Advanced Engineering Mathematics

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Lecture - 30
Conformal Mappings from Half Plane to Disk and Half Plane to Half Plane - I

Hello friends. Welcome to my lecture on conformal mappings from half plane to disk and half plane to half plane. So there will be 2 lectures on this topic. This is first of those 2 lectures. Let us first define inverse points with respect to a line. Two points P and Q are said to be inverse points with respect to a line say AB if Q is the image of P in AB. That is if the line AB is the right bisector of PQ.

So if you take a line let us say AB, then two points P and Q are said to be inverse points with respect to the line AB if AB is the right bisector of PQ okay.

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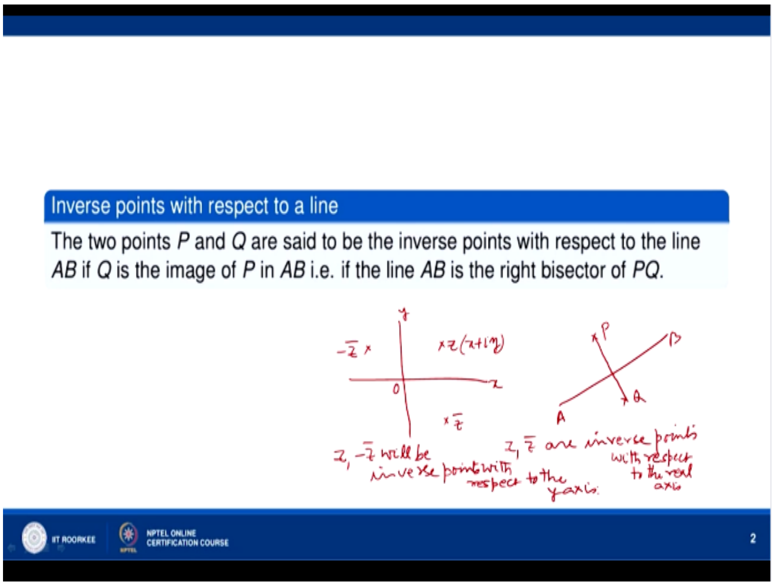

For example, if you take in the z-plane if you take the real axis okay x-axis the real axis, then you take any complex number z, the inverse point of the complex number z with respect to real axis will be z conjugate okay. So with respect to the real axis, the z and z conjugate will be inverse points and with respect to y-axis we shall have if z is z+iy then we will have –z conjugate okay.

So that will be the inverse points okay for z with respect to y-axis. So with respect to x-axis, z and z conjugate are inverse points with respect to x-axis that is the real axis in the z-plane, z and –z conjugate will be inverse points in the z-plane with respect to the imaginary axis the y-axis okay. Now let us see how we define the inverse points with respect to a circle.

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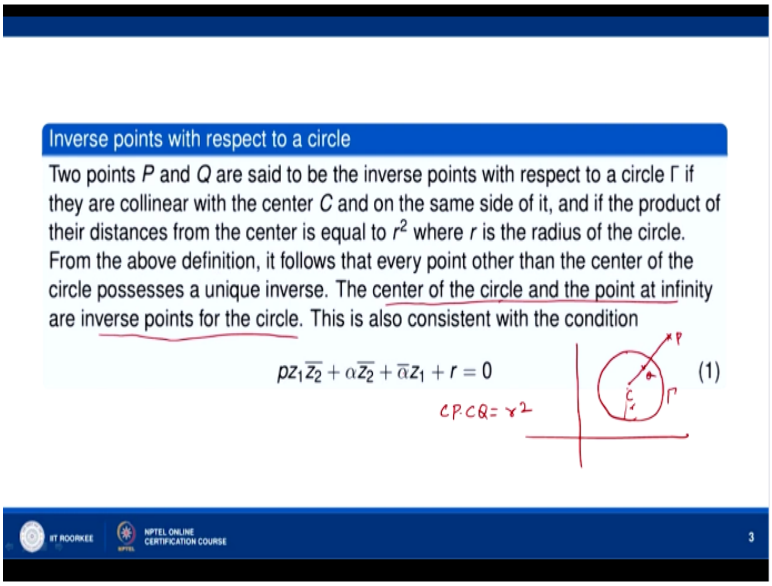

Two points P and Q are said to be the inverse points with respect to a circle gamma if they are collinear with the center. So let us take any circle okay. So let us say C is the center, circle is gamma, then two points P and Q are called inverse points with respect to the circle gamma if they are collinear with the center okay, if they are collinear with the center C and on the same side of it okay.

And if the product of their distances from the center is equal to r square, that means CP*CQ is=r square where r is the radius of the circle. So from the above definition, if follows that every point other than the center of the circle, you take any point other than the center of the circle it possesses a unique inverse okay. The center of the circle and the point at infinity are inverse points for the circle.

Now this center of the circle and the point at infinity are inverse points for the circle. This is also consistent with the condition pz1 z2 bar+alpha z2 bar+alpha bar z1+r=0.

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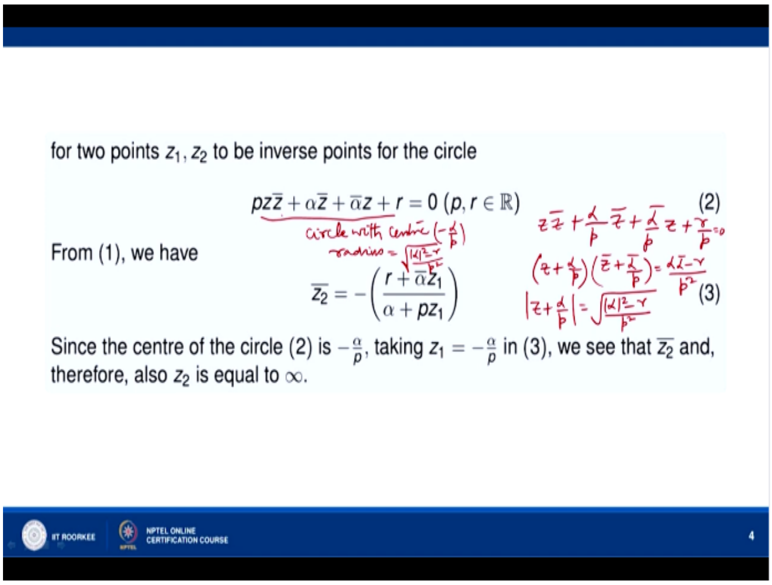

For two points z1, z2 to be inverse points for the circle pz z bar+alpha z bar+alpha bar z+r=0 where p and r are real numbers. This is the equation of any circle or straight line in the z-plane. If p is not equal to 0, it represents a circle. If p is=0, then it represents a straight line in the z-plane. So you can also see that pz z bar+alpha z bar we can write as if you divide this because p is not equal to 0 we can divide this equation by p.

So then we get zz bar+alpha/p*z bar+alpha bar/p*z+r/p=0. So that we can write it as z+alpha/p*z bar+alpha bar/p okay. So this gives you zz bar+alpha/pz bar+alpha bar/p*z+alpha alpha bar/p square. So I can write it as alpha alpha bar-r/p square okay. So this gives you mod of z+alpha/p=square root of mod of alpha square-r/p square okay because left hand side is mod of z+alpha/p square okay, p conjugate we are not writing because p is a real number.

So this equation represents a circle with center –alpha/p and radius square root of mod of alpha square-r/p square okay. So for this circle if we want z1 and z2 to be inverse points for this circle, then the condition is this one, this is the condition. We are going to prove that this is the condition for z1, z2 to be inverse points of this circle okay. Now let us assume for this on time being that for z1, z2 to be inverse points of the circle, the condition is this.

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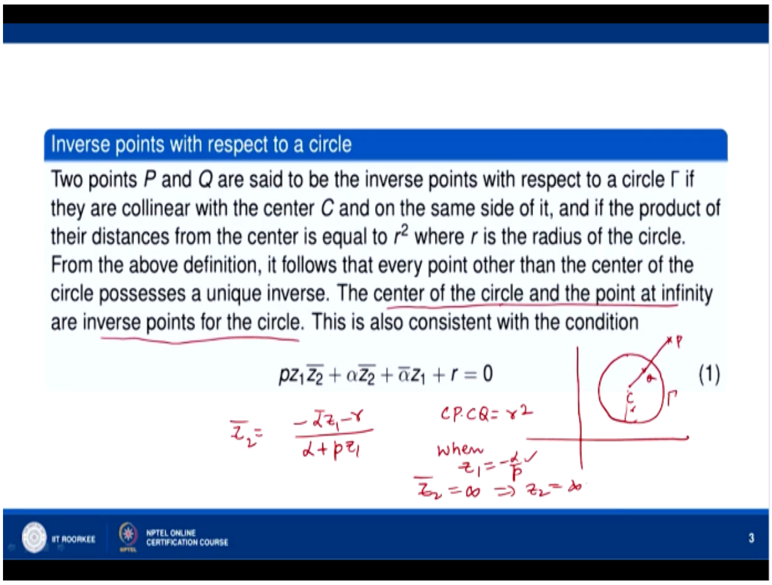

Then from this condition we notice that z2 bar is=-alpha bar z1-r/p z1+alpha okay. So that when z1 is=-alpha/p, z2 bar is=infinity which says that z2 is=infinity. So the center, center is from the circle this one, center is –alpha/p okay and when we put –alpha/p here okay in this condition, then we get for z1 then we get z2 bar to be=infinity and get z2=infinity.

So that means that the center z1=-alpha/p and z2=infinity are inverse points with respect to the circle pz z bar+alpha z bar+alpha bar z+r=0. So the condition which we have here, this condition for z1, z2 to be inverse points for the circle is consistent with the fact that if the center –alpha/p and the infinity are also inverse points with respect to this circle okay. So since the center of the circle 2 is –alpha/p taking z1=-alpha/p in 3 we get z2 bar equal to infinity and so that z2 is also equal to infinity.

So that condition now let us prove this condition that z1, z2 are inverse points for the circle when we have that condition. So let us prove that.

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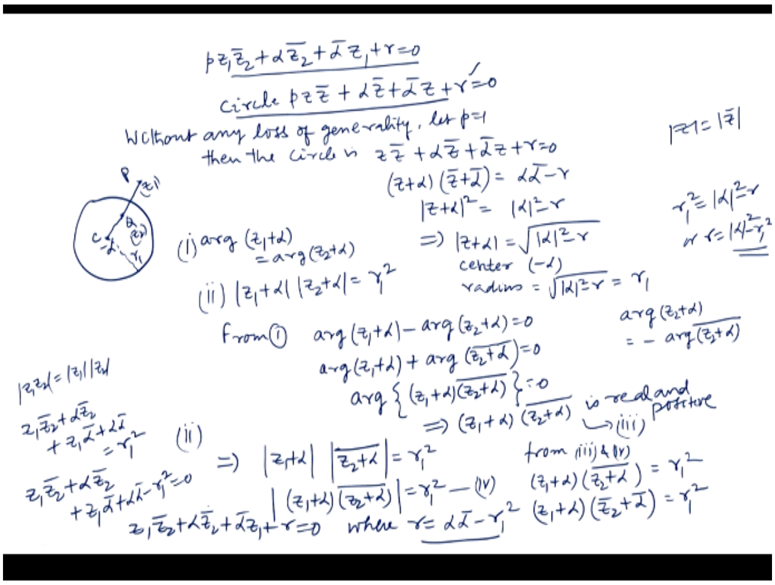

So p z1 z2 bar+alpha z2 bar we have this condition p z1 z2 bar and then we have alpha z2 bar and then we have alpha bar z1+r=0. This is the condition which we have to prove and the circle is this one pz z bar and then we have alpha z bar+alpha bar z+r=0. So we have to prove that for this circle okay. If z1 and z2 are inverse points, then this condition holds okay. Now without any loss of generality, we can assume that p=1 okay.

Because if p is not equal to 1 we can divide this equation by p and let the coefficient of zz bar=1. So without any loss of generality let us take p=1, then the equation of the circle is zz bar+alpha z bar+alpha bar z+r=0 and this we can write as z+alpha*z bar+alpha bar=alpha alpha bar-r. So this is now mod of z+alpha square=mod of alpha square-r which imply that mod of z+alpha=square root mod of alpha square-r okay.

So center of the circle is at center is at –alpha okay and radius is under root mod of alpha square –r. Now let us take a circle okay. Say this is –alpha center okay, P and Q are these points which are inverse points with respect to the circle. Let us say P is complex number z1 and Q is complex number z2 okay. Then, if P and Q are inverse points with respect to the circle, then P and Q must be on the same side of the center and collinear with the center okay.

So what we have argument of z1+alpha must be same as argument of z2+alpha that is the first condition because they are collinear with center and on the same side of it and moreover that this is let us say C okay. CP*CQ=radius square, so this means that CP that means modulus of z1+alpha*modulus of z2+alpha okay is=radius square. So we have modulus of z1+alpha*modulus of z2+alpha=radius of the circle square.

So that is r square okay. So there are two conditions, this is condition number 1, this is condition number 2 okay. Now let us notice that from condition 1, argument of z1+alpha-argument of z2+alpha=0. Now we can also say that argument of z2+alpha, argument of z2+alpha is=-argument of z2+alpha conjugate okay. If z is any complex number, then argument of z is same as –argument of z conjugate okay.

So making use of that I can write it as argument of z1+alpha+argument of z2+alpha conjugate=0 okay or I can say argument of z1+alpha*z2+alpha conjugate=0. Now if argument of a complex number is 0, it will mean that z1+alpha*z2+alpha conjugate is a real positive number, is real and positive okay. Now modulus of z1+alpha modulus of z2+alpha=r square. The condition 2 gives us modulus of z1+alpha, now modulus of z is same as modulus of z conjugate okay.

So modulus of z2+alpha is same as modulus of z2+alpha conjugate=r square or I can say that modulus of z1+alpha*z2+alpha conjugate is=r square because modulus of z1*z2 is=modulus of z1*modulus of z2 okay. So now there is a complex number whose modulus is r square and that complex number is real and positive, so from this condition this one this is condition number 3 and this is condition number 4 okay.

From these two conditions, it follows that okay from 3 and 4, here we are saying that z1+alpha*z2+alpha conjugate is real and positive and here its modulus is r square. So this complex number itself is r square okay. So z1+alpha*z2+alpha conjugate=r square and this gives you what, z1+alpha*z2 conjugate+alpha conjugate=r square.

So then this will be z1 z2 conjugate+alpha z2 conjugate+z1 alpha conjugate z1 z2 conjugate+alpha z2 conjugate+z1 alpha conjugate+alpha alpha conjugate=actually this r should be taken as radius of the circle which I have taken, it should not be taken as r, it should be some r dash it will be better because here this r is not the radius of the circle, this r is actually a certain real number and the radius is mod of alpha square-r.

So this r you can write as say r1 here it will be better okay. So this I can write as r1, so then what we will have here r1 square okay. So then if we do that so then this will be equal to z1 z2 conjugate+alpha z2 conjugate+z1 alpha conjugate and then we have+alpha alpha conjugate-r1 square=0 and this is then z1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=0, so this r is where r is your alpha alpha 1-r1 square okay.

So alpha alpha conjugate-r1 square, so what we are getting z1 z2 conjugate okay P we have taken as 1 so z1 z2 conjugate alpha z2 conjugate alpha conjugate z1+r=0. So this radius of the circle which we are writing here as under root mod of alpha square-r, this is actually r1 okay. So r1 square is mod of alpha square-r or we can say r is=mod of alpha square-r1 square okay, so this r is alpha alpha conjugate-r1 square.

So radius of the circle is r1 okay and r1 is under root mod of alpha square-r, so the condition for the circle pz z bar+alpha z bar+alpha bar z+r=0 to have the inverse points z1, z2 as inverse points is that p z1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=0. So this is how we prove this condition that is p z1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=0 is the condition for the circle pz z conjugate+alpha z conjugate+alpha conjugate z+r=0 to have the inverse points at z1 and z2.

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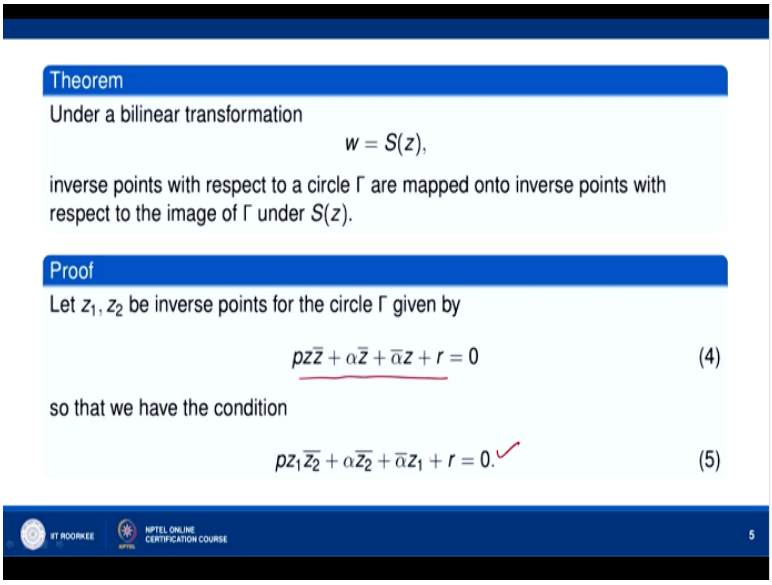

Now let us go to this theorem which says that under a bilinear transformation w=Sz, inverse points with respect to a circle are mapped onto inverse point with respect to the image of the circle gamma under Sz. So let us say z1, z2 be inverse points for the circle pz z conjugate+alpha z conjugate+alpha conjugate z+r=0. Then, we have this condition as we have proved just now p z1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=0.

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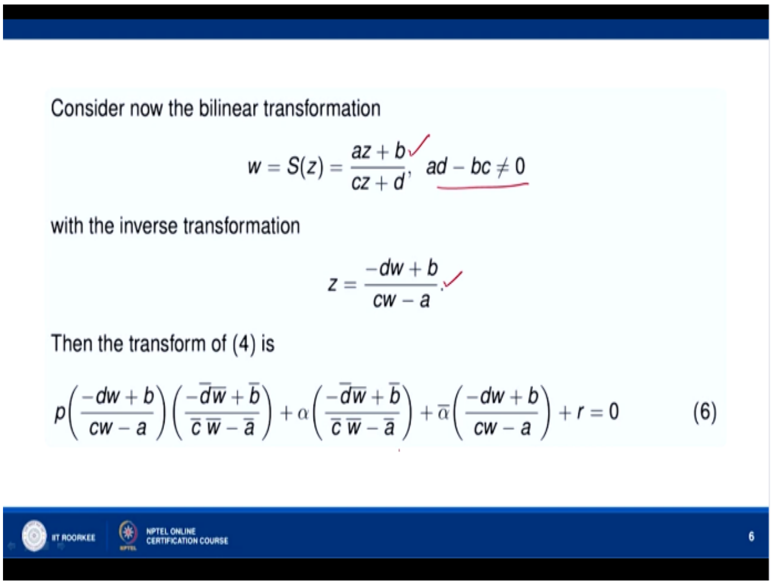

Now let us consider the transformation w=Sz to be az+b/cz+d where ad-bc is nonzero. Then, we can write the inverse while in a transformation z=-dw+b/cw-a. Let us put the value of z as –dw+b/cw-a in the equation of this one in the condition okay pz1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=-0. So let us put in this and see what is the condition that we get okay.

So then we get p times transform of 4 okay. First, we are transforming the circle okay. First, we are transforming the circle under the bilinear transformation by putting the value of z, so when you put the value of z we get p times –dw+b/cw-a this is the value of z, then z conjugate. So –d conjugate w conjugate+b conjugate/c conjugate w conjugate-a conjugate because a, b, c, d are complex constants, so we have to take their conjugates here.

So alpha times z conjugate then alpha conjugate z+r=0. So this is the equation that we get under the bilinear transformation w=Sz for the given circle in the w-plane okay.

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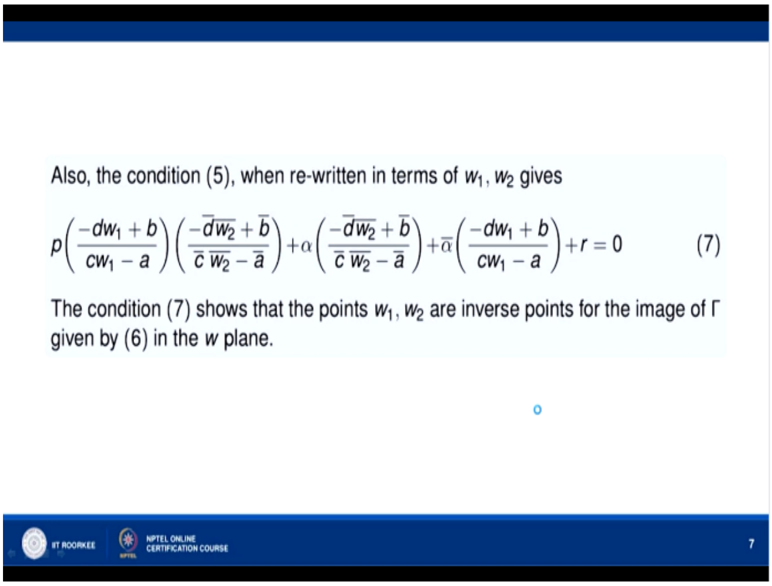

Now also the condition 5 okay, when the condition 5 is written in terms of w1, w2. So let us see okay so this condition okay. This condition let us write w1 is the image of z1 under the bilinear transformation w=Sz and w2 is the image of z2. So we have this transformation p times z1 then here z2 conjugate+alpha times z2 conjugate+alpha conjugate z2+r=0 and you can see that this condition okay shows that the points w1, w2 are inverse points for the image of gamma given by 6 okay.

This is the image of gamma okay so when we want to write the condition for w1, w2 to be the inverse points with respect to this image, the condition is this one okay so this condition we are using. So this condition you can see we are getting here p z1 z2 conjugate+alpha z2 conjugate+alpha conjugate z1+r=0, it is of that type okay. So this condition tells us that w1, w2 are inverse points for the image of the circle under w=Sz.

The image is given by the equation 6 okay. So this condition shows that w1, w2 are inverse points for the image of gamma under this w=Sz.

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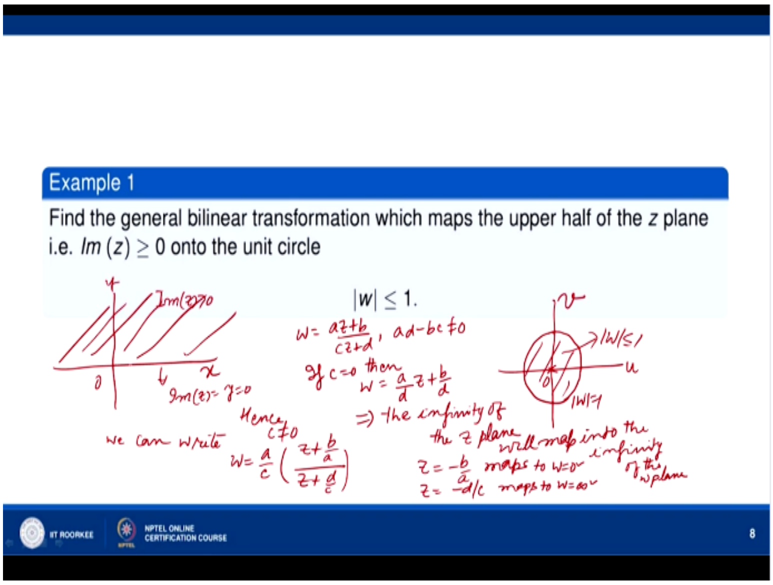

Now let us find the general bilinear transformation which maps the half plane upper half plane upper half of z-plane that is imaginary part of z>=0. So the boundary of the upper half of the z-plane, this is imaginary part of z>0 boundary is real axis okay. On the real axis, imaginary part of z=0 which is y, y=0 okay so we want to map it onto the unit disk in the w-plane into this region okay.

Now let us consider the general bilinear transformation w=az+b/cz+d. We know that it represents a bilinear transformation when ad-bc is not equal to 0, a, b, c, d are any complex constants okay. Now first thing that we notice is that if c is=0 then ad is not equal to 0, so a and d cannot be 0, so then w=a/d*z+b/d we get a linear mapping okay. Under a linear mapping, infinity z=infinity goes to w=infinity okay.

So then this implies that the infinity is in the two planes will correspond. The infinity of the z-plane will map into the infinity of the w-plane okay. So this means that z=infinity of the z-plane will go into infinity so in that case what will happen the imaginary part of z>0 will not be bounded. We want the imaginary part of z to go to a bounded region okay, mod of w<=1.

So mod of w<=1 is a bounded region and therefore it does not contain infinity w=infinity. So z=infinity if goes to w=infinity then the imaginary part of z>=0 will not be mod of w<=1 okay. So hence c cannot be 0 okay. So when c is not equal to 0, we can write w=a/c*z+b/a/z+d/c okay. Now let us see this transformation okay. Here what happens is z=-b/a goes to maps into w=0 which is the center of the circle mod of z=1 and z=-d/c maps to w=infinity okay.

Now we want the imaginary part of z>0 to map into the interior of mod of w=1. Now this w=0 and w=infinity are inverse points with respect to mod of w=1. So they must be the images of the inverse points with respect to the real axis okay because real axis is the boundary here okay.

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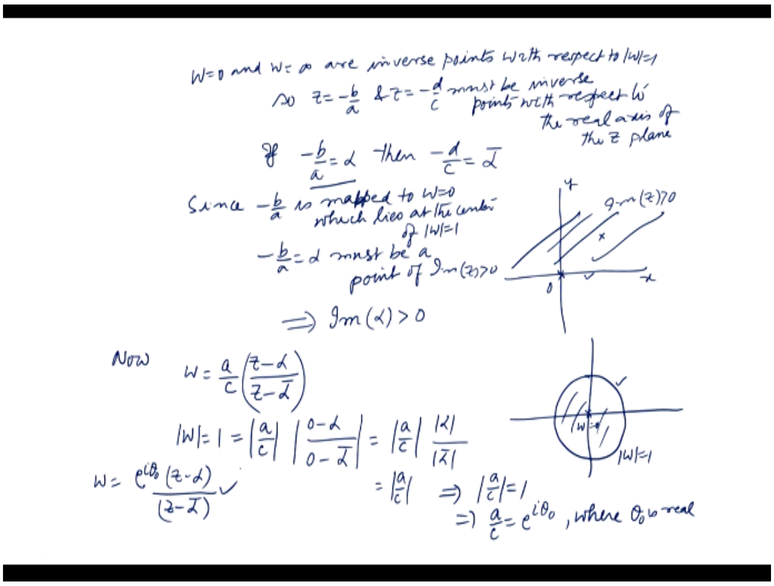

W=0 and w=infinity are inverse points with respect to mod of w=1. So z=-b/a and z=-d/c must be inverse points with respect to the real axis of the z-plane which means that if z=-b/a if this is alpha if –b/a=alpha then –d/c must be=alpha conjugate. Now one more thing we notice that –b/a okay, this is z-plane, this is w-plane okay. We have this mod w=1, so the interior imaginary part of z>0 okay, this we want to map to interior here of mod of w=1.

And –b/a is going to w=0 okay, this is w=0. So –b/a must be a point of the imaginary part of z>0 okay. So since –b/a is mapped to w=0 which lies at the center of mod w=1-b/a=alpha must be a point of imaginary part of z>0 okay which imply that imaginary part of alpha must be>0 okay or imaginary part of –b/a must be>0 okay. So now we can write w=a/c*z+b/a means z-alpha and z+d/c means z-alpha conjugate okay.

So about transformation w=a/c*z+b/a/z+d/c now transforms to w=a/c*z-alpha/z-alpha conjugate. Now we want this boundary of the region imaginary part of z>0 to map to the boundary of mod of w<1. So here boundary is mod of w=1, here boundary is imaginary part of z=0 that is the x-axis. So let us take the point z=0 on the real axis that z=0 must be mapped onto some point of mod of w=1.

So mod of w=1=a/c modulus of a/c*mod of z you put 0, 0-alpha/0-alpha conjugate okay. When z=0, it should map to some point where mod of w=1. So this is equal to mod of a/c mod of alpha/mod of alpha conjugate okay. Now mod of z=mod of z conjugate, so mod of z alpha/mod of alpha conjugate is 1, so we get mod of a/c okay. So 1 is=mod of a/c, this imply that a/c is=e to the power i theta 0 okay where theta 0 is a real number okay where theta 0 is a real quantity okay.

So we get the most general bilinear transformation that maps the upper half plane into mod of w<=1 is w=e to the power i theta 0*z-alpha/z-alpha conjugate. So this is the transformation which maps the upper half of the z-plane to the unit disk mod of w<=1 okay.

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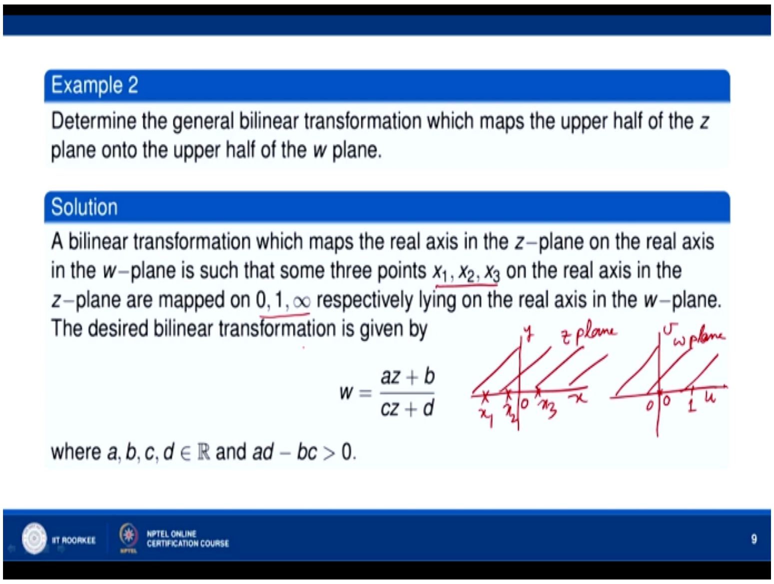

Now we go to the next example determine the general bilinear transformation which maps the upper half of the z-plane onto upper half of w-plane. So here we want the upper half of the z-plane to map to upper half of the w-plane okay. So this is z-plane, this is w-plane, so let us first find the bilinear transformation which maps the real axis of the z-plane to the real axis of the w-plane okay.

So a bilinear transformation which maps the real axis in the z-plane on the real axis of the w-plane is such that some 3 points you take 3 points x1, x2, x3 here, they go to say 3 points, let me take one point 0 and then another point 1 and the third point at infinity okay. So x1, x2, x3 if they map to 0, 1, infinity, we get a unique bilinear transformation which that it is. So let us map these points x1, x2, x3 to 0, 1, infinity okay, x1, x2, x3 are points on real axis x-axis.

And 0, 1, infinity are points on the u-axis, the real axis of w-plane okay. Now let us find the bilinear transformation under which x1, x2, x3 go to 0, 1, infinity.

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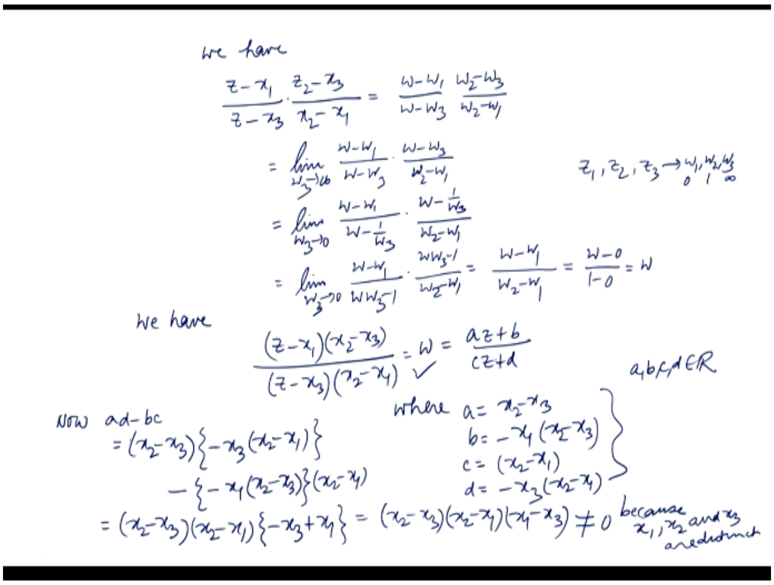

So we have the bilinear transformation which maps x1, x2, x3 to w1, w2, w3 that is given by we have z-x1/z-x3 then z2-x3/z2-x1 okay. When the z1, z2, z3 are mapped to w1, w2, w3 okay, we know what is the bilinear transformation which does this. So here z1, z2, z3 are x1, x2, x3. So this is equal to w-w1/w-w3 w2-w3/w2-w1 okay. Now w1, w2, w3 is w1 is 0, w2 is 1, w3 is infinity, so let us find the right hand side cross ratio right hand side it gives us 2.

Limit w3 goes to infinity, w-w1/w-w3*w-w3/w2-w1 okay. So this is equal to limit w3 goes to 0, w-w1/w-1/w3 then w-1/w3/w2-w1 okay. So this is equal to limit w3 goes to 0 w-w1/ww3-1*ww3-1/w2-w1 okay. So this is equal to w-w1/w2-w1, w1 is=0 okay and w2=1, so 1-0 so the right hand side is w okay. So we have z-x1*x2-x3/z-x3*this should be x2-x1 so x2-x1=w okay.

Now this is of the form az+b/cz+d where a=x2-x3, b=-x1*x2-x3, c=x2-x1 and d=-x3*x2-x1 okay. Now if we want to say that this is bilinear transformation then we should show that ad-bc is not 0 okay. So ad-bc is how much? So x2-x3*-x3*x2-x1-ad-bc okay, so –x1*x2-x3 and then x2-x1, so this is what I can take x2-x3 and x2-x1 common, then what we get is –x3+x1 okay. So what we get x2-x3 x2-x1 and x1-x3 and which is not equal to 0.

Because x1, x2, x3 are distinct, they are distinct okay and moreover that we notice that x1, x2, x3 are real numbers, so a, b, c, d belong to R okay; a, b, c, d belong to R because x1, x2, x3 are real okay and ad-bc is nonzero okay.

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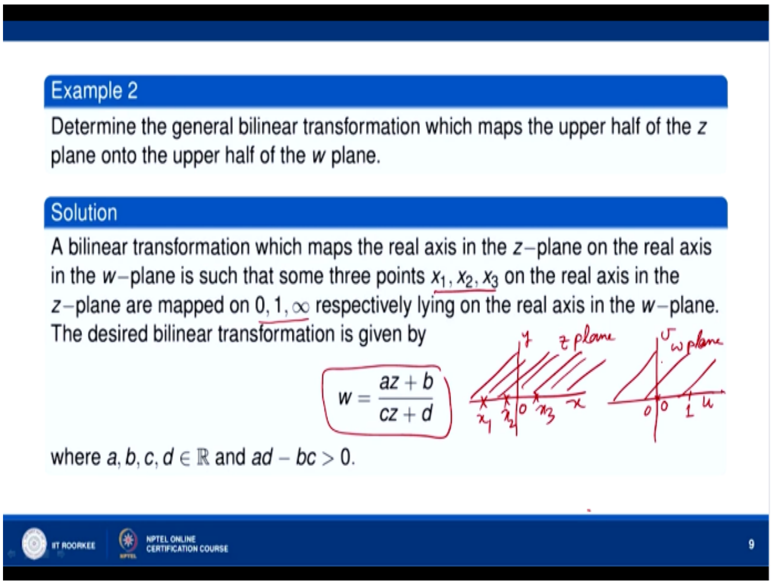

So we get the transformation w=ax+b/cz+d where a, b, c, d, is not 0 okay where a, b, c, d are real numbers and ad-bc is nonzero. Now let us show that the upper half of the z-plane goes to upper half of the w-plane.

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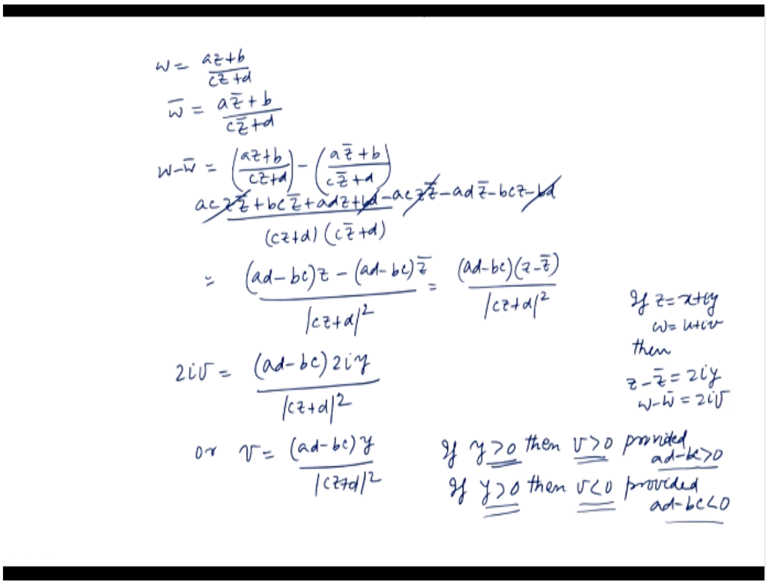

For that we have to consider w=az+b/cz+d okay. So then w conjugate is az conjugate+b/cz conjugate+d okay. We are not taking conjugates of a, b, c, d because they are real. Then w-w conjugate=az+b/cz+d-az conjugate+b/cz conjugate+d. So we take the LCM cz+d cz conjugate+d and then we get let us multiply, so ac zz conjugate okay then bc z conjugate, then we get ad z and then we get bd okay.

And then we get –ac zz conjugate, then we get –ad z conjugate and then we get –bc z and –bd. So this bd cancel out, ac zz conjugate cancel out and what we get, let us collect the coefficient of z and z conjugate. So the coefficient of z is ad-bc okay and z conjugate coefficient is what, -ad-bc*z conjugate/now cz+d*cz conjugate+d we can write as mod of cz+d square okay, so this is ad-bc*z-z conjugate okay/mod of cz+d square okay.

Now so what we have w-w conjugate is=ad-bc*z-z conjugate/mod of cz+d square okay. So if z=x+iy and w=u+iv okay, w=z, z=x+iy, w=u+iv, then z-z conjugate is 2 iy okay and w-w conjugate is 2 iv okay. So 2 iv=ad-bc*2 iy/mod of cz+d square okay or v=ad-bc*y/this. So if y is>0 okay if y is>0, then v is>0 provided ad-bc is>0. We can also can say that if y is>0, then v is<0 provided ad-bc is<0 okay.

So the upper half of the z-plane will map to the upper half of the w-plane if ad-bc is>0 and upper half of the z-plane will map to lower half of the w-plane if ad-bc is<0 okay. So the bilinear transformation which maps the upper half of z-plane to the upper half of the w-plane is given by w=az+b/cz+d where a, b, c, d are real numbers and ad-bc is>0. If you put the condition on a, b, c, d that which they are real and ad-bc is<0 then upper half of the z-plane will map to lower half of the w-plane okay.

So with that we come to the end of this lecture. Thank you very much for your attention.