**Advanced Engineering Mathematics**
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**Lecture - 29**
**Conformal Mapping - II**

Hello friends. Welcome to my second lecture on conformal mapping.

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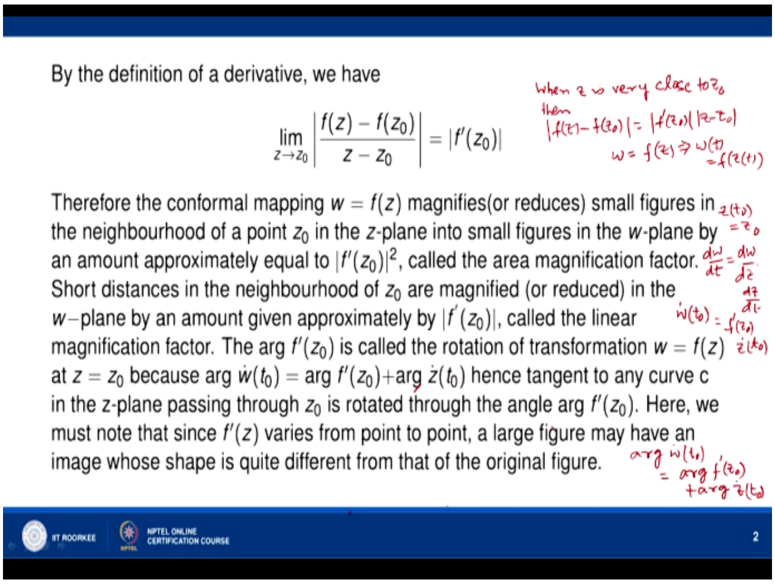


We know that by the definition of derivative limit z tends to z0 mod of fz-f z0/z-z0 is=mod of f prime z0 and therefore we can see here that when z is very close to z0, mod of fz-f z0 is=mod of f prime z0*mod of z-z0. When z is very close to z0, then mod of fz-f z0 is=approximately mod of f prime z0*mod of z-z0.

Therefore, the conformal mapping w=fz magnifies or reduces small figures in the neighborhood of a point z0 in the z-plane into small figures in the w-plane by an amount equal to mod of f prime z0 square called the area magnification factor. In the case of area, it will be mod of f prime z0 square and in the case of distances like here mod of fz-f z0=mod of f prime z0*mod of z-z0.

The length will be magnified or reduced by the factor mod of f prime z0. This is here the length of the amount between fz and f z0 and here mod of z-z0 is the length between z and z0. So the lengths in the z-plane are magnified or reduced in the w-plane by an amount mod of f prime z0 and the areas are magnified or reduced by an amount approximately equal to mod of f prime z0 square.

So in the case of area, mod of f prime z0 square is called the area magnification factor and in the case of lengths, mod of f prime z0 is called the linear magnification factor. The argument of f prime z0 is called the rotation of the transformation w=fz at z=z0 because argument of w dot t0=argument of f prime z0+argument of z prime t0. This we have seen in the lecture on conformal mapping in the previous lecture.

We had w=fz and when the curve C is given by its parametric representation with this imply that w t=f of zt. Now let us say z t0=z0. Then, this equation gave us w t=f zt gives us by chain rule dw/dt=dw/dz*dz/dt. So we can say that w dot t0 is=dw/dz is f prime z, so f prime z0*z dot t0 okay and this gives us argument of f w dot t0=argument of f prime z0+argument of z dot t0.

Now this equation tells us that the tangent to the curve C at the point t=t0 that is z=z0 is rotated by an angle or equal to argument of f prime z0 in the w-plane. So the tangent to any curve C in the z-plane passing through z0 is rotated through the angle argument of f prime z0 and that is why f prime z0 is called the rotation factor, rotation of the transformation w=fz at z=z0. Now here we have to notice that since f prime z varies from point to point a large figure.

This actually is valid because limit of mod of fz-f z0/z-z0 is=mod of f prime z0, so this is actually valid for values of z which are very close to z0 that is in a sufficiently small neighborhood of z0, the lengths are magnified or reduced by a factor mod of f dash z0, the areas are magnified or reduced by a factor mod of f prime z0 square but in the case of a large figure, we may have an image whose shape is quite different from that of the original figure so that we have to make a note of.

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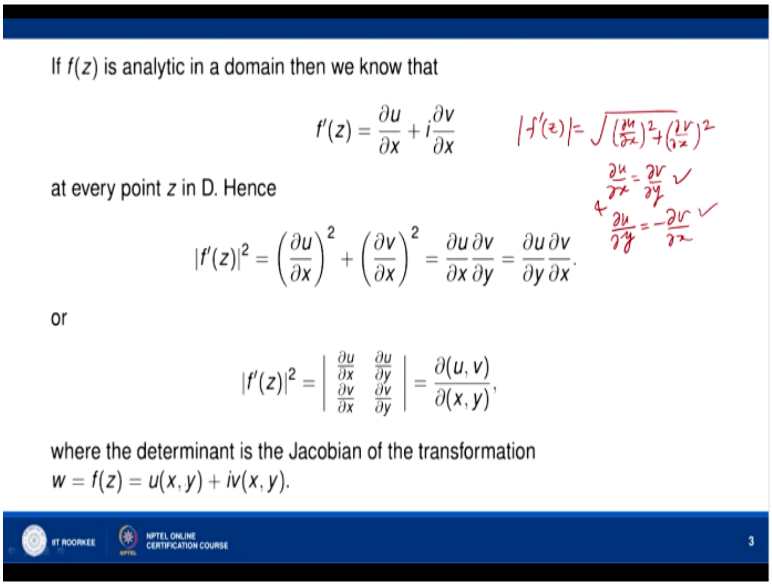


Now if fz is analytic in a domain then we know that the derivative of fz at a point z in D is given by derivative of u with respect to x partial derivative and then+i times partial derivative of v with respect to x. Now from here mod of f prime z is=square root ux square+vx square so that mod of f prime z square is ux square+vx square and we also know that when the function is analytic in a domain D, it satisfies Cauchy-Riemann equations at every point in D.

So ux is=vy and uy is=-vx okay, so let us make use of these Cauchy-Riemann equations. Then, we can write ux square as ux*vy and vx square we can write as vx*-uy okay. So ux*vy-uy*vx or we can say mod of f prime z square we can write this value ux vy-uy vx in the form of the determinant ux uy vx vy and this determinant is nothing but the Jacobian of u, v with respect to y, so where this determinant is the Jacobian of the transformation w=fz=u x, y+i v x, y okay.

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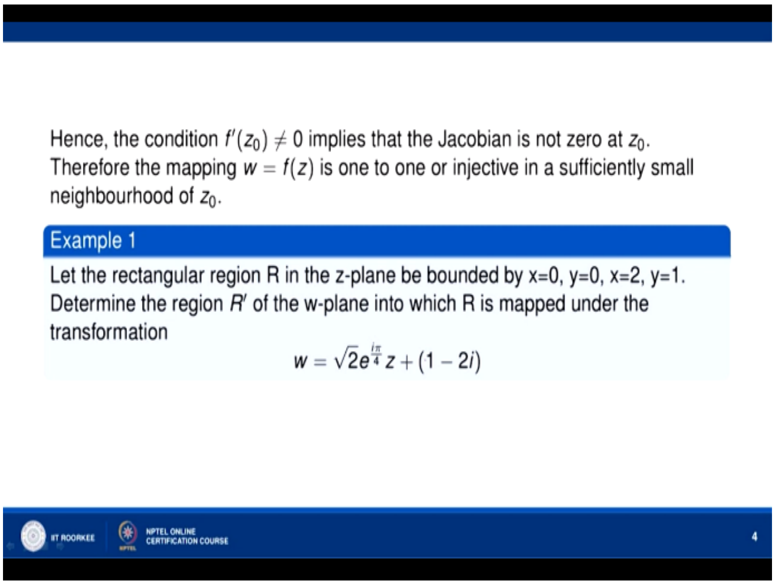


Now hence the condition f prime z0 is not equal to 0. If f prime z0 is not equal to 0, then what do we get, the Jacobian of u, v with respect to x, y at the point x0, y0 will be nonzero and this implies that the mapping w=fz is one to one or injective in a sufficiently small neighborhood of z0. Now let us consider an example. Let us say take the rectangular region R in the z-plane which is bounded by the lines x=0, y=0, x=2, y=1.

And we have to determine the region R dash of the w-plane into which R is mapped under the transformation w=root 2*e to the power i pi/4*z+1-2i.

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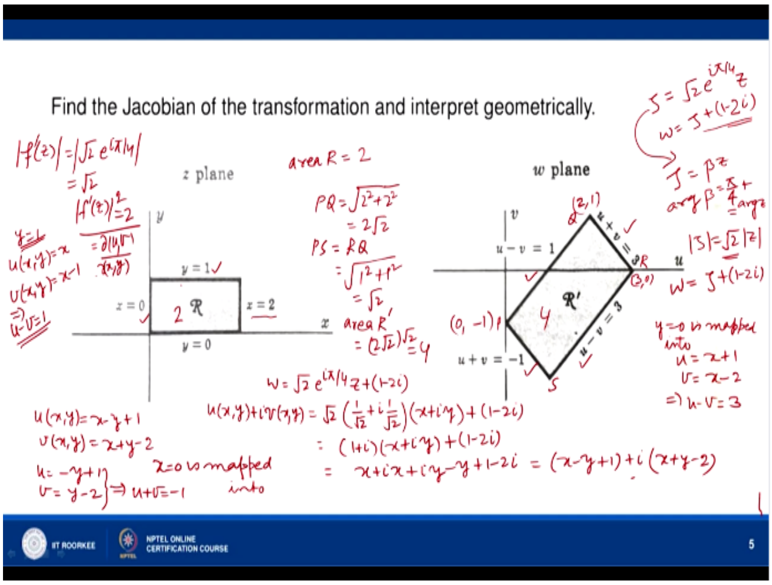


Now we can see the figure here. We have to also find the Jacobian of the transformation and interpret the result geometrically. Now this is your y-axis, this is x-axis, so this is line y=0, this line x=0, we have x=2 here, y=1 here, so it is a rectangular region, bounded by x=0, y=0, x=2, y=1 okay. Now we have the transformation w=root 2*e to the power i pi/4*z+1-2 i. Now we have w=u x, y+i v x, y and we have z=x+i y, so we have root 2 e to power i pi/4 is cos pi/4+i sin pi/4 so 1/root 2+i*1/root 2.

And here we have x+iy+1-2i, so we have 1+i*x+iy+1-2i. Now we can write it as x+ix, then iy, then we have i square y so –y+1-2i. Now collecting the real parts and imaginary parts on the right side, we have x-y+1+i times x+y-2 okay. So equating real and imaginary parts what we get u x, y=x-y+1 and v x, y=x+y-2 okay. Now let us see where y=0 is mapped under the transformation w=root 2 e to the power i pi/4*z+1-2i.

So let us first see the image of x=0 okay, so x=0 is mapped into okay so x=0 gives you u=-y+1 and v=y-2 okay so that u+v is=-1. This gives you u+v=-1 okay. So x=0 this is mapped into u+v=-1 and similarly we can see y=0 is mapped into u=x+1 and v=x-2 okay. So this gives you u-v=3. So y=0 goes to u-v=3 okay and similarly we can see the image of y=1. When y=1 okay we get u x, y=x okay and v x, y is=we are looking at the image of y=1.

So x-1 okay and now eliminating x here what we get, u-v=1 okay so u-v=1 okay. This is the image of y=1 and then the image of x=2 is u+v=3. So we get a rectangular region again in the w-plane but you can see under the transformation w=root 2*e to power i pi/4*z+1-2, we actually have a composition of 2 transformations, one is of the type let us say zeta=root 2 e to the power i pi/4*z and the other one is w=zeta+1-2i okay.

Now let us look at first zeta=root 2*e to the power i pi/4*z okay. This transformation is of the type zeta=beta z okay. So if you see the argument of beta, argument of beta will be root 2 is a positive real number, so its argument is 0, e to the power i pi/4 has argument pi/4 okay+argument of z okay. Argument of beta is pi/4+argument of z that means the figure in the z-plane is rotated in the anti-clockwise direction by an angle pi/4.

So this y=0 will be in the direction of the ray theta=pi/4, so this rectangular region is rotated in the anti-clockwise direction by an angle pi/4 and this root 2 will play the role in the magnification of the figure. So mod of zeta we can say is=root 2 times mod of e to the power i pi/4 is 1, so mod of zeta is=root 2 times mod of z. This means that the figure in the zeta plane will be magnified by root 2 times the figure in the z-plane.

That means each side of the z-plane will be magnified by the factor root 2 in the zeta plane okay. So this zeta plane the figure in the zeta plane that we have is then translated by the number alpha=1-2i. So which means that w=zeta+1-2i okay. There will be translation in the direction of 1-2i/the magnitude of mod of 1-2i that is mod of 1-2i is square root of 5. So we will translate the figure that we obtain by zeta=root 2 times e to the power pi/4*z/the magnitude of 1-2i in the direction of 1-2i.

So this is the figure that we get ultimately in the w-plane. So this means that the figure in the w-plane is the resultant of the rotation and magnification and translation okay, w=root 2 times e to power i pi/4*z rotates the figure by angle pi/4, magnifies it by the factor root 2 and then the w=zeta+1-2i translates the figure by this complex number 1-2i in the direction of 1-2i.

So we will translate the figure by the magnitude of 1-2i in the direction of 1-2i and we get this figure okay. Now let us see the area of the rectangular region in the z-plane is area R is=this length is 2 okay and width is 1, so area is=2 okay and here you can see the area u+v=-1, so we can find this point, this point is this is u=0, so this is 0, -1 point, this point is let me call it P, this point is 0, -1 and here u+v=3, v=0 so this is 3, 0 point okay.

And u-v=1, u+v=3 gives you when you add the two let us find this point also, so u-v=1, u+v=3 gives you 2u=4, so u=2 okay and when u=2, v=1. So this point is let me call it as Q and this point as R and this point as S okay. So PQ is=the distance between 0, -1 and 2, 1 so that is under root 2 square+2 square. So this is 2 root 2 okay. Now you can see this u-v=1, u-v=1 corresponds to y=1 okay.

This corresponds to y=1 this line okay and the length of y=1 is 2 okay. The length of y=1 is 2 and this length is then magnified by root 2 factor, it becomes 2 root 2 here okay and similarly here the length PS, PS is=same as your RQ, so this one is 3-2 square that is 1 square and then +1 square, so that is root 2 and you can see that this u+v=3, u+v=3 we got as corresponding to x=2 we have u+v=3 and the length of this side of the rectangular region is 1 okay and here this length RQ is root 2.

So this length this side of the rectangular region is magnified by root 2 okay. So this length of RQ becomes root 2. Now the area of the rectangular region R dash is area R dash is=you can see 2 root 2*root 2, so this is=4 okay. So here the area is 2, here area becomes 4 okay. Now when we discussed this article we have seen that when the mapping is conformal okay here the mapping is conformal.

And so the area is magnified by a factor mod of f prime z0 square okay, mod of f prime z0 square and here let us find f prime z, mod of f prime z square is=the Jacobian of u, v with respect to x, y, so let us find this Jacobian and see what we get. So here mod of f prime z okay f prime z=root 2*e to power i pi/4 okay, f prime z=root 2*e to power i pi/4, so this is mod of e to power i pi/4 is 1, so root 2 okay and mod of f prime z square is=2 okay.

So area of the region R is then magnified by mod of f prime z square that is 2 times the area of R, so R dash has area 2 times the area of R and lengths we have seen they are magnified by mod of f prime z which is=root 2, area is magnified by mod of f prime z square. So we have found the Jacobian here, Jacobian of the transformation, Jacobian is nothing but Jacobian is mod of f prime z square okay. So Jacobian is=this Jacobian okay Jacobian is=mod of f prime z square, so this is=2.

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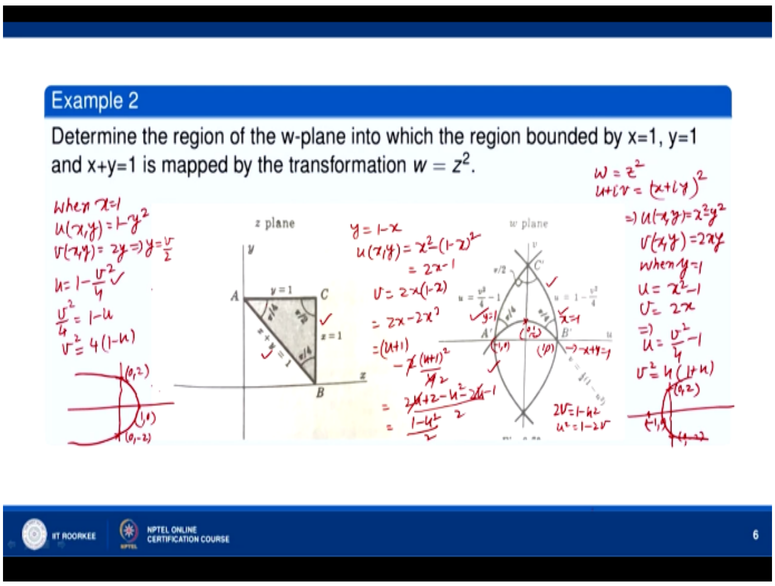


Now let us go to the next question. Determine the region of the w-plane into which the region bounded by x=1, y=1 and x+y=1 is mapped by the transformation w=z square. So w=z square gives you u+iv=x+iy whole square which gives you u x, y=equatorial and imaginary parts x square-y square and v x, y=2xy. Now the region in the z-plane is bounded by x=1, y=1 and x+y=1, so this is the line by BC.

The line BC is x=1, AC is y=1 and this line is x+y=1 is this AB, so this is the triangular region okay and we can see that this angle is pi/4, AC and BC are equal, so this angle is same as this angle, so this is pi/4, this is pi/4 okay. Now let us see the image of these sides of the triangle into the w-plane where they are mapped okay. So when x=1 what do we get, u x, y=1-y square, v x, y=2xy so that gives 2y okay.

So y=v/2, so u is=1-v square/4 okay. So this gives you a parabola okay. So x=1 is mapped into a parabola u=1-v square/4. So this is the parabola u=1-v square/4. We can easily draw this parabola v square/4=1-u okay or v square=4 times 1-u. So you can see that we can easily trace this parabola. If u is>1, then v square is negative, so v will be imaginary, so the curve does not exist for u>1 and when u is=1, it is v=0 so vertex of the parabola is at 1, 0.

And when your u=0 we get v square=4, so v=+-2 so it crosses the v axis at 0, 2 and 0, -2 okay So it opens leftwards, so we can easily draw this parabola okay. This is v square=4 times 1-u, this is the parabola okay u=1-v square/4. Now let us take the other side say y=1, so this is mapped into this one okay and then y=1 if you take when y=1 we get u=x square-1 and v=2x okay, so this gives you u=v square/4-1 okay.

So this is u=v square/4-1 okay. This parabola then we can easily draw v square=4 times 1+u okay. So this will open rightwards okay because if u is<-1 okay then v square will be negative, so v will be imaginary. So this is your -1, 0 point okay and then it crosses the v axis at again points 0, 2 and 0, -2. This is 0, 2 and here we have 0, -2. So it opens like this okay, so this is the parabola okay, u=v square/4 okay.

And then u=v square/4-1 and then we have x+y=1, so when x+y=1 what do we get, u x, y, y=1-x okay for this y=1-x, so u x, y=x square-1-x whole square. So this gives you 2x-1 and v is=2x*1-x. So this is 2x-2x square. 2x is=u+1, so we get u+1 here and then we have -2 times u+1/2 whole square, so u+1 whole square/4 so this gives you 2 and this is what 2u+2-u square-2u-1/2.

So what do we get, 2u will cancel and we get 1-u square/2 okay. So v is=1-u square/2 and this also we can draw very easily okay because when u is 0 okay v=1/2, so this is 0, 1/2 point and when it crosses the and moreover if u square this is 2u=1-u square okay so we can say u square=1-this is 2v so u square=1-2v okay. So this means that if v is more than 1/2 okay then u square will be negative, so u will be imaginary so v cannot be more than 1/2 okay.

That means it opens downwards okay. So when u is=0, v=1/2 and when v=0 u square=1, so it crosses the u axis at this is 1, 0 point and this is -1, 0 point okay. So this is parabola okay. Now let us see so this one when u=1-v square/4 is the image of x=1 okay. This is image of x=1, this one is image of y=1 and this parabola is image of x+y=1. Now let us see the angle between x=1 and y=1 is pi/2 and here also you can see the angle between the images of x=1 and y=1 is pi/2.

How we can see that? The point of intersection of u=1-v square/4 and u=v square/4-1, we have to see that by finding the slopes of the tangents, so u=let me do it on this one.

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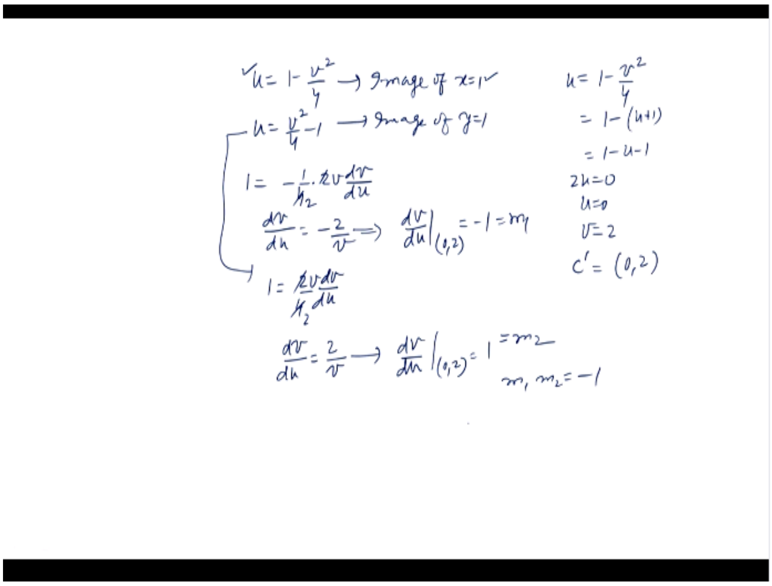


So u=1-v square/4, this is the image of x=1 and u=v square/4-1 that is the image of y=1. So u=v square/4-1, this is the image of x=1 and this is the image of y=1. Let us see the angle of intersection between the two. So what do you get, dv/du let us find, so when you differentiate this with respect to u what you get, du/du is 1 so then we have -1/4*2v dv/du so what we get here, dv/du=-4/2 so that means -2/v okay.

This is for the image of x=1 okay. So for the slope of the tangent is this one and for this one the slope of the tangent is similarly 1=2v dv/du/4 okay. So what we get here dv/du=2/v okay. Now let us see v square/4 you can simplify u=1-v square/4 okay and v square/4=u+1 here so let us put that, so 1-u+1 okay, so what do we get, 1-u-1 okay. So this means that 2u=0, so u=0, u=0 means v=2 and +-2 but v=+-2 so that means because we are in the upper half plane so C dash is having coordinates 0, 2 okay.

So the coordinates of C dash are 0, 2 okay, so C dash denotes the curve okay the point of intersection. This point of intersection is 0, 2 okay. So this means let us put here so here this gives you dv/du at 0, 2=-1 and this gives you dv/du at 0, 2=+1 okay. So product of this is m1 say this is m2. So m1*m2=-1, so they are perpendicular to each other that means this angle of intersection is pi/2 okay.

So here we are getting a curvilinear triangle and we see that the angles between the sides okay of the triangle and the angle between the corresponding arcs here corresponding sides are curvilinear triangle are same. They are equal in magnitude and also in sense because you see that when this is x=1, from x=1 to y=1 when we go we go clockwise okay, clockwise direction, in this direction.

And here also from the image of x=1 to the image of y=1 when we move we go clockwise okay. So angle of intersection is preserved in magnitude as well as in sense. So this also we can see. These angles are pi/4 by considering the slopes of the tangents to the curves which are the images of the other sides of the triangle. So these angles are also pi/4, so angles are preserved in magnitude as well as in sense.

So this is because w=z square is conformal at each point where dw/dz is not equal to 0 and dw/dz is equal to 0 at z=0. So because of conformality of w=z square the angles are preserved in magnitude as well as in sense.

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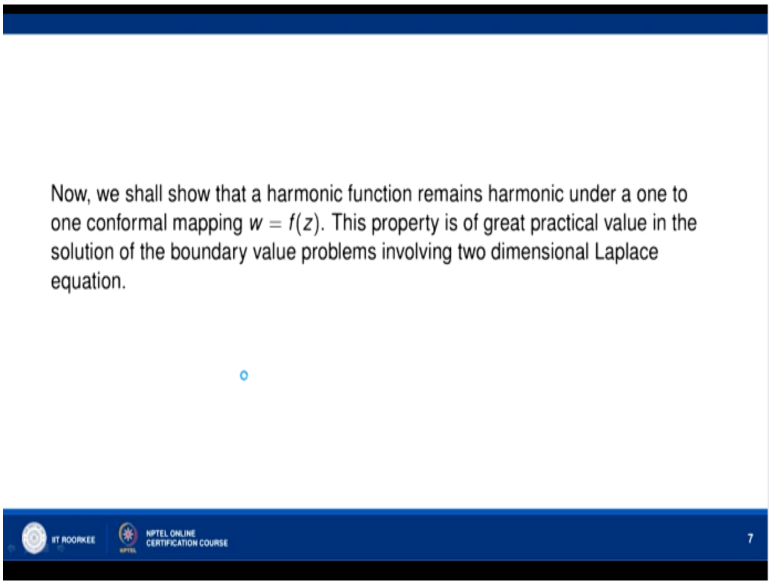


So now we shall show that harmonic function remains harmonic under one to one conformal mapping w=fz. This property is of great practical value in the solution of boundary value problems involving two-dimensional Laplace equation.

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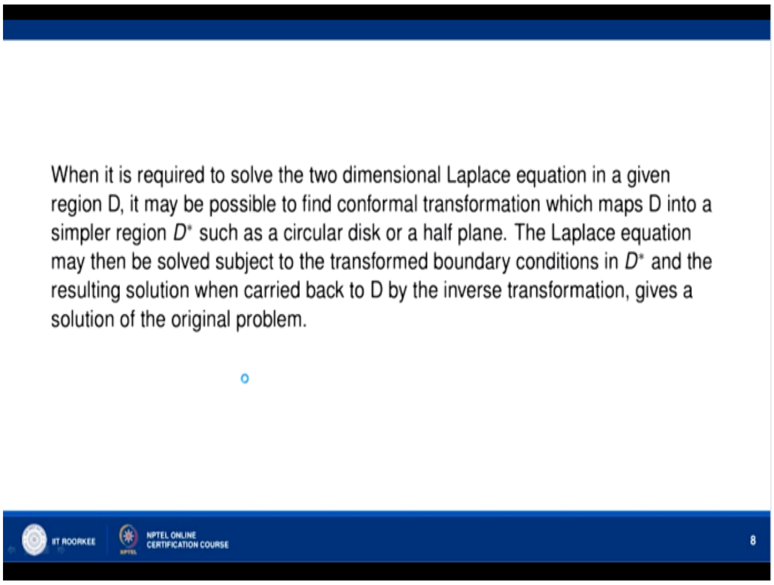


So when it is required to solve the two-dimensional Laplace equation in a given region D, it may be possible to find conformal transformation which maps D into a simpler region D star such as a circular disk or a half plane. The Laplace equation can then be solved subject to the transformed boundary conditions in D star and the resulting solution when carried back to D by the inverse transformation gives a solution of the original problem.

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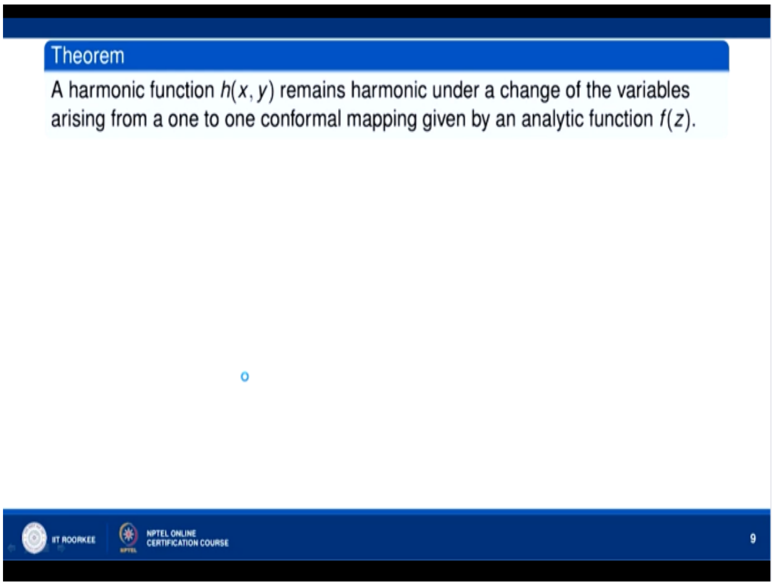


So let us show that a harmonic function h x, y remains harmonic under a change of the variables arising from a one to one conformal mapping given by an analytic function fz.

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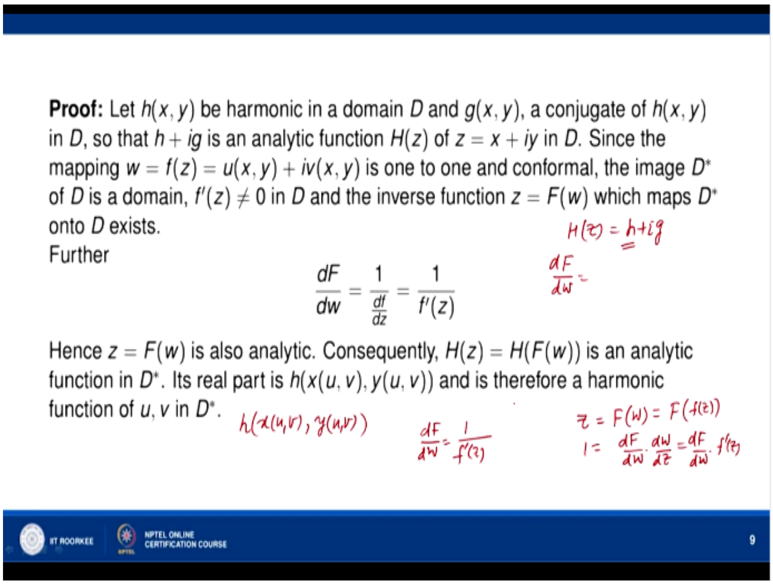


So let say h x, y be harmonic function in a domain D and g x, y is a conjugate of h x, y in D, so that h+ig is an analytic function. Let us denote it y as Hz so Hz is an analytic function of z in D okay. Hz is=h+ig, g is a conjugate harmonic function of h. Now since the mapping w=fz is one to one and conformal okay which D star of D is a domain and f prime z is not equal to 0 in D okay.

And the inverse function z=Fw which maps D star onto D exists okay. Now dF/dw okay, dF/dw is=okay z=F of w and w is fz okay, w=fz, so we have okay so by chain rule we can say this is 1=dF/dw*dw/dz or dF/dw*f prime z. Since f prime z is not equal to 0, we can write dF/dw=1/f prime z. So dF/dw is written as 1/f prime z and f prime z is not equal to 0, it is analytic function, so z=Fw okay, dF/dw is also not 0.

So z=Fw is also analytic and consequently Hz, Hz=H of Fw okay, Hz=H of Fw so analytic function of an analytic function is analytic and so this will be Hz will be analytic in D star and the real part of Hz is h okay so real part of Hz is h, h x, y okay so that x now will depend on u and v. So real part is h x, y and x, y are functions of u and v, so this real part is a harmonic function okay of u, v in D star.

So harmonic function remain harmonic under a one to one conformal mapping. This fact will be used when we solve the boundary value problems.

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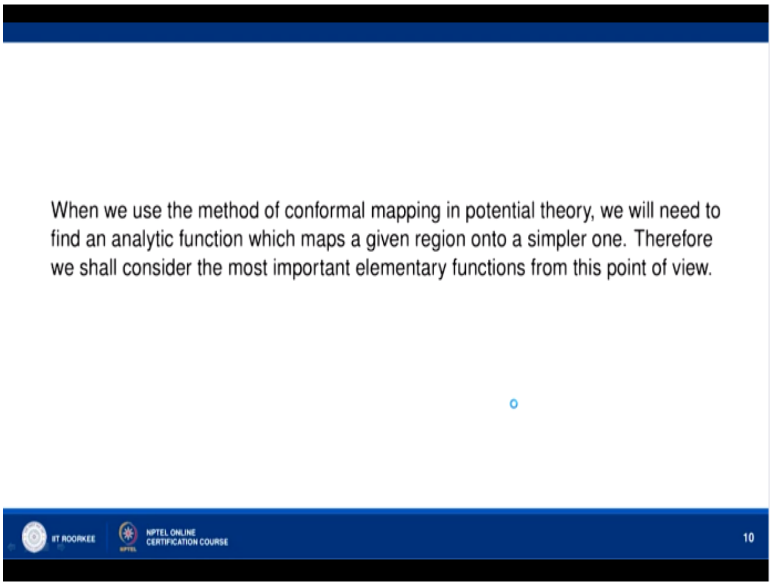


So when we use the method of conformal mapping in potential theory we will need to find an analytic function which maps the given region onto a simpler one okay. Therefore, we shall consider the most important elementary functions from this point of view. So in the next lecture, we shall be discussing bilinear transformations which are a special class of conformal mappings.

And then we will take up the various transformations which map half plane onto a disk or half plane into a half plane or we will consider the mappings where disk is not done to a disk. So such transformations will be very helpful when we solve the boundary value problems. With this I would like to end my lecture. Thank you very much for your attention.