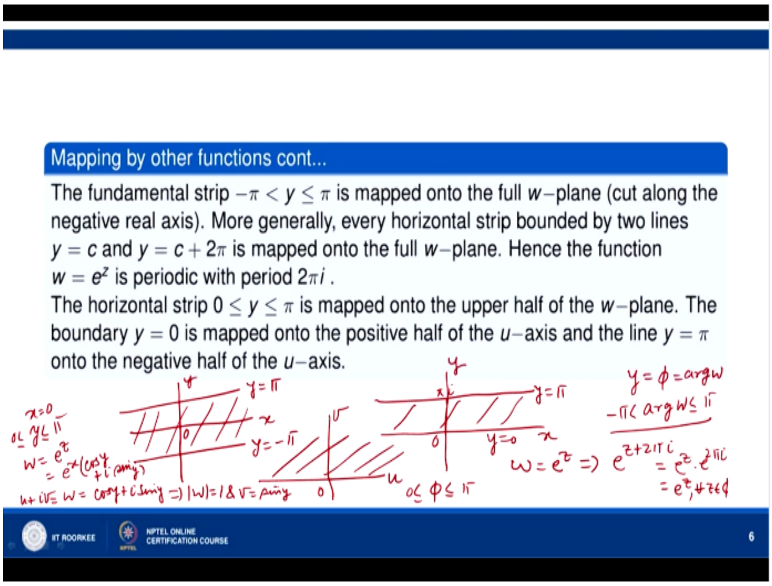

So let us first look at the segment from 0 to pi i along the y-axis okay. This pi i here okay so here along this segment x=0 y varies from 0 to pi. So now w=e to the power z okay, so this is e to the power x*cos y+i sin y okay. Now x=0, so w=cos/+i sin y and this gives you mod of w=1 and v=sin y equating real imaginary parts, w=u+iv, w=u+iv=cos/+i sin y gives you v=sin y.

Now y varies from 0 to pi, so sin y is>=0 and therefore the segment 0 to pi i is mapped onto mod of w=1 and v>=0.

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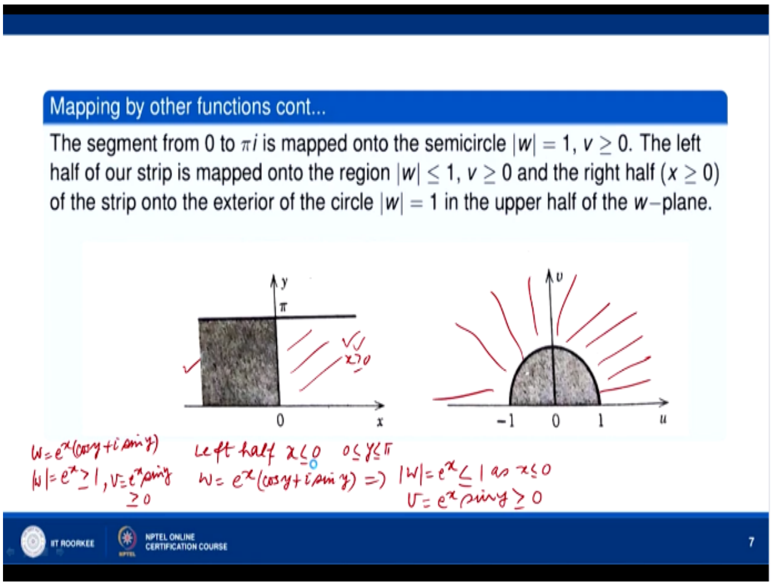

So the segment from 0 to pi i is mapped onto the semicircle mod of w=1, v>=0. The left half of our strip, the left half of the strip this one okay, this left half, this one okay, the left half of the strip let us see where does this go okay. The left half of the strip means this portion okay, this portion shaded portion. So in this part what happens is x is negative in the left half, x is<=0 and 0 is<=y<=pi okay.

Now w=e to the power x*cos y+i sin y okay. So mod of w is=e to the power x okay. Now x is<=0, so mod of w is<=1 okay. So left half is mapped onto the region mod of w<=1 and v is>=0 again because v is=e to the power x sin y. When x is a real number, e to power x is always>=0 and sin y is positive because 0 is<=y<=pi so v is>=0. So left half of the strip goes to mod of w<=1, v>=0.

That means that the shaded region goes to this shaded region here okay which is defined by mod of w<=1 and v>=0 and let us now look at the image of the right half of the strip that is x>=0. So when x is>=0 what happens, again w=e to the power x*cos y+i sin y, so mod of w=e to the power x and x is>=0 here okay, x is>=0 so this is>=1, so mod of w is>=1 and v is e to the power x sin y okay.

So v is>=0 because x is real number and 0 is<=y<=pi, so mod of w is>=1 and v is >=0 that means the right half of the strip between y=0 to y=pi is mapped into this portion okay and thus the region bounded by this one 0<=y<=pi is mapped onto the upper half of the w-plane. So with this I would like to end my lecture. Thank you very much for your attention.