Advanced Engineering Mathematics
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Lecture – 28
Conformal Mapping - I

Hello friends. Welcome to my lecture on conformal mappings. A complex linear function is defined as a function of the form fz=az+b where a and b are complex constants.

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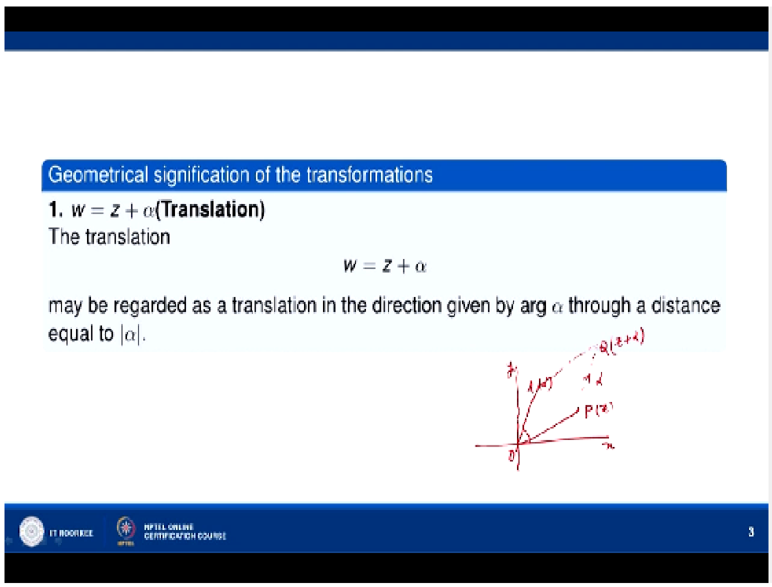

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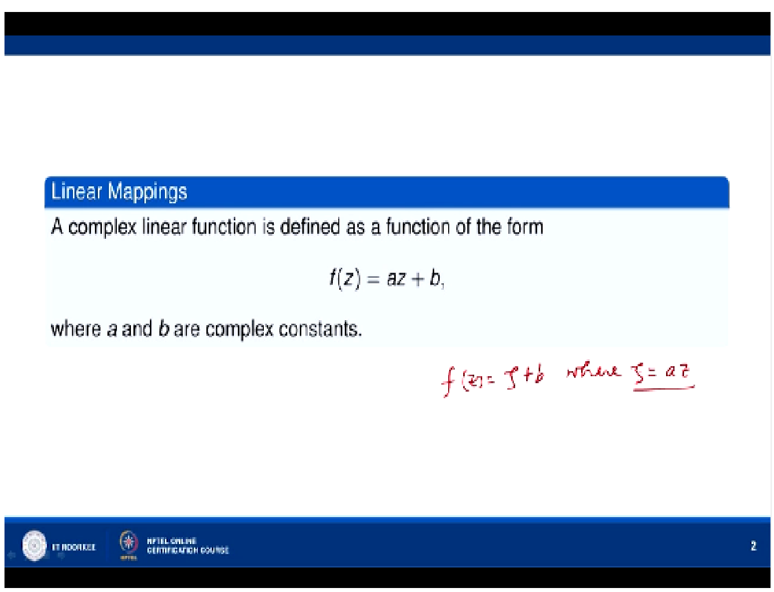

If you look at this function fz=az+b, then it consist of 2 transformations. The 2 transformations are, first transformation is the translation.

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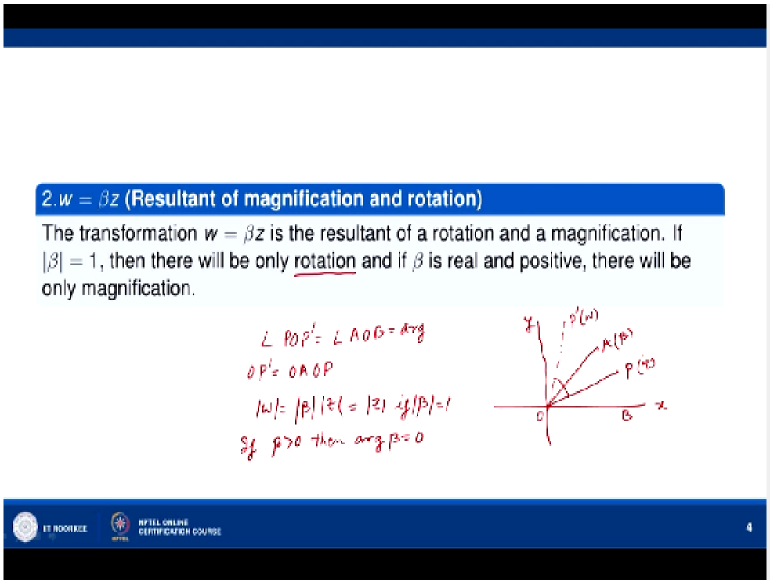

And the next transformation is the resultant of magnification and rotation. So let us first discuss the translation, okay. You can see this fz=az+b is actually a composition of 2 transformations. One is w=z+alpha. The other one is w=beta z. So let us see what is the geometrical significant of w=z+alpha. In the complex z plane, let us say P and A represent the complex numbers z and alpha where alpha is fixed, z is a variable point.

Let us join them to the origin, okay. And then we complete the parallelogram OPQA, okay. Then this vector PPQ, okay. The vector PPQ represents the complex numbers alpha and so this Q becomes z+alpha, okay. So you can see that this transformation w=z+alpha is actually a translation of the point P in the direction of the argument of alpha through a distance mod of alpha, okay.

This is the angle that the vector OA makes with the x axis. This is argument of alpha, okay. So this OA is parallel to PQ, and therefore, we have to move the point P to get to the point Q. If we move P in the direction of the argument of alpha and through a magnitude=mod of alpha because the magnitude of PQ=alpha.

So we can say that the transformation w=z+alpha is regarded as a translation in the direction given by argument of alpha through a distance equal to mod of alpha and when we look at the transformation w+beta z, suppose we have this point P here and then we the complex numbers beta. Beta is represented by the point A. P represents the complex numbers z.

Then P dash, okay, P dash represents w, w=beta z where the angle P dash OP, okay, the angle POP dash=angle AOB, which is the argument of beta. Moreover that you can see from w=beta z, that is OP dash=mod of OA*OP. So OA*OP. That is to get to the point P dash, we have to turn the vector OP anticlockwise through an argument of beta and magnify or contract the vector OP by the magnitude of beta to get to the point P dash.

So we can say that w=beta z is the resultant of a rotation, we have to rotate the vector OP through argument of beta and then magnify, okay, this vector OP by mod of beta times to get the vector OP dash. So it is a resultant of rotation and magnification. Now in case mod of beta=1, then we can see mod of w=mod of beta*mod of z, okay. So if mod beta=1, this will be equal to mod of z, then there will be no magnification, okay.

The length of OP and the or you can say the magnitude of OP will be same as the magnitude of OP dash, so there will be only rotation. And if beta is real and positive, then argument of beta=0. If beta>0, then argument of beta=0, so there will be no rotation. There will be only magnification. The vector OP will simply be, will only be magnified. There will be no rotation in this case.

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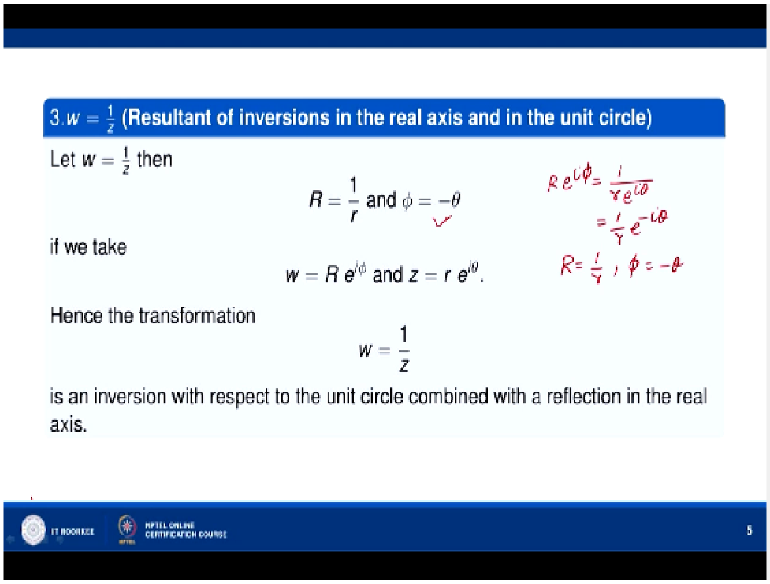

Now let us consider the reciprocal function w=1/z, okay. It is the result of inversions in the real axis and the unit circle. So let us take w=1/z where w is Rei phi and z=rei theta, then what will happen? w=1/z will give us Rei phi=1/z. So 1/re to the power i theta and which is equal to 1/re to the power -i theta. So R will be equal to, equating absolute values, R=1/r and equating arguments, we get phi=-theta. So R is 1/r and phi=-theta. Now let us show that how w=1/z gives us an inversion with respect to the unit circle combined with the reflection in the real axis.

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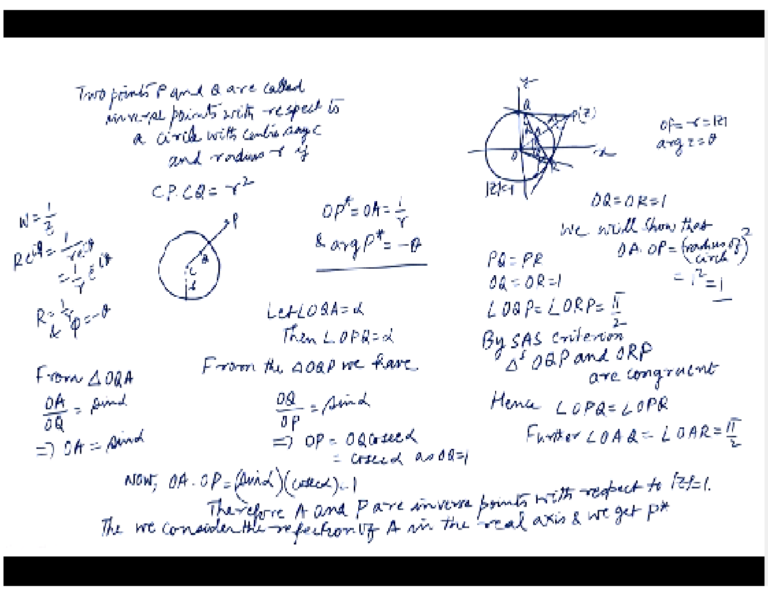

So let us look at this figure. Let us consider this unit circle. Suppose we have P point here. This is P, okay. OP=r and argument of, this P represents the complex numbers z. So this is mod of z. An argument of z=theta, okay. So this angle is theta. Let us drop the tangents from this point P on the circle, okay. Say the point of contacts are Q and R, okay. So join Q to R, okay, like this and then join Q to the origin and R to origin.

Then this is mod of z=1. So OQ=OR=1. Let me call this point of intersection as A, okay. We will show that P and A are inverse points with respect to the unit circle, okay. So we will show that OA*OP=radius of the circle square, that is 1 square=1, okay. So let us see how we define the inverse points with respect to a circle? 2 points P and Q are called inverse points with respect to a circle with center say C and radius r if CP*CQ=r square.

So let us say we have this circle of radius r. This is center. Radius of the circle is r. Suppose P is here, Q is here, okay. Then CP*CQ, should be equal to r square. So here we were going to show that A and P are inverse points with respect to the unit circle. Therefore, we have to show that OP*OA=1, okay. Now from the construction, it is clear that PQ=PR, length of the 2 tangents are same, okay.

And moreover, OQ=OR+1 because the radius of the circle OQ and OR are the radius of the circle which is 1. Now we also notice that this angle, the angle OQP=angle ORQ=pi/2. Because PQ is tangent and OQ is radius. Similarly, PR is tangent and OR is radius, okay. Now by SAS criterion, okay, PQ=PR, OQ=OR, the angle between OQ and QP that is, is pi/2 and angle between OR and PR is also pi/2.

So angle OQP=angle ORP. So by SAS criterion, triangles OQP and ORP are congruent. And therefore, this angle, okay, OPQ is same as the angle OPR, okay. Hence angle OPQ is same as angle OPR. Furthermore, the angle OAQ=angle OAR=pi/2, okay. Now let us consider the triangle OQP, okay. So let us consider this, let us say this angle is, suppose I take it as say alpha, okay.

Let us say angle OQA=alpha, okay. Then because OQP is pi/2, so AQP is pi/2-alpha. And this angle is pi/2, so this angle is also alpha, okay. So then angle OPQ=alpha. Now let us consider the triangle OQP. From the triangle OQP, we have OQ/OP=sin alpha. And therefore, OP=OQ cosec alpha. But OQ=1, so this is cosec alpha, okay. Now let us consider the triangle OQA. From triangle OQA, what do we notice?

OA/OQ, okay,=sin alpha. So this gives you OA=sin alpha, okay. Now OA*OP, we can see is equal to sin alpha*cosec alpha and therefore, it is equal to 1. So OA*OP=1 and therefore, A and P are inverse points with respect to the unit circle, with respect to mod z=1, okay. Now what we do? Let us consider the reflection of this point A, okay, in the real axis. So this is your point P*, okay.

And this P* is obtained by reflecting the point A in the real axis. So OP* is same as OA, the length of OP* is same as the length of OA and OP* makes theta angle with the real axis. So then reflect, then we consider the reflection of point A in the real axis, okay and we get P*, okay. So OP*=OA=, this is 1/r, okay. And argument of P*=-theta, okay. So this means that what we found was we had w=1/z, so we found that if w is Rei alpha e and z is rei theta, then 1/rei theta we have, this is equal to 1/re to the power -i theta.

So R=1/r and phi=-theta. So we see that OP*=1/r and argument of P*=-theta and therefore, the point A under the point P, under the transformation w=1/z gives us P* which P* is obtained by considering the inversion of P with respect to the unit circle, we get the point A and then we consider the reflection of the point A in the real axis and we get P*. So w=1/z is regarded as an inversion with respect to the unit circle combined with the reflection in the real axis.

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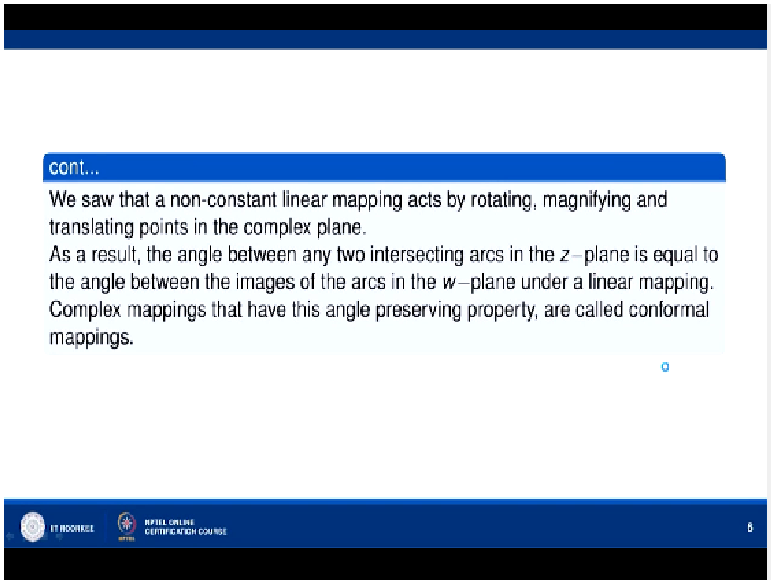

Now we have seen that non-constant linear mapping, okay, we consider the non-constant linear mapping w=az+b. It consist of 2 transformations. So you can regard this fz=az+b as say fz=, say, zeta+b where zeta=az. So this gives you the mapping of the type beta z. This w=beta z type, okay. So this is of beta z type and this is the mapping of the type w=z+alpha. So fz=az+b consist of 2 transformations, okay.

Translation and this is translation and this one is rotation and magnification. So a non-constant linear mapping acts by rotating, magnifying and translating points in the complex plane. As a result, the angle between any 2 intersecting arc in the z-plane is equal to the angle between the images of the arcs in the w-plane because the linear mapping does not change the shape of the curve, okay. It only rotates or magnifies and translates it.

So the angle between 2 intersecting arcs will remain preserved under a linear mapping. Now the complex mappings that have this angle preserving property are called conformal mappings. So let us discuss conformal mappings in detail.

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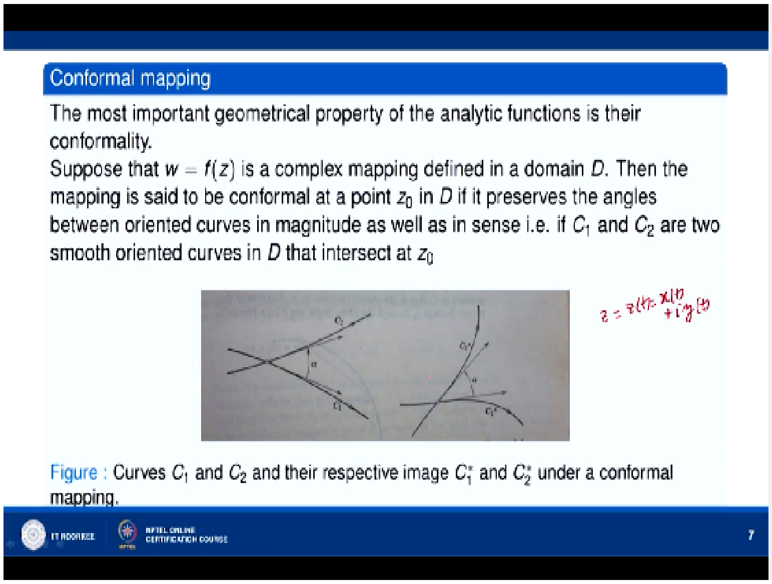

The most important geometrical property of the analytic functions is their conformality. Suppose that 2w=fz is a complex mapping defined in a domain D. Then the mapping is called conformal at a point z0 in D if it preserves the angles between oriented curves in magnitude as well as in sense. That is if C1 and C2 are 2 smooth oriented curves in D that intersects. Suppose C1 is this curve, okay.

This is C1 and this is C2, okay. The angle between the curve C1 and C2 is the angle between their respective tangents. So let us say this is the tangent to the curve C1. This one is the tangent to the curve C2. Alpha is the angle between them. Now the oriented curves means the; when we write the equation of the curve in the parametric form, suppose it is z=zt=xt+iyt. Then the curve is said to have a positive sense in the direction in which t increases, okay.

So and the sense along which the curve has a positive sense, the tangent in that same direction will also said to have a positive sense, okay. So these are the tangents, oriented tangents to the curve C1 and C2. And similarly, this C1 and C2 under a transformation, under the transformation w=fz, they are mapped into say this curve, C1* and C2*. These oriented tangents to the curve C1* and C2* at their point of intersection, if this point is z0, okay, in the z plane, then this point is fz0 under the transformation w=fz.

So this is fz0 at the point of intersection fz0, the 2 tangents, oriented tangents to the curves C1* which is the image of C1 and C2* which is the image of C2, okay, make the same angle alpha and in the same direction. The direction from C1 to C2. If we are going from C1 to C2 here, here also from C1* to C2* when we go, we should be moving in the same direction. So then if the angle is preserved in magnitude as well as in sense, we say that the mapping is conformal.

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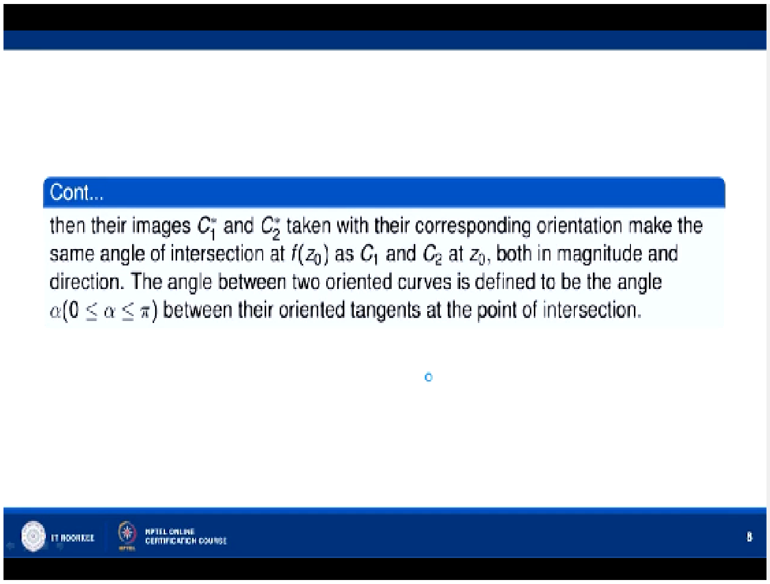

So the images C1*, C2* taken with their corresponding orientation should make the same angle and the angle of intersection is the angle between their respective tangents, that angle if it is alpha, the alpha lies between 0 and pi.

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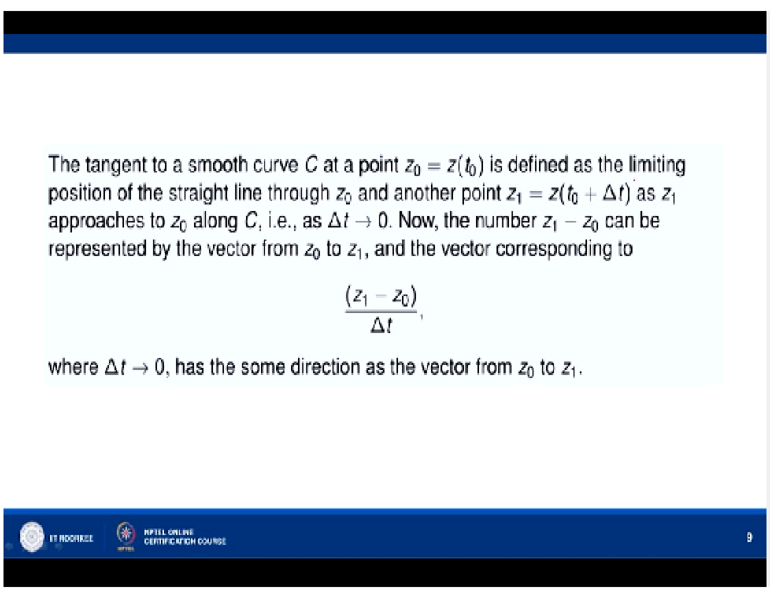

So the tangent to a smooth curve C at a point z0=zt0 is defined as the limiting position of the straight line through z0 and another point z1, z1=zt0+delta t as we can see in this figure, okay.

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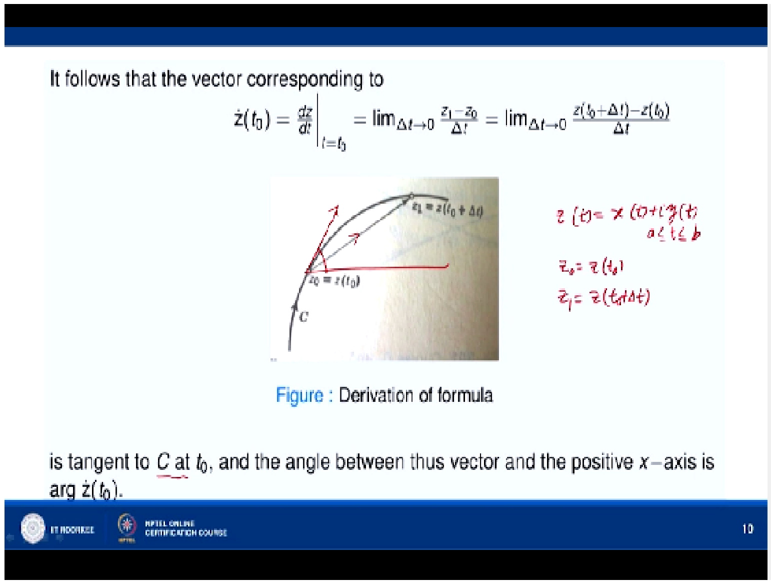

This is any curve C in the complex plane. The curve C is defined by zt=xt+iyt, t varies from some a to b. When we take t=t0, it gives us a point on the curve, okay. Let that be z0. So z0=zt0. Now let us take another point on the curve, okay, for the value t0+delta t. So zt0+delta t is let us say z1, okay. Then we can see that z1-z0/delta t, okay, this is z1, this is z0, okay. So the number z1-z0 can be represented by the vector from z0 to z1, okay.

This vector can represent the complex number z1-z0, okay and z1-z0/delta t will have the same direction as the vector z1-z0. So as delta t goes to 0, that is as the point z1 moves along the curve to the point z0, okay, then z.t0, okay, z.t0 is dz/dt at t=t0 which is limit of z1-z0/delta t as delta t goes to 0 and this is limit delta t goes to 0, zt0+delta t-zt0/delta t. So z.t0, now this, so this limit of this expression zt0+delta t-zt0/delta t, is same as along the tangent, this limit gives us the direction of the tangent to the curve at the point t0, that is at the point z0.

So z.t0 is tangent to the curve C at the point t0. When this point z1 will move along the curve, as delta t goes to 0, z1 will move along the curve to the point z0. And this vector joining z0 to z1, will move to the tangent to the curve, they will approach to the tangent to the curve at the point z0. So this z.t0 gives us the direction of the tangent to the curve at t0. And the angle between this vector and the positive x axis, okay, angle between this vector, this vector and the positive x axis, okay, this angle, okay, this is the argument of z.t0.

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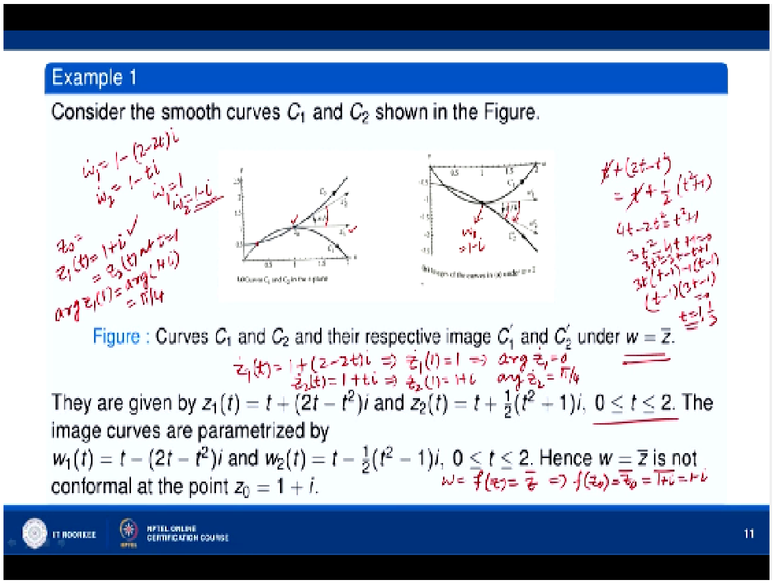

Now consider the smooth curve C1 and C2, let us consider 2 curves C1 and C2, which are shown in this figure. The curve C1 and C2 and their respective C1 dash and C2 dash, okay. C1 dash is the image of this C1. And C2 dash is the image of C2 under the transformation w=z conjugate, okay. The curve C1 is given by in the parametric form by z1t=t+2t-t square*i. And the curve C2 in the parametric form is given by z2=t+1/2 t square+1*i.

0 less than or equal to t less than or equal to 2. Let us find the point of intersection of z1t and z2t, okay. So at the point of intersection, z1t and z2t will be same. So t+2t-t square=t+1/2 t square+1, okay. So this cancels with this and what we get? 4t-2t square=t square+1. So what we get? 3t square=3, okay. So this means t square=, no 3t square-4t+1=0, okay. And when we factorize this, what we get?

We can see that t=1 satisfies this equation. 3-4+1. So this is t-1*t-1, we can multiply by 3t. So 3t square-2t-1, so we get -t+1, okay. So t-1 and 3t-1, okay, equal to 0. So t=1 and 1/3. Now 0 is less than or equal to t, less than or equal to 2. So this is z0, this z0 corresponds to t=1 and this point of intersection corresponds to t=1/3. We are considering the point of intersection at t=1.

So when t=1, what is z1t? z1t will be 1+i and similarly will be z2t at t=1 and this is z0. z0 is the point of intersection of z1t and z2t at t=1, okay. Now so this at the point z1, we have this curve. At the point z0, this is the curve C1, this is the curve C1. So argument of z1, argument of 1+i. Argument of 1+I we know, it is pi/4, okay. Let us find z1.t, okay. So z1.t=1+2-2ti, okay. So z1., at t=t0, that is, so this implies that z1.t0, t0=1, okay.

And when we find z2., what we get? z2.=1+2t/2, okay. So 1+t*i, okay, which implies that z2.=1+i, okay. So the angle which the tangent to the vector at C1, okay, z1., okay, at z1 dash, okay, it makes with the x axis is 0, okay. This means that argument of z1.=0 and argument of z2.=i/4, okay. So argument of z1. gives the angle which the tangent to the curve at C1 makes with the x axis.

So angle that the curve, tangent to the curve C1 makes with the x axis is 0 while the tangent to the curve C2 at the point z0 makes angle pi/4. So this angle pi/4 is the angle between the tangents to the curve C1 and C2 at the point z0. Now let us come to the image curves, C1 dash and C2 dash. C1 dash becomes this one, under the mapping w=z conjugate, C curve is given by C1 curve is given by z1t=t+2t-t square*i.

So image curve will be given w1 will be given by w1t=t-2t-t square*I and the image curve C2 dash will be given by w2t=t-1/2 t square+1*i, okay. So now let us see w1., okay. At the point, where do they intersect w1t and w2t? w1t and w2t intersect at the point fz0, okay. So fz0, w=fz=z conjugate. So this gives you fz0=z0 bar, okay. And fz0=, z0=1+i, okay. So this is 1+i bar, okay.

So 1-i, okay. So they intersect, this is your w0=1-i point, okay. And if you find w1., w1.=1-, 2t-t square means 2-2t*i, okay, in to complex number i. And w2. will be what? w2. will be 1-1/2*2t=1-t*i, okay. And t=1, okay. So you can see t=1, so this means that w1.=1, t=1 means this is 0. So w1.=1. w2.=1/i, okay. So w1. means the curve C1 dash, okay. You can see this curve C1 dash, okay.

It makes angle 0 with the u axis, okay with the real axis. w1. is parallel to the u axis. So it makes angle 0 and w2.=1/i, okay. It makes angle this one pi/4 but its argument will be -pi/4 when you associate the direction, okay. So -pi/4. So here you can see that the angle remains preserved, angle between C1 and C2 at the point of intersection z0 is pi/4. At the point w0 also, the angle between C1 dash and C2 dash is pi/4.

But here the sense of the angle from C1 to C2 is in the anticlockwise direction while here the angle from C1 dash to C2 dash is in the clockwise direction, okay, in this direction. So angle is preserved in the magnitude but not in sense and therefore, the mapping w=z conjugate is not conformal at the point of intersection z0, that is at the point zw=1+i.

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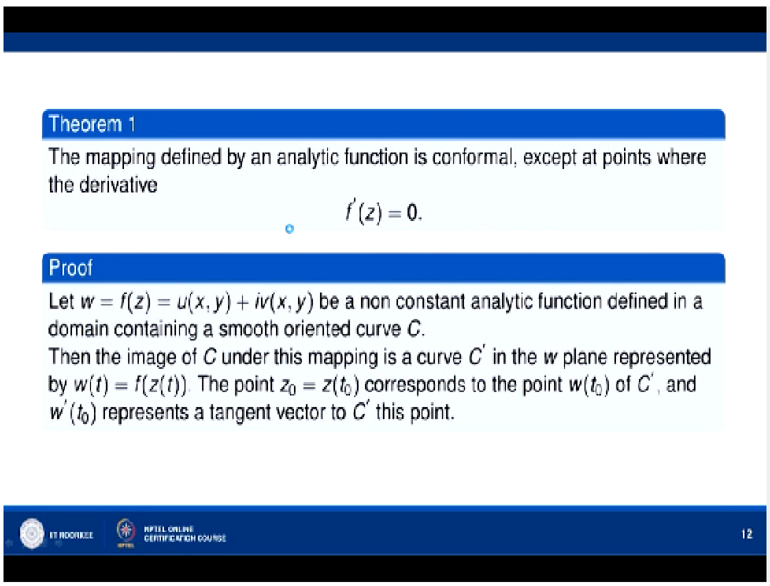

So in fact treat the mapping w=z conjugate is not conformal at any point of the complex plane. Here we have taken 2 particular curves and showed that it is not conformal at their point of intersection but it is not-conformal at any point of the z plane. Now let us consider the theorem which says that the mapping defined by an analytic function is conformal, at all points except at those points where the derivative is 0.

So this is very important theorem because from this theorem, we can easily test the points where while giving mapping is a conformal mapping by finding its derivative. So the points where the derivative vanishes are called the critical points of the mapping. The points where the derivative does not exist of a complex function, those points are also called the critical points. So critical points actually consist of those points where the derivative, either the derivative of the function w=fz does not exist or it is 0.

So here let us consider the analytic function. An analytic function is differentiable infinitely, so we can say that it is conformal at all points except at the critical points, that is the points where its derivative is 0. So let us say, let us prove this. Let w=fz=uxy+ivxy be a non-constant analytic function defined in a domain containing a smooth oriented curve C. Then the image of C under this mapping is a curve C dash in the w plane represented by wt=fzt.

The point z0=zt0 corresponds to the point wt0 of C dash and w dash t0 as we have already discussed represents the tangent to the curve C dash at the point t=t0.

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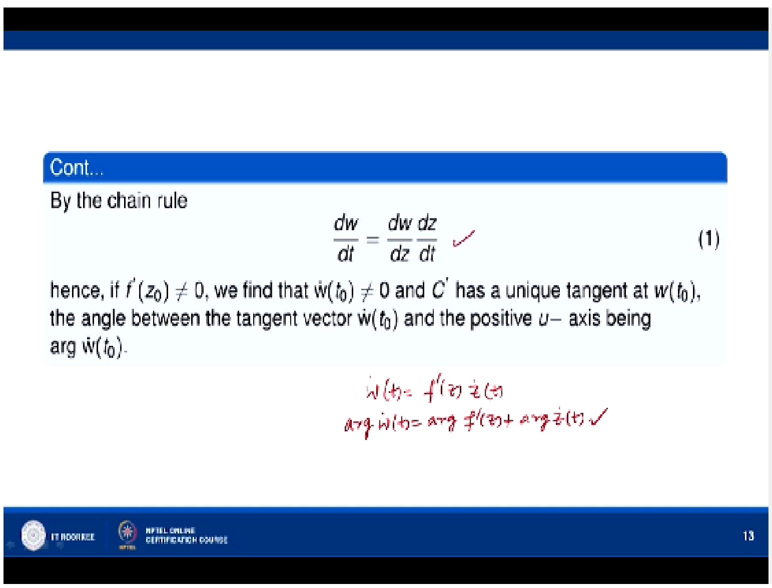

Now by the chain rule, dw/dt=dw/dz*dz/dt, okay. So if f prime z0 is not equal to 0, then from here you can see w.t0 is not equal to 0, okay. And therefore, C dash has a unique tangent at the point wt0. Now the angle between the tangent vector w.t0 and the positive u axis is given by argument of w.t0.

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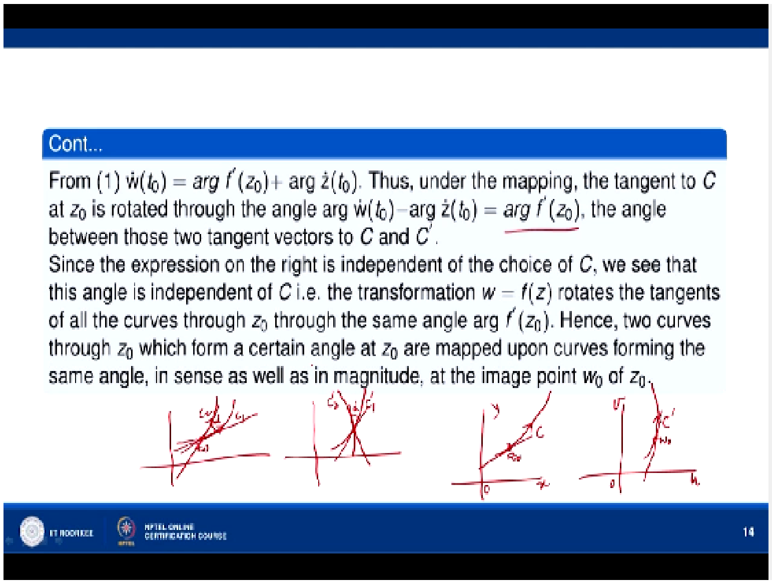

From this equation, we find that argument of w.t0=argument of w.t0. This is w.t, okay. w.t=f prime z*z.t, okay. So we know that argument of z1*z2 is argument of z1+argument of z2. So argument of w.t=argument of f prime z + argument of z.t. So from here, it follows that argument of w.t0=argument of fz dash z. + argument of z.t0. Now under the mapping, the tangent to C at the point z0 is rotated through the angle, okay.

We can see. It is rotated through the angle. Argument of w.t0-argument of z.t0, okay. So suppose you have this curve, let us say, in the z plane. Let us say this curve we have curve C, okay. So this curve will be rotated by argument of f prime z0, okay, in the w plane, okay. So the tangent to the curve, let us say you take this tangent to the curve at the point z0, okay. Then this is your w0 suppose.

The tangent to the curve at the point w0 will be rotated by angle given by argument of fzt0. Argument of w.t0-argument of of z.t0, this gives the angle by which the tangent to the curve C at the point z0 is rotated, okay. So that is given by argument of f prime z0, the angle between those 2 tangent vectors to C and C dash, okay. So since the expression on the right, now this expression is independent of the choice of C.

This angle is independent of C. The transformation w=fz therefore rotates the tangents of all curves through z0 through the same angle argument of f prime z0. Hence, 2 curves through z0, if you take 2 curves through z0. Let us say we have 2 curves through z0, C1 and C2, okay. This is C1 and this is C2, okay.

So this is the tangent to the curve C1 and this is the tangent to the curve C2, okay. If this angle is say alpha, okay, then both these curves, okay, are rotated by the same angle, okay given by argument of f prime z0. So this is C1 dash and this is C2 dash for example, okay. Then this is the tangent to the curve C1 dash. Then this is the tangent to the curve C2 dash.

The angle between the 2 curves will remain alpha because C1 is rotated by argument of f prime z0 and C2 is also rotated by argument of f prime z0, so the angle between C1 and C2 does not change, okay. So hence 2 curves through z0 which form a certain angle at z0 are mapped upon curves forming the same angle in sense as well as in magnitude at the image point w0 of z0.

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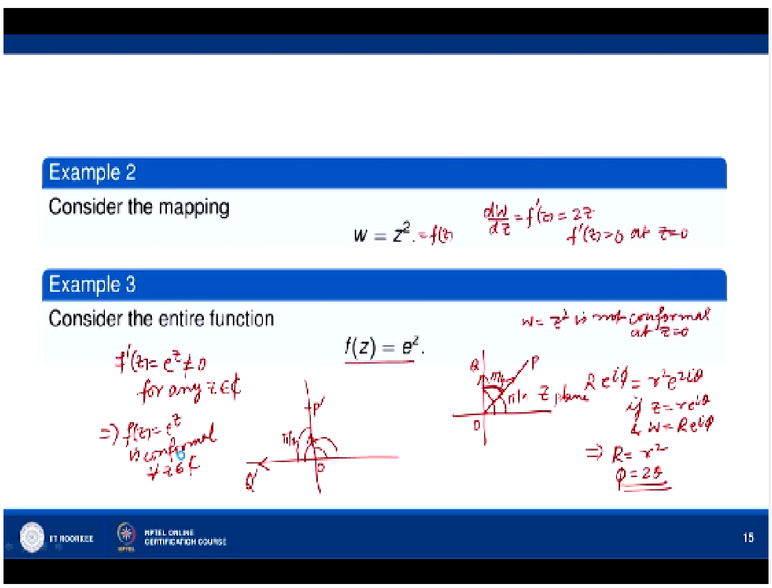

Now let us consider the mapping w=z square. So we can see that dw/dz or you can say f prime z if I take it fz, then this is equal to 2z, okay. So f prime z=0 at z=0. This means that w=z square is not conformal, okay at z=0. At any other point, any point other than the origin, it is a conformal mapping. Now let us see how it is not conformal at z=0. It will be easy to take say OP which makes angle pi/4, okay and this y axis, okay OQ which makes angle pi/2, okay.

So let us take these 2 arcs, these 2 rays, let us take these 2 rays. Then OP, 2=z square, so Rei phi=r square e to the power 2i theta, if z=rei theta and w=Rei phi. So R=r square and phi will be 2 theta, okay. Now this is theta=pi/4, okay. So theta=pi/4 will become pi/2, okay. So this is z plane, okay. So OP will be mapped on to OP dash here, okay. OP will go to OP dash and OQ, this makes angle pi/2, okay.

So this will be mapped on to negative u axis, okay. This is OQ dash, okay. And we can see that angle between OP and OQ is pi/4, okay. Here the angle between OP dash and OQ dash becomes pi/2, okay. Because phi=2 theta, so OP makes angle pi/4 will go to OP dash which makes angle pi/2 and OQ makes angle pi/2 with the x axis, so OQ dash will make an angle pi with the u axis, okay. And therefore, the angle between OP dash and OQ dash is pi/2.

So angles are doubled at the origin. And therefore, the angle between 2 curves, okay, here they are rays, okay, OP and OQ. The angle between 2 rays is not preserved at the origin and so w=z square is not conformal at the origin. Now if you consider the other example, fz=e to the power z, we know that it is an entire function because it is differentiable for all finite complex numbers z. So and if you find f prime z, f prime z=e to the power z.

And we know that e to the power z is not equal to 0 for any complex number z. And therefore, fz=e to the power z is conformal for all z belonging to C.

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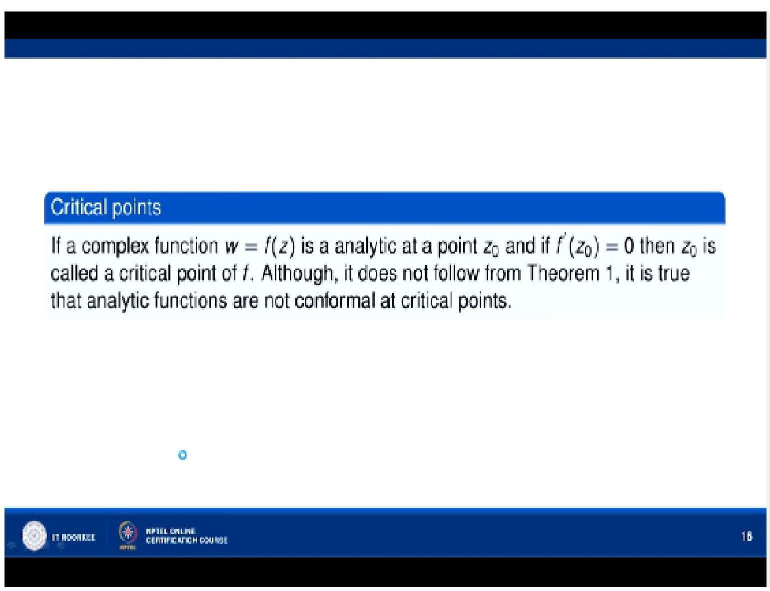

Now let us consider critical points as I had already said if a complex function w=fz is analytic at a point z0 and if the derivative at z0 vanishes, then z0 is called a critical point of f. Now it does not follow from theorem 1 that analytic functions which are not conformal, it does not follow that, if f prime z0=0, then the analytic function cannot be; see from this theorem, it follows that if you look at this theorem, okay.

It says that the mapping is conformal at every point z where f prime z is not equal to 0. Now if f prime z=0, then why the mapping is not conformal? Why it cannot be conformal? See it does not follow from theorem 1 that analytic functions are not conformal at critical points. The point where f prime z=0. It only says that wherever derivative is not 0, the mapping is conformal. But why it is not conformal at the points where f prime z=0, it follows from the second theorem.

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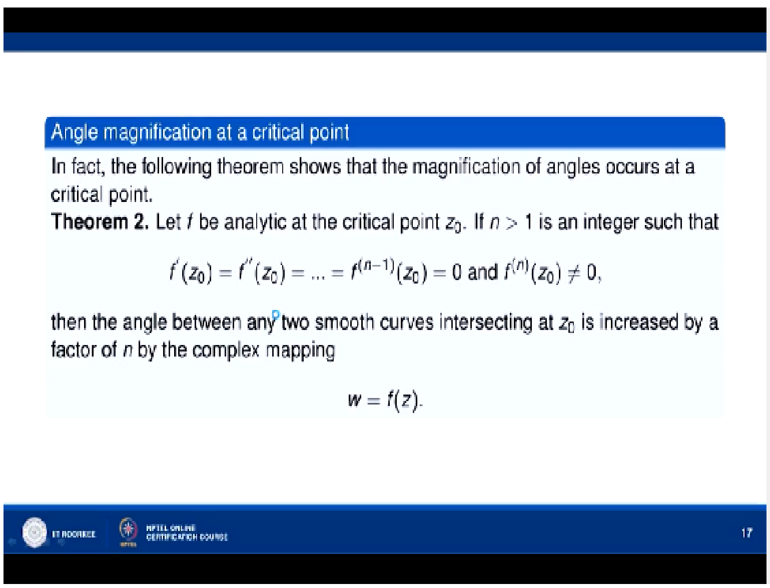

This theorem tells us that the magnification of angles occurs at a critical point, okay. So angles are not preserved actually at a critical point and therefore, wherever the function, analytic function has derivative 0, it cannot be conformal at those points. So let f be analytic at the critical point z0. Suppose n>1 is an integer such that f prime z0, f double prime z0 and -one of the derivative of fz at z0 is 0 but the nth derivative is non-0, then the angle between any 2 smooth curves intersecting at z0 is increased by a factor of n by the complex mapping.

So if you have angle between any 2 intersecting curves at the point z0 alpha, then at the point w0 in the w plane, the angle between the corresponding images will be n*alpha. So the angle is not preserved and therefore, we can say that at the critical point, the analytic function fz does not have, is not conformal. So let us prove this result by our hypothesis.

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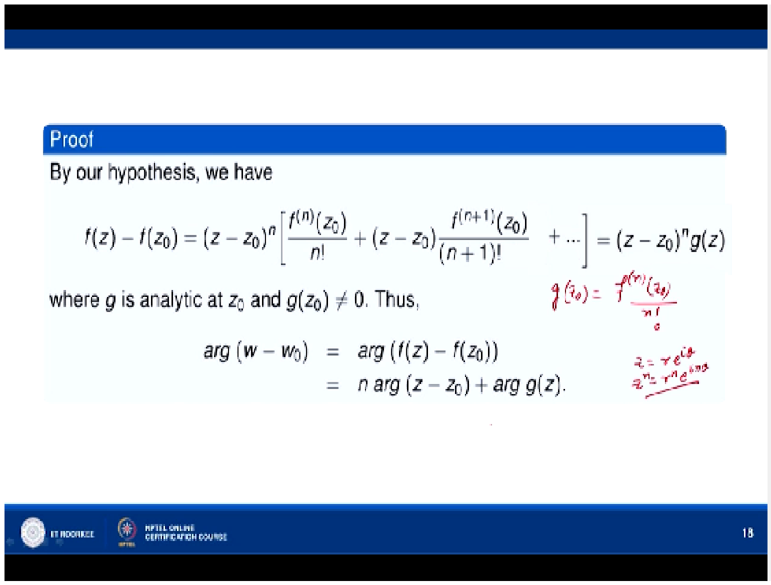

We have assumed that fz is analytic at the point z0. So we can, by our hypothesis, we can write its Taylor series expansion fz=fz0+f prime z0*z-z0 and so on. And then because f prime z0, f double prime z0, fn-1th derivative at z0 is 0 but the nth derivative is non-0, the power series reduces to this, okay. fz=fz0+z-z0 to the power n*fnz0/n factorial z-z0 to the power n+1 fn+1 z0/n+1 factorial and so on.

And this I can write in this form. z-z0 to the power n we can take as a common factor and then the remaining expression inside the bracket will represent an analytic function gz and this gz analytic function at z0 is not equal to 0 because gz=fnz0/n factorial. fnz0 is not equal to 0. So gz0 is not equal to 0. And therefore, argument of w-w0, w=fz, w0 is fz0. So argument of fz-fz0=n*argument of z-z0, okay, +argument of gz.

Argument of z, we know that if z=rei theta, so theta is the argument of z, then z to the power n is r to the power n e to the power i and theta. So argument of z to the power n becomes n*argument of z. So here argument of z-z0 to the power n is n*argument of z-z0+argument of gz, okay.

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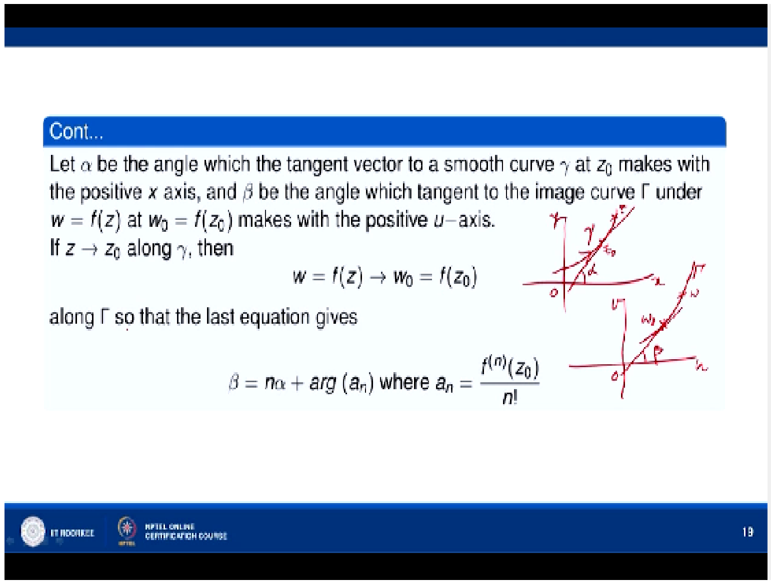

Now let alpha be the angle which the tangent vector to a smooth curve gamma at z0 makes with the positive x axis, let us take this, okay. Suppose we have this curve, okay, a smooth curve, okay, and suppose this is your point z0, the curve, the tangent at the point z0, okay, makes an angle; this is your curve, a smooth curve we have denoted by gamma. So this smooth curve by gamma and here this angle is alpha, okay.

This angle which the tangent to the curve at the point z0 makes with the positive x axis. Beta be the angle with the tangent to the image curve gamma, okay. Let us take this w plane. So in the w plane, suppose this is the curve, okay, capital gamma and this is your point w0=fz0, okay. So tangent to this curve at this point makes this angle. This angle is beta let us say, okay. Beta be the angle which is the tangent to the image curve gamma at w=fz, okay and the w=fz at w0 makes with the positive u axis.

Then if z tends to z0 along gamma. If z tends to z0, let us say z be any point here. If z tends to z0 along gamma, then w tends to w0, okay. This is w, okay, tends to w0=fz0 along gamma. So that the last equation gives; from the last equation, what do we notice? As z tends to z0, this equation will give you this one, okay. Beta=argument of n alpha + argument of an, okay. This is fz-, argument of fz-fz0. fz-fz0 is this vector, okay, this vector, okay. You can join w-w0 and this is z-z0.

So as z tends to z0, you will get the direction of the tangent here. And when z tends to z0, w will tend to w0. So it will give you the direction of the tangent here. So this when z tends to z0, argument of w-w0 tends to beta and argument of z-z0 tends to alpha, okay. And this becomes argument of gz0. So what we have? Beta will be equal n alpha + argument of gz0. gz0 is an. an is fnz0/n factorial.

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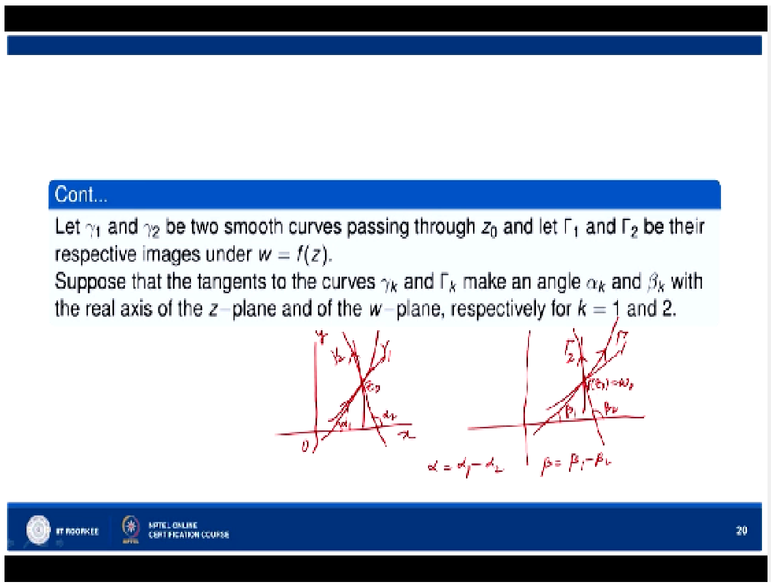

Now let gamma 1 and gamma 2 be 2 smooth curves passing through z0. Take 2 curves. Suppose this is 1 curve and this is another curve, okay. This is gamma 1, this is gamma 2, okay in the z plane. And let gamma 1 gamma 2 be 2 smooth curves passing through z0, okay. And let gamma 1, capital gamma 1, capital gamma 2 will be corresponding images. This is gamma 1 and this is gamma 2, okay.

This is z0, so this is fz0 or w0, okay. Gamma 1 gamma 2 with their respective images and the w=fz, suppose that the tangents to the curves gamma k and capital gamma k make an angle alpha k and beta k with the real axis of the z plane. So let us say gamma 1 makes angle alpha 1, okay. Gamma 2 makes angle at the point of intersection alpha 2, okay. Here gamma 1, capital gamma 1 makes angle beta 1.

This capital gamma 2 makes angle beta 2, okay. Then so gamma k and gamma k makes angle alpha k and beta k with real axis of the z plane and w plane respectively for k=1 and 2, okay.

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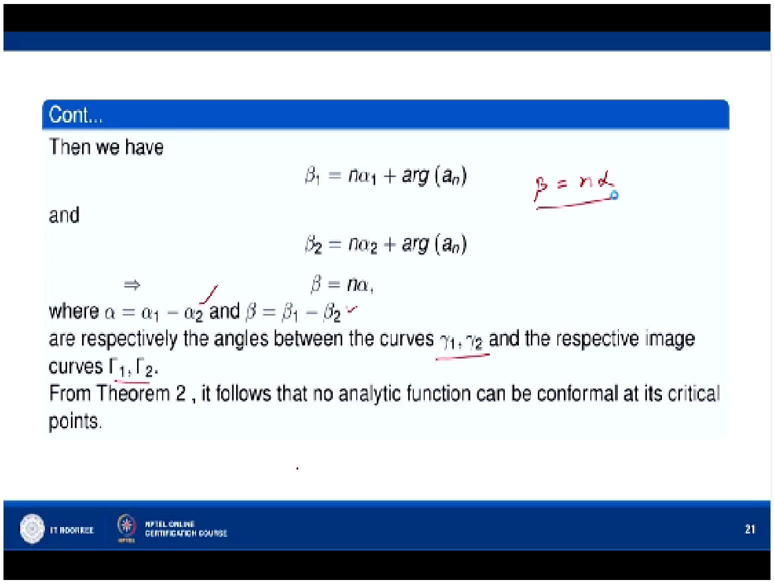

Then beta 1 will be equal to n alpha 1+argument of an. Beta 2 will be n alpha 2+argument of an. And this will imply that alpha, if you take alpha=alpha 1-alpha 2, so alpha=alpha 1-alpha 2. And beta=beta 1-beta 2, okay. So then what happens? If we take alpha=alpha 1-alpha 2, beta=beta 1-beta 2 respectively. Then the angle between gamma 1, gamma 2, okay and their images, capital gamma 1, capital gamma 2, okay, will be like this.

So beta 1-beta 2 will be beta=n*alpha 1-alpha 2, that is alpha. So you will have argument of an will cancel. So beta=n alpha, okay. So you can see the angle between the 2 curves, this angle, okay, beta, this angle is alpha and this angle is beta, okay. Beta becomes n alpha. So from this theorem, it follows that; so the angle at the point of intersection is not preserved at a critical point. So from this theorem, it follows that no analytic function can be conformal at its critical points. The angle gets magnified. It becomes n*the angle between the curves in the z plane.

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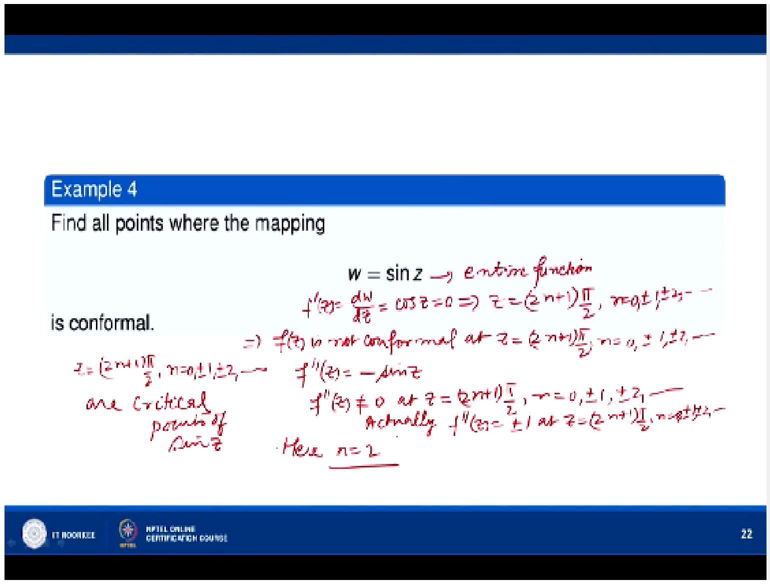

Now let us consider the mapping w=sin z, okay. So we see that it is an entire function. And if you find its derivative, then dw/dz=cos z, okay. And cos z=0 gives z=2n+1*pi/2 where n=0, +-1, +-2 and so on, okay. And this f prime z. So f prime z is 0 at z=2n+1*pi/2 where n=0, +-1, +-2 and so on. So this means that fz is not conformal at z=2n+1*pi/2, n=0, +-1, +-2 and so on. At all other points, it is conformal, okay.

Now if you find second derivative f double prime z. f double prime z is -sin z, okay. So f double prime z is not equal to 0 at z=2n+1*pi/2, okay, n=0, +-1, +-2 and so on. Actually f double prime=+-1 at z=2n+1*pi/2. So the theorem 2 tells us that at these critical points, these are critical points, z=; because the derivative of fz vanishes at z=2n+1*pi/2. So z=2n+1*pi/2 are the critical points of sin z.

And since f double prime z is not equal to 0, the theorem 2 tells us that at the critical points z=2n+1*pi/2, the angle between any 2 intersecting curves in the z plane if it is alpha, it will be doubled in w plane. Because the theorem 2 tells us that the angle between any 2 intersecting curves at the point critical point in the z plane is multiplied by n, okay, where n is the number which is greater than, n is the positive integer greater than 1 such that f prime z0 equal to 0, f double prime z0 equal to 0, n-1th derivative of fz0 is 0 but nth derivative is non-0.

And here we notice that f double prime z0 is not equal to 0. So n=2 here, okay. So here n=2. So let us now show that the angle at critical points between 2 intersecting curves doubles here.

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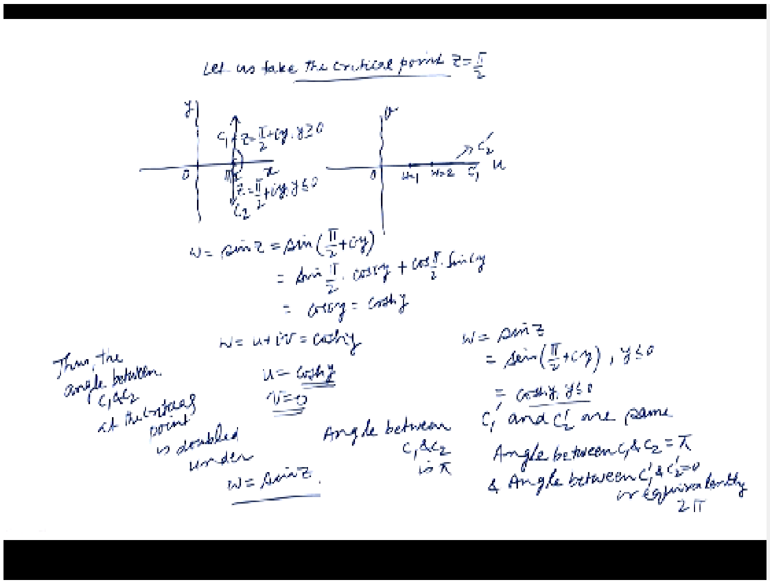

Now let us show that, let us take the critical point z=pi/2. We see that critical points are, z is given by z=2n+1*pi/2 where n can take value 0, +-1, +-2 and so on. So let us take n=0. So let us consider the critical point z=pi/2, okay. Let us consider the z plane. Here this is, so let us consider the point pi/2 here, okay. Let us consider the ray, this ray, okay. Any point z on this ray will have equation z=pi/2+iy, okay.

So let us consider the ray C1 emanating from z=pi/2 and then let us see under the mapping w=sin z, what happens to the image of C1, okay? So w=sin z gives sin of pi/2+iy, okay. So this is sin pi/2*cos iy+cos pi/2*sin iy. Sin pi/2 is 1, cos pi/2 is 0. So we have cos iy. And cos iy=cos hyperbolic y, okay. So w is equal to u+iv=cos hyperbolic y. So this means that u is cos hyperbolic y and v=0, okay.

So this means that the ray C1 is mapped on to the real axis. This point pi/2 goes to; when your z=pi/2, okay, this goes to w=1, okay. So this is w=1 here and here you have w=2 and so on, okay. So this cos hyperbolic y, okay, and w=cos hyperbolic y gives you u=cos hyperbolic y, v=0, so this is mapped into this ray, C1 dash, okay. This ray given by this w=pi/2+iy where y is greater than or equal to 0, okay, is mapped to the ray emanating from w=1 along the u axis, okay.

Because v=0, okay. Only u is there, u is cos hyperbolic y, okay. And y is greater than or equal to 0. So when y=0, we get here pi/2. That pi/2 maps into cos; this y=0 means this pi/2 point. Pi/2 here, it goes to 1. So this w=1 is the image of z=pi/2, okay. Now let us take another ray C2, okay. So C2 now will have equation, it is emanating from pi/2 and going in the y direction just opposite of C1, okay.

So this is z=pi/2-iy, okay. Or you can say pi/2+iy, y less than or equal to 0, okay. So z=pi/2+iy where y is less than or equal to 0. So this C2 will map into C2 dash and w=sin z will again give the same value, sin pi/2+iy where y is less than or equal to 0. This is cos hyperbolic y where y is less than or equal to 0, okay. But cos hyperbolic y always assumes positive values, okay. It is an even function of y.

So whatever values it takes for y greater than or equal to 0, same values it takes for y less than or equal to 0. And therefore, what happens? C2 is also mapped into the same image, C2 dash and C1 dash are same, okay. Now angle between C1 and C2 is pi, okay at the critical point pi/2. But here this point, the critical point z0 is mapped into w=1. And the angle between C1 dash and C2 dash is 0 which is same as 2pi, okay.

So angle between C1 dash and C2 dash is 2pi. So the angle between C1 and C2 which is pi is doubled here. It becomes 2pi, okay. So angle between C1 and C2 is equal to pi and angle between C1 dash and C2 dash is 0 are equivalently 2pi. Thus the angle between C1 and C2 at the critical point pi/2 is doubled under w=sin z. So this verifies the theorem 2. With that, I would like to end this lecture. Thank you very much for your attention.