Advanced Engineering Mathematics
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Lecture – 03
Harmonic Functions, Harmonic Conjugates and Milne's Method

Hello friends. Welcome to my lecture on Harmonic Functions, Harmonic Conjugates and Milne's Method. Let us first define a harmonic function. A real valued function fxy of 2 real variables, x and y is said to be harmonic in a domain D if it is a solution of the Laplace equation and has continuous partial derivatives of second order in D.

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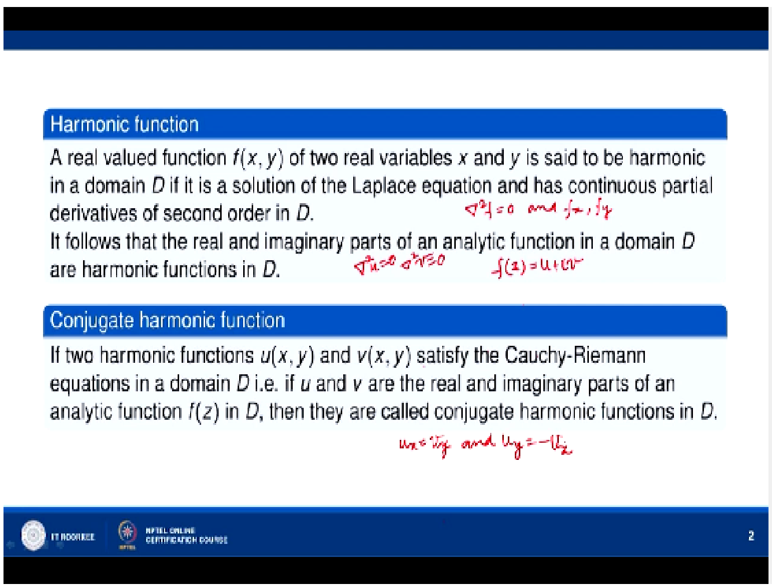

This means that del square f=0 and fx fy, they are continuous functions of x and y. It follows that real and imaginary parts of an analytic function in a domain D are harmonic functions in D. In our previous lecture, we have seen that the real imagine, if you take an analytic function fz=u+iv, then fz=x+iy, u and v are in general functions of x and y and u and v satisfy del square u=0, okay.

That is u and del square v=0, that is u and v are solutions of Laplace equation. And moreover, u and v have continuous second order partial derivatives. So the real and imaginary parts of an analytic function in a domain D are harmonic functions in D. Now let us look at conjugate harmonic function. If 2 harmonic functions uxy and vxy satisfies the Cache-Riemann equations in a domain D, that is ux=vy and uy=-vx, then u and v are said to be conjugate harmonic functions in D.

The real and imaginary parts of an analytic functions are therefore conjugate harmonic functions. Because they are harmonic functions and they are related by Cache-Riemann equations. So 2 functions are called conjugate, 2 harmonic functions are said to be conjugate harmonic functions if they are related by Cache-Riemann equations.

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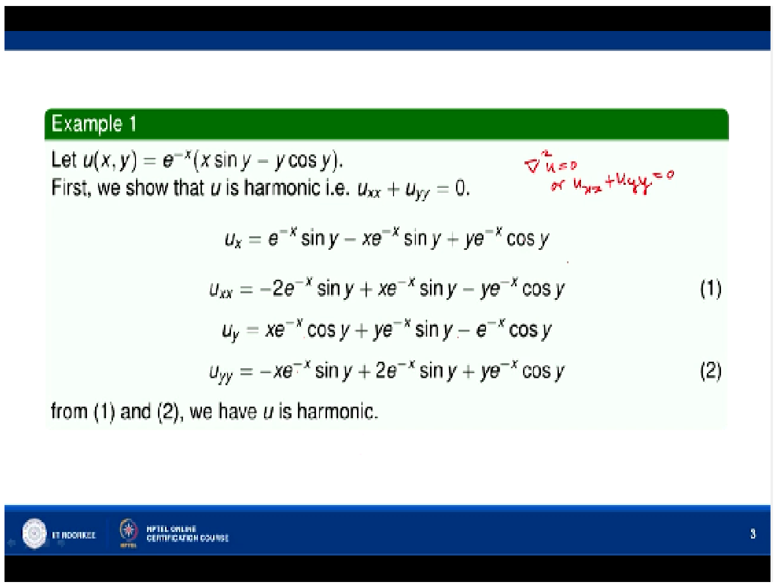

Now let us take an example if we know 1 harmonic function, the other harmonic function can be found out using the Cache-Riemann equations. And we can then find the corresponding analytic function. So if you have uxy=e to the power -x*x sin y-ye cos y, then first we see that this function uxy is a harmonic function of x and y. Now you can find the, in order to prove that u is harmonic, we need to show that u is a solution of the Laplace equation that is del square u=0 or we can say uxx+uyy=0.

And moreover the second order partial derivatives are continuous. So here you can see u is e to the power -x*x sin y-y cos y, it is infinitely differentiable, okay. All order derivatives of u exist, okay. And with respect to x and y, therefore, second order partial derivatives of u are continuous. So we just have to show that u is a solution of Laplace equation. So in order to prove that uxx+uyy=0, let us first find the partial derivative of u with respect to x.

If you differentiate u with respect to x, what you get is, ux=e to the power -x*sin y-x e to the power -x*sin y+y e to the power -x cos y. If you again differentiate ux with respect to x, you get second order partial derivatives of u with respect to x that is uxx=-2*e to the power -x sin y+x*e to the power -x sin y-y e to the power -x cos y. Similarly, if you differentiate u with respect to y, you find the partial derivative of u with respect to y.

Keeping x constant, we see that it is x e to the power -x cos y+y e to the power -x sin y-e to the power -x cos y. And then again if you differentiate uy with respect to y, you get uyy. So it comes out to be -x e to the power -x sin y+2*e to the power -x sin y+ y e to the power -x cos y. Now you can see if you add this second equation, uxx equal to this and uyy equal to this expression, then uxx+uyy, =this term will cancel with this term, this term will cancel with this and this term will cancel with this and we will get uxx+uyy=0.

So u is a harmonic function of x and y, okay in the whole complex plan.

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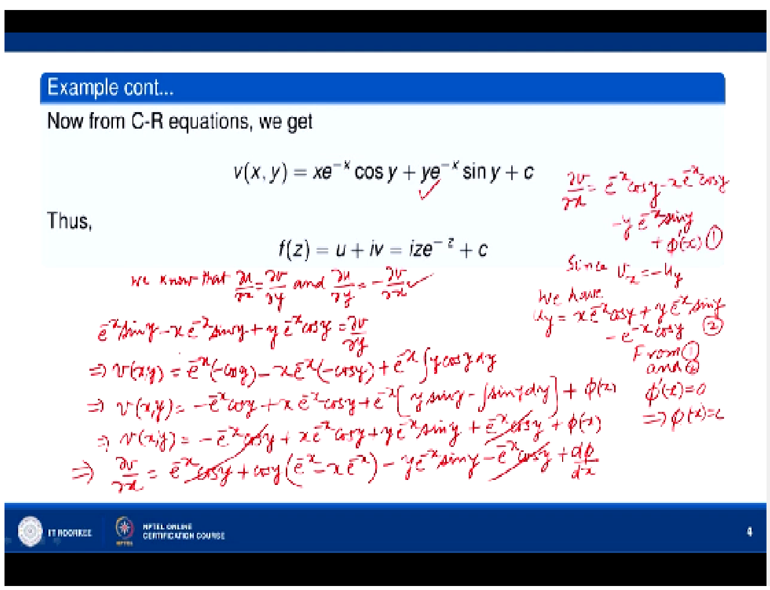

Now let us find the corresponding harmonic function vxy. So we will have to use the Cache-Riemann equations. So we know that ux=vy and uy=-vx, okay. So we have the value of ux with us, okay. Let us put the value of ux. So we get ux=e to the power -x sin y. So e to the power -x sin y, then we have -x e to the power -x sin y. Then we have y e to the power -x cos y. So this is the value of ux and ux=vy, okay.

Now in order to find v, let us integrate this equation with respect to y but this is partial derivative of v with respect to y. So while integrating v with respect to y, this equation with respect to y, we will have to keep x constant, okay. So this implies vxy=, since we are keeping x constant, e to the power x-x, e to the power -x will remain unchanged. Integral of sin y is -cos y, okay. And then we have -x e to the power -x.

Integral of sin y is -cos y. Then we have e to the power -x integral of y cos y dy, okay. This gives us vxy=-e to the power -x cos y+x e to the power -x cos y+e to the power -x. Now let us integrate this. So we have integral of, integration by parts we are doing. So y*sin y-integral, derivative of y is 1 and we have sin y+a function of, because we are integrating with respect to y assuming x as a constant.

So constant of integration will be a function of x. Let us write phi x. So this will be vxy=-e to the power -x cos y+x e to the power -x cos y. Then we have y e to the power -x sin y. Integral of sin y is -cos y. So we will have +e to the power -x cos y+phi x, okay. Now we will have to use in order to determine this unknown function phi x, we will have to use this second equation, okay. So in the second equation, we need the derivative of v with respect to x.

So let us differentiate this equation with respect to x. So when we differentiate this with respect to x, we get, derivative of e to the power -x is e to the power -x*-1. So we get e to the power -x cos y. And here, cos y*derivative of x is 1. So we get e to the power -x. Then derivative of e to power -x is e to the power -x*-1. So we get -x e to the power -x. And here, derivative of e to the power -x is e to the power -x*-1, so we get -y e to the power -x*sin y.

Here also we get -e to the power -x cos y and we get d phi y dx, because phi depends only on x, okay. This e to the power -x cos y and e to the power -x cos y will cancel and we get the partial derivative of v with respect to x as e to the power -x cos y, -x e to the power -x cos y and then we get -y e to the power -x sin y+phi dash x. This d phi/dx, let us write as phi dash x. Now this vx, vx=-uy, okay.

And since vx=-uy, okay, the value of vx=-uy and uy we have found here. So -uy is what? -x e to the power -x cos y, okay, let us put the value of uy, okay. We have uy=x e to the power -x cos y. Then we have y e to the power -x sin y and then we have -e to the power -x cos y, okay. So we have uy=this. Now let us put the values of, this is the equation, let us call it as 1 and this as 2, okay.

So vx=this value and uy=this value. From 1 and 2, what do you notice? vx=-uy, okay. So this vx=-uy will give you what? This term will cancel with this term, okay. This term will cancel with this term. When you use vx=-uy and this term will cancel with this term and what we get? phi dash x=0. So from 1 and 2, phi dash x=0, which implies phi x=some arbitrary constant c, okay.

So what do you get? vxy=, vxy is this, okay. This vxy, this was the value of vxy. This term you can cancel here itself, okay. So vxy=x e to the power -x cos y+y e to the power -x sin y+c, okay. So we get this value of vxy using phi x=c. Now fz=u+iv. Let us find the corresponding analytic function.

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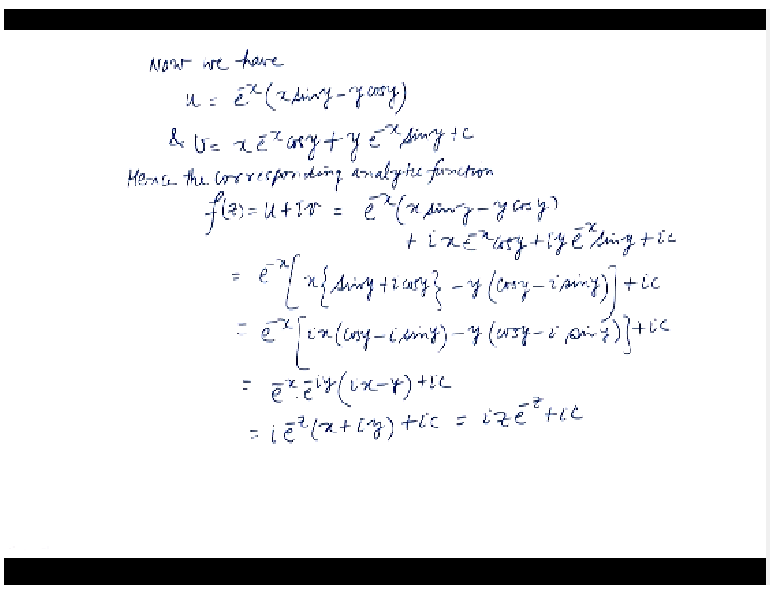

So now we have the value of u=e to the power -x, x sin y-y cos y, this is u, okay. And what is v? v we have found to be x e to the power -x cos y+y e to the power -x sin y+c, okay. So hence the corresponding analytic function fz=u+iv=e to the power -x*x sin y-y cos y+i*x e to the power -x cos y+iy e to the power -x sin y+i*c, okay. So now what do we get? Let us combine, let us write e to the power -x here and then see what we have here.

So we have x sin y, x*sin y+i cos y and here we have -y cos y. So -y*cos y-i sin y. Let us look at it again, e to the power -x here. Then x we take common from these 2 terms. So we get x*sin y+i cos y and here we write -y*cos y-i sin y, right, okay. So we have e to the power -x and here what I do? I take an i outside. So i*x and then this is cos y+1/i sin y. So cos y-i sin y it will be. And here this is already cos y-i sin y.

So -y cos y-i sin y. What do we get? So this will be e to the power -iy. So e to the power -x*e to the power -iy we get and what we have here? ix-y+ic. So this e to the power -x -iy e to the power -z e to the power -z, z is x+iy. i we can write here then x-1/i*y. So that is x+iy and this we get as iz*e to the power -z+ic. So we get the corresponding analytic function fz as iz*e to the power -z+i*c. Thus we get the corresponding analytic function fz as iz*e to the power -z+ic where c is a arbitrary constant, real constant, okay.

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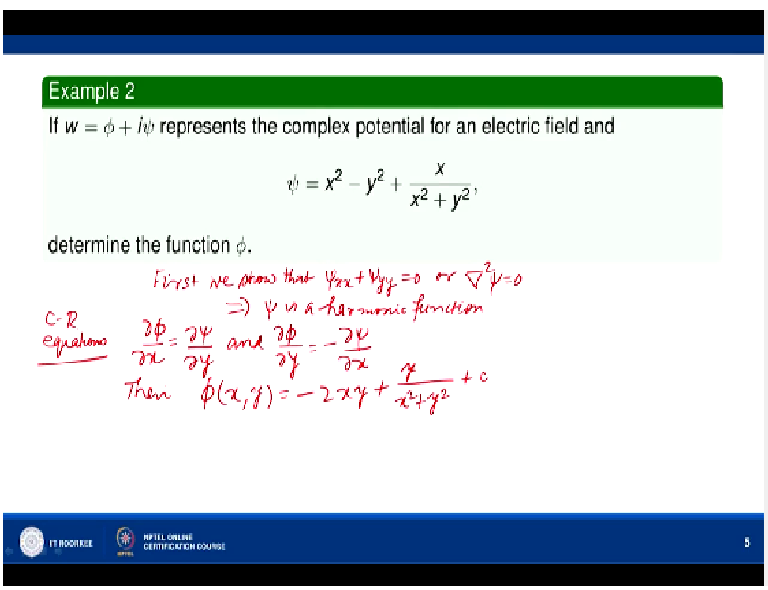

Now let us take 1 more example on this. Suppose w=phi+i psi represents the complex potential for an electric field and psi is given to be x square-y square+x/x square+y square. We want to determine the function phi, so first thing you can show that psi xx+psi yy=0. First we can show, first we show that psi xx+psi yy=0 or we can say del square psi=0, okay. Now psi can be here is differentiable infinite number of times with respect to x and y.

So the second order partial derivatives of psi are continuous functions of x and y, okay. And therefore, psi is a harmonic function of x and y, okay. Now what we do? We want to find the corresponding harmonic function, conjugate harmonic function phi. So we use the Cache-Riemann equations phi x=psi y and phi y=-psi x. So using these Cache-Riemann equations, we can determine the corresponding function phi.

It comes out to be phi xy=-2xy+y/x square+y square. Phi xy=-2xy+y square. So that is the correspond, the +a constant function, constant c, okay. So first we show that psi is a harmonic function, then use the CR equations. By using CR equations, we have phi x=psi y and phi y=-psi x. Then it turns out that, we follow the same process as we have done in the case of example 1 and we see that phi xy=-2xy+y/x square+y square+real constant c, okay.

So phi xy=-2xy+y/x square+y square+c. So we can follow the same process as an example 1 to arrive at this function phi.

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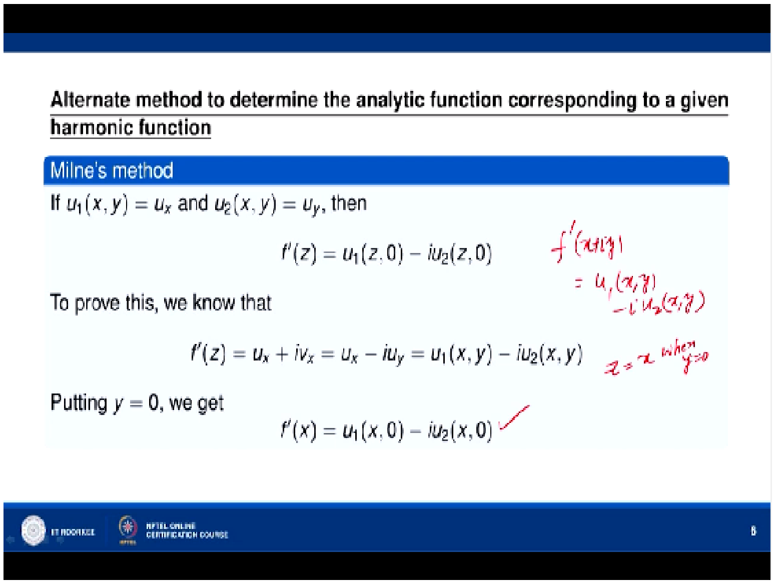

Now there is an alternate method to determine the analytic function corresponding to a given harmonic function. It is known as Milne's method. Suppose we denote the partial derivative of u with respect to xy the function u1xy and the partial derivative of u with respect to y by another function u2xy, then f prime z is given by u1z0-i*u2z0. This is the formula to determine the function, analytic function fz.

So f prime z=u1z0-iu2z0, let us see how do we get this? So to prove this formula, we know that f prime z we have seen in the theorem on Cache-Riemann equations where we proved that if fz is differentiable at a point z, then f prime z is given by ux+ivx, okay. So we know that f prime z is ux+ivx and it satisfies Cache-Riemann equations because this is analytic. So ux+ivx is also equal to ux-iuy because uy=-vx.

So ux-iuy and the ux be denoted by u1uy. So we have u1xy here. And uy=u2xy. So f prime z=u1xy-iu2xy. Now let us put y=0 in this equation, okay. If you put y=0, then z=x+iy, so f prime x+iy, this is equal to u1xy-iu2xy, okay. So putting y=0, we get f prime x=u1x0-iu2x0, okay. Now when y=0, z=x. Z=x when y=0, okay. So f prime x=u1x0-iu2x0. Now let us replace x by z.

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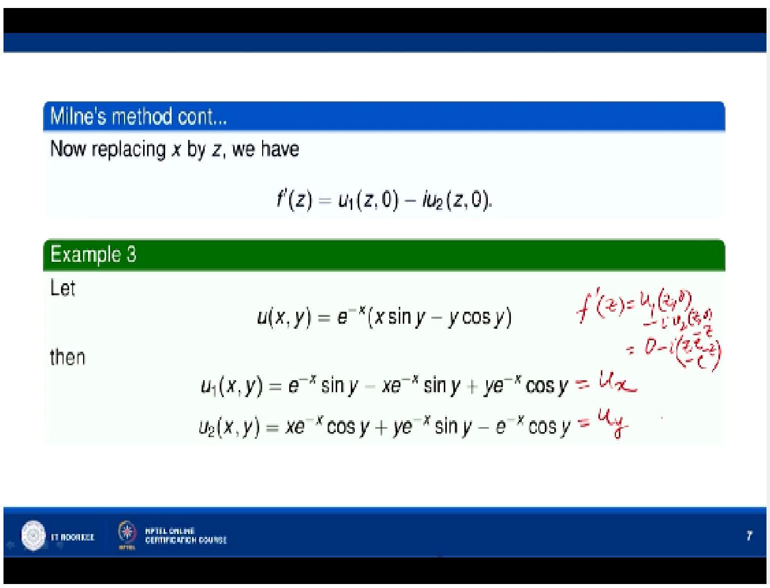

So we get f prime z=u1z0-iu2z0, okay. So this is how we prove this result. Now this result can be used to determine the corresponding analytic function when the harmonic function u is known, okay. So let uxy=e to the power -x x sin y-y cos y, let us take the example which we have earlier considered, okay. So we found that the partial derivative of u with respect to x, this is ux, okay.

And partial derivative of u with respect to y came out to be this expression, okay. By our notation, ux is u1xy. So u1xy equal to this and u2xy equal to this which is equal to uy, okay. Now what we do? f prime z=u1z0-iu2z0. So in this expression, replace y by 0 and x by z. What do we get? u1z0 is e to the power -z*sin 0 which is 0 and here also we have sin 0, so it is 0. Here y is 0, so this is also 0.

So u1z0=0. And then -i*u2z0 is what? When y=0, this term becomes 0 and this term becomes z*e to the power -z. So z e to the power -z*cos 0 is 1. This term becomes 0 when y is 0. This term becomes -e to the power -z, okay. So we get f prime z=-i*z e to the power -z-e to the power -z. Now we can find the integral of this function, okay.

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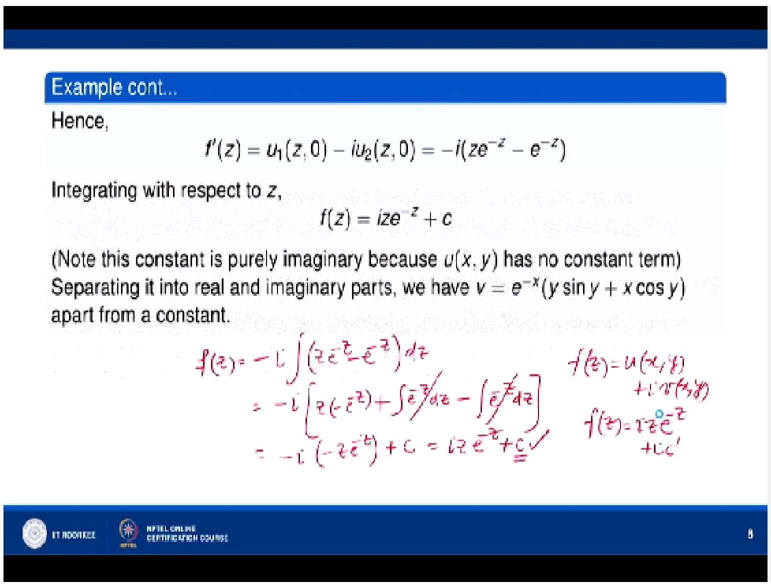

So f prime z is this, okay. We can integrate that. So fz=-i*integral of z e to the power -z-e to the power -zdz, okay. And this is -i*integral of z e to the power -z will be z*-e to the power -z -integral of e to the power -zdz. And then we have second term -integral of e to the power -zdz, okay. So this is -i*-z e to the power -z, okay, and here what we get when we integrate? This will be - here, so this will become + when we put - e power -z and this will cancel with this, okay.

So this +some constant, okay. Now what is it? iz*e to the power -z+some constant c. Now let us look at the nature of this constant. Whether it is real or it is complex or what. So this constant is purely imaginary. Why? Because uxy, fz is uxy+ivxy. fz=, okay. So this quantity, this quantity, must be equal to uxy+ivxy. Now this means that this constant c does not have a real part. Because uxy does not have a constant term, okay. And therefore, this c is purely imaginary.

So we can write this c as some i*another constant, okay. So we have fz=iz*e to the power -z+some constant i*c dash, okay. So this is how we get the corresponding analytic function fz. Now if you want to find the harmonic function, conjugate harmonic function of u, then that you can find from fz=iz*e to the power -z+ic dash/separating it into real and imaginary parts, okay. By writing z=x+iy and equating real and imaginary parts, we can find the function vxy.

So this Milne's method helps us in finding the analytic function associated with a given harmonic function.

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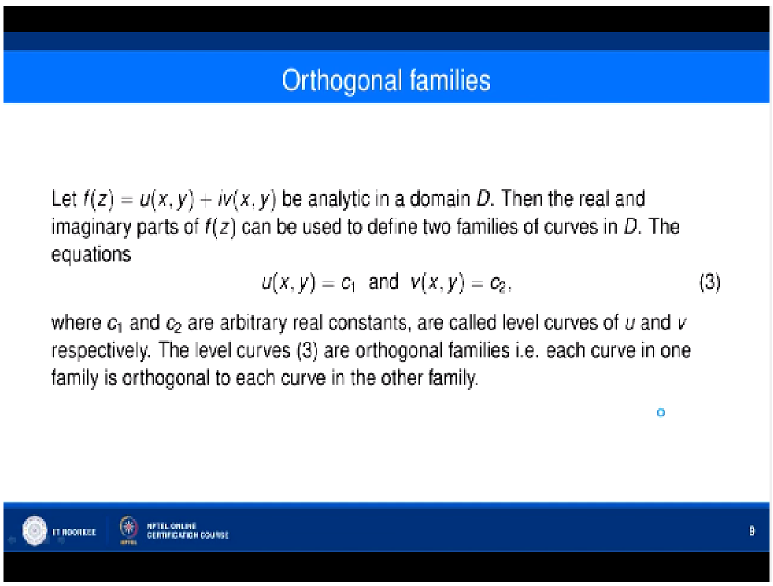

Let us now discuss orthogonal families. This is a very important property of analytic functions. Let fz=uxy+ivxy be analytic in a domain D. Then the real and imaginary parts of fz can be used to define 2 families of curves in D. They are called as level curves. So the equations uxy=c1 and vxy=c2 where c1 and c2 are arbitrary real constants, are called level curves of u and v respectively.

The level curves of these equations are defined orthogonal families. That is each curve in one family is orthogonal to each curve in the other family. That means wherever they intersect at the intersection point, the tangent to one curve is a perpendicular to the tangent to the other curve. Say for example let us look at this.

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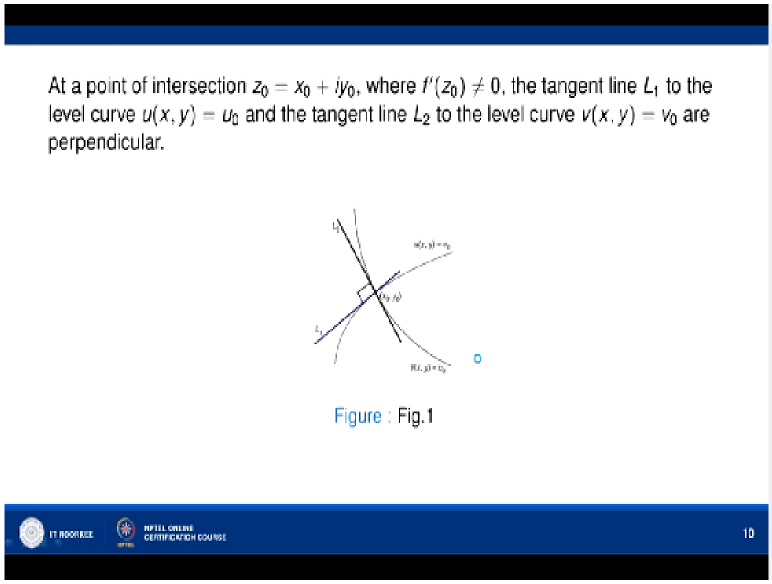

Suppose we have one family of curves uxy=u0 and this is another curve, vxy=v0. u0 and v0 are obtained from the fact that uxy=c1 and vxy=c2, they cut each other at x0y0. So from that we can determine u0. u0 is the value of uxy at x0y0, okay. So at a point of intersection, z0=x0+iy0 where f prime z0 is not equal to 0. This is essential that at the point z0, f prime must not be 0. The tangent line L1, this is tangent line L1, okay.

This is tangent line L1 to the curve uxy=u0 and the tangent line L2, this is tangent line L2 to the other curve vxy=v0. They are perpendicular. So each curve of the one family is orthogonal, cut each curve of the other family at right angles. So they are called orthogonal families.

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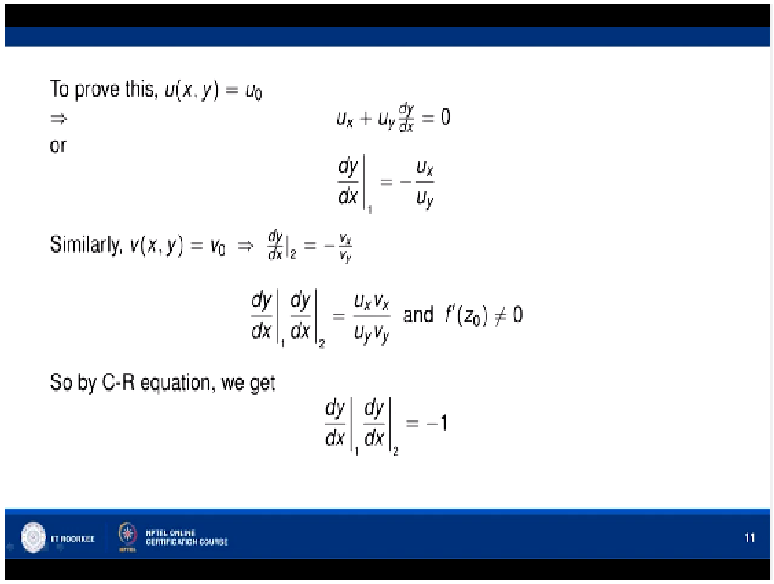

Now let us prove this. So how we prove this? We have 1 curve, uxy=u0, so okay. So uxy=u0 gives us, when we differentiate this with respect to x partially, we get ux+uydy/dx=0. dy/dx1 means this is the slope of the tangent to the curve uxy=u0 at the point x0y0. So this is -ux/uy. These partial derivatives are being calculated at the point x0y0, that is at the point of intersection of the curves uxy=u0 and vxy=v0.

So similarly, vxy=v0 gives in the same manner when we differentiate it with respect to x, we get dy/dx2=-vx/vy. So again here vx and vy are calculated at the point of intersection x0y0. Now the product of the 2 slopes, dy/dx1 dy/dx2 is uxvx/uyvy. Now the function fz is analytic. So Cache-Riemann equations will hold, okay. So ux will be equal to vy, so this ux and vy will cancel but uy=-vx. So uy=-vx gives us that uxvx/uyvy=-1.

So by CR equations, we get dy/dx1*dy/dx2=-1. And therefore, the curve uxy=u0 cuts the curve vxy=v0 at right angle at the point of intersection x0y0.

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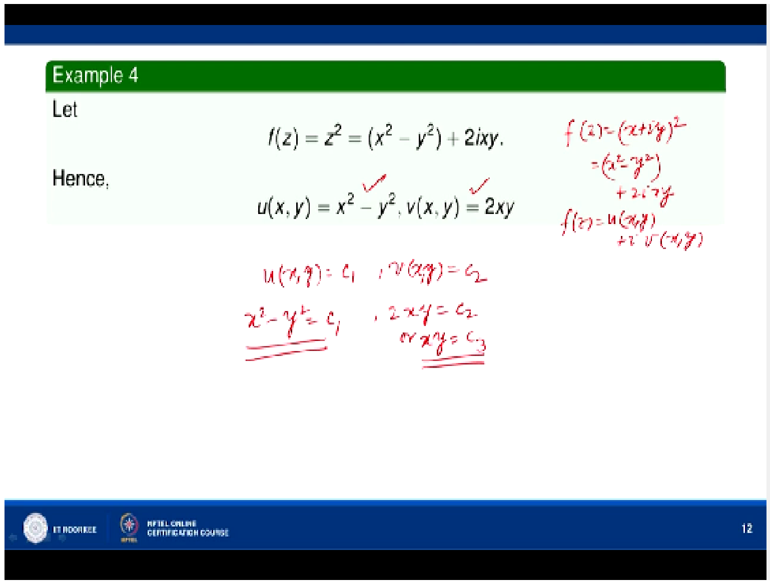

Now let us consider the function fz=z square. It is an analytic function for all z, okay. And here you can see when we put z=x+iy and then when x+iy whole square gives you using iota square=-1, we get x square-y square+2ixy. So here fz=uxy+ivxy when we write, okay, equating real and imaginary parts, we get uxy=x square-y square and vxy=2xy. Now when you take uxy=a constant, say c1 and vxy=another constant c2.

That is you find the level curves given by the equation uxy=c1 and vxy=c2. You can see that this is x square-y square=c1 and this 2xy=c2 or you can say xy=some other constant c3, okay. So this is the family of hyperbolas and this is also a family of hyperbolas, okay. And each member of one family cuts each member of the other family at right angles. So let us see that.

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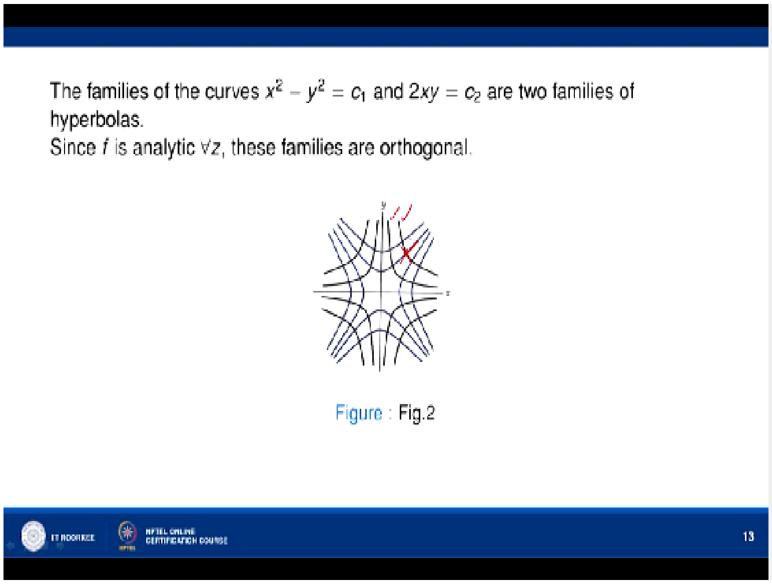

This you can see the families of curves x square-y square=c1, 2xy=c2 are two families of hyperbolas. Since fz is analytic for all z, these families are orthogonal, we have just now shown. Now these curves, okay, if you look these curves, so this one, okay, black curves, they are at xy=constant, okay. While this other blue curves, they are given by x square-y square=c1. So at each point of intersection, you can see tangent to one is orthogonal to tangent to the other, okay.

So the two families level curves are orthogonal to each other. With this, I would like to end my lecture. Thank you very much for your attention.