Advanced Engineering Mathematics
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Lecture – 27
Cross Ratio

Hello friends welcome to my lecture on cross ratios of a bilinear transformation. Now suppose first we will discuss fixed points of a transformation. So let us say suppose we have a mapping w = fz, then by a fixed point of w = fz we mean a point whose image under the function w = fz is the same complex number.

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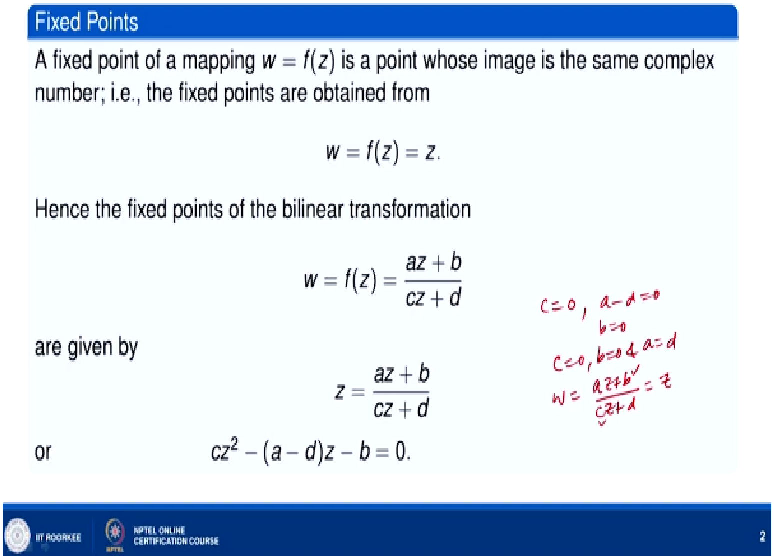

That is the fixed points of w = fz are obtained from the equation fz = z. Hence in particular the fixed points of the bilinear transformation w = az + b/cz + d are given by az + b/cz + d = z okay. This can be written as cz square – a – d * z – b = 0.

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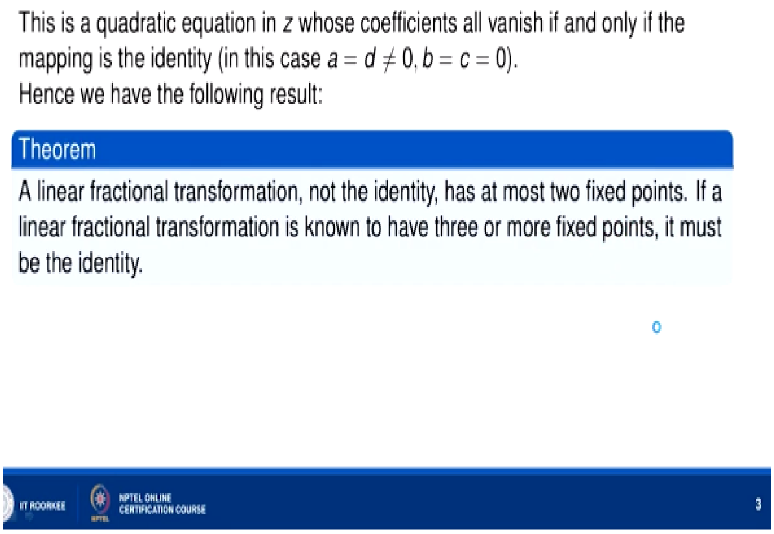

Now this is a quadratic equation in z, you can see this is a quadratic equation in z whose coefficients all vanish if and only if the mapping is the identity. You can see if the coefficients all are 0s, c = 0 a – d = 0 and b = 0, okay, then c = 0, b = 0 and a = d gives you w = az + b/cz + d. So b = 0, this means this is = 0, c = 0 means this is = 0, then a = d. So it is = z, so when c is 0, b is 0, a = d we get w = z, that is the identity transformation.

So the coefficients all vanish, the quadratic equations coefficients all vanish if and only if the mapping is the identity in this case a = d not = 0, b = c =0, because we want ad – bc to be nonzero. So we have to take a = d0 = 0. Now therefore we have the following result. A linear sectional transformation not the identity has utmost 2 fixed points okay. If you do not want this fractional transformation to be an identity, then it will have utmost 2 fixed points okay.

If it has more than 2 fixed points that is 3 fixed points or more fixed points, then it must be the identity transformation okay.

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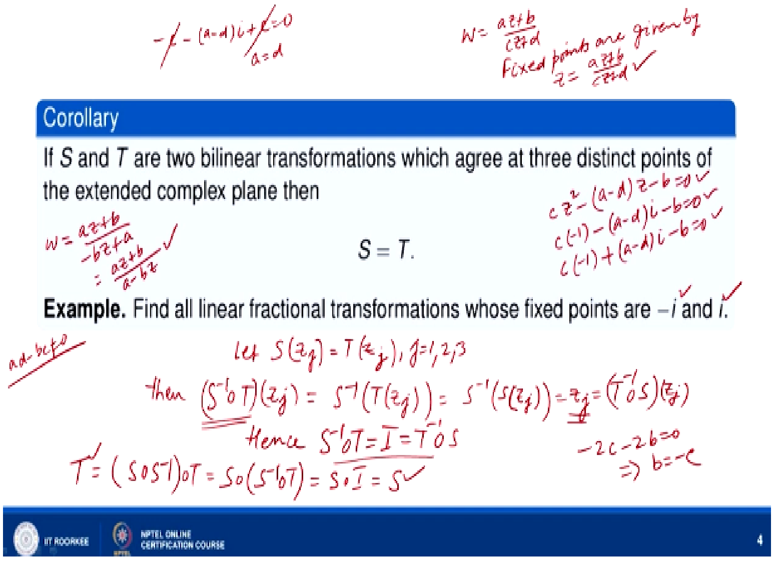

So if S and T are 2 bilinear transformations we in particular in as a corollary we have if S and T are 2 bilinear transformations which agree at 3 distinct points of the extended complex plane then S must be same as T, okay. So we can easily prove this, suppose let us say Szj = Tzj okay for j = 1, 2, 3 okay.

Then let us take 3 distinct points z1, z2, z3 in the extended complex z plane and suppose that S and T agree on these 3 points z1, z2, z3, then S inverse let us say this composition mapping. S inverse OT, because when S is the bilinear transformation then S inverse is also bilinear transformation and composition of 2 bilinear transformations is also bilinear transformation. So s inverse OT is a bilinear transformation and this will be = S inverse Tzj.

So this is = Tzj = Szj, so S inverse Szj, S inverse Szj = zj and this also equal to similarly T inverse os, in a similar manner we can say that T inverse os at zj = zj because this is T inverse Szj, Szj = Tzj and T inverse Tzj = zj. So we see that S inverse/Tzj = T inverse os zj okay and now, so this is bilinear transformation okay, S inverse OT which maps zj * zj. So this by the previous theorem S inverse OT must be identity map okay.

So hence S inverse OT will be identity map and same will be true for the bilinear transformation T inverse OS. Now let us we want to show that S = T okay. So we can see that S = S O S inverse okay, S = S O S inverse this is identity okay, OT okay. We can write T as SOS inverse OT because S O S inverse is identity map okay. So this is equal to this and then SOS inverse OT okay = SO identity map which is = S okay.

So T can be written as SOS inverse OT but this is = SOS inverse OT and S inverse OT = identity map, so SOi be = i so T = S. So if 2 bilinear transformations agree at 3 distinct points of the extended complex plane then they must be same okay. Now let us see in example find all bilinear transformations whose fixed points are –i and i. We have seen that if W = fz = az + b/cz + d, then the fixed points of this transformation are given by z = az + b/cz + d.

Okay which gives you cz square – a-d * z – b = 0 okay. This equation gives us this okay. Now if fixed points are i and –i, fixed points are given by this equation okay. So if i and –i are fixed points then let us put them here. So let us put first i okay. When you put i here C i square means – 1, -a, -d * i – b = 0 okay. Now let us put –i. So we get – i whole square is i square. So again -1, -i we put here, so we get + a-d * i – b = 0.

So let us solve these 2 equations okay. Adding these 2 equations what do we get – 2C, when you add them then this middle term will cancel and we will get -2C – 2b = 0, so we get b = -C okay, b = -C, now let us put b = -C in this equation what do we get –C, -a –d * i and b is –c. So + C = 0. So this cancels with this and what do we get A = D okay. So the bilinear transformation W = az + b/cz + d whose fixed points are i and –i has to be w = az + b, b=-c okay.

So az + b okay c is –b, so – bz and d = a, so we get az + b/a – bz okay where a and b are arbitrary but we have to remember that ad – bc must be nonzero okay. So this is the set of all linear fractional transformations whose fixed points are –i and i, a and b are any real numbers, but a and b are related to c and d by this equation ad – bc not = 0.

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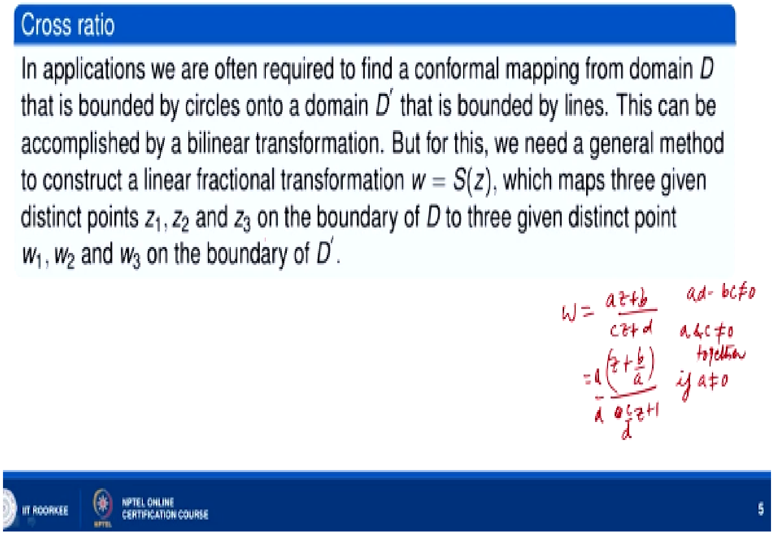

Okay now let us come to cross ration. In applications we often required to find a conformal mapping from domain D that is bounded by circles onto a domain D dash that is bounded by lines. We can accomplish this by a bilinear transformation. But for this we need a general method to construct a linear fractional transformation w – Sz, which maps 3 given distinct points z1, z2, z3, because to draw a circle we need 3 non-collinear points in the plane okay.

So which maps 3 given distinct points z1, z2, z3 on the boundary of D to the 3 given distinct points w1, w2, w3 on the boundary of d dash. You can see w = az + b/cz + d okay, because of ad - bc not = 0 okay, a and c cannot be both 0 okay. So if a is not = 0 we can divide by a okay and write it as z + b/a, a and c cannot be both 0 simultaneously okay, because then ad – bc will be 0 okay.

So suppose if a is not 0, okay, a is not 0, we can write it as ad z + b/a and here so this will be what, a z + b/a and here when a is not 0 we can have cz + d okay. So we will have to write it as a is nonzero we are taking d can be 0 okay, a is not 0, so if a is not 0 then c is not 0 okay. Either C is not 0, if C is 0 then d cannot be 0 okay. So we can take then D, we can divide by D and write it as a/d and then c/d * z + 1 okay.

So then you can see there are only 3 ratios, a/d, b/a, c/d they can be replaced by 3 constants. So to determine you need transformation, you need bilinear transformation, we need to have 3 distinct points okay. So if we have 3 distinct points then we can determine a unique bilinear transformation which maps 3 distinct points z1, z2, z3 to 3 distinct points w1, w2, w3. So we will construct a general method to which will take us from 3 distinct points z1, z2, z3 in the z plane to 3 distinct points w1, w2, w3 in the w plane okay.

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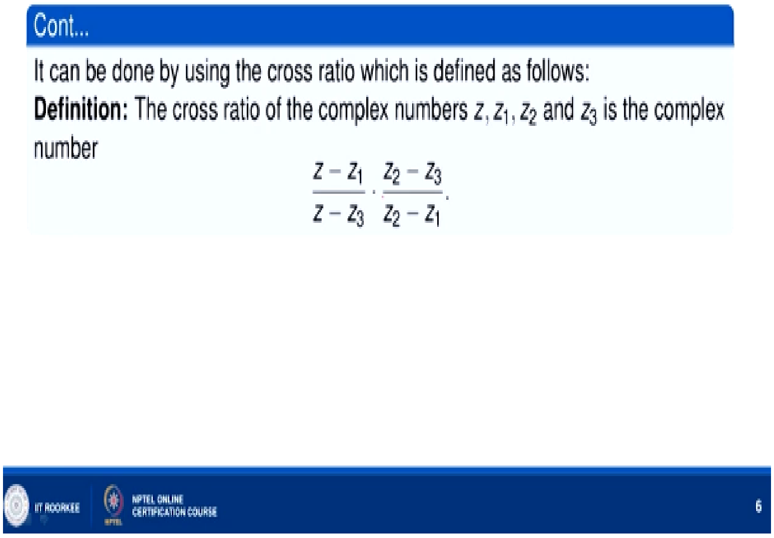

So let us, it can be done by using the cross ratio which is defined as follows. The cross ratio of the complex number let us take z, z1, z2, z3 okay, z, z1, z2, z3, the complex, then this cross ratio of these 4 numbers is given by this complex number, z – z1/z- z3, z2- z3/z2- z1 okay. This cross ratio will help us in finding the transformation which maps the 3 points z1, z2, z3 into the 3 points w1, w2, w3 in the complex plane.

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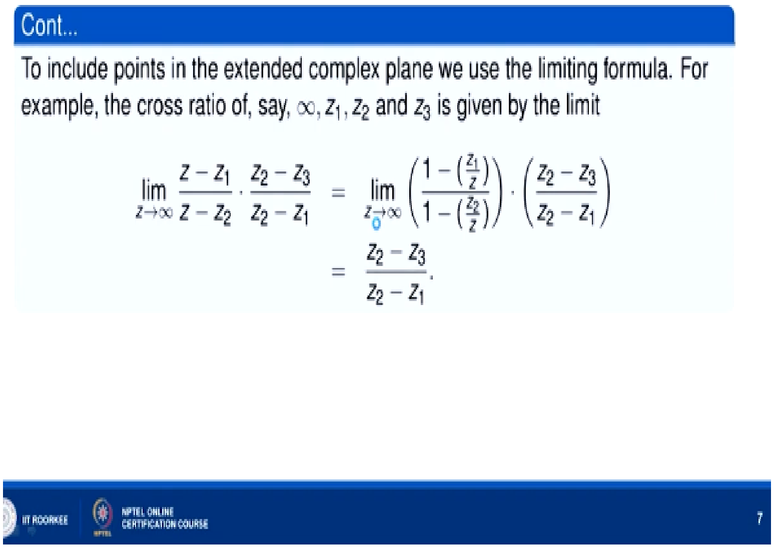

So to include points in the extended complex plane because we want the linear transformation, the bilinear transformation to be defined for the extended complex plane. So to include the points in the extended complex plane we use the limiting formula, that is for example the cross ratio of say z, z let us take at infinity, infinity z1, z2 and z3, it is given by limit z tends to z-z1/z-z2 z2-z3/z2-z1.

Now the limit z tends to infinity z-z1/z-z2 * z2-z3/z2-z1 = limit z tends to 0, we replace z/1/z. So 1/z-z1/1/z-z2 okay. So this is independent of z okay. So we can write this like this. So this is = limit z tends to 0 1- z z 1/1- z z 2 okay * z2 –z3/z2- z1. So this is = z2-z3/z2-z1 okay. So this cross ratio will then reduce to z2 – z3/z2 – z1.

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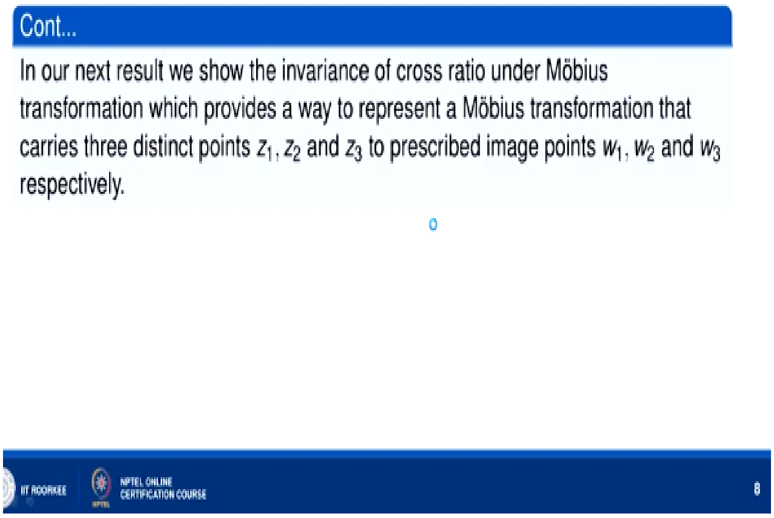

Now in our next result we show that the invariance of cross ratio under Mobius transformation, we are going to show that the cross ratio remains invariant under bilinear transformation, so we prove this, we show that the cross ratio remains invariant under Mobius transformation, which provides way to present Mobius transformation that carries 3 distinct points z1, z2, z3 to prescribed 3 image points w1, w2 and w3.

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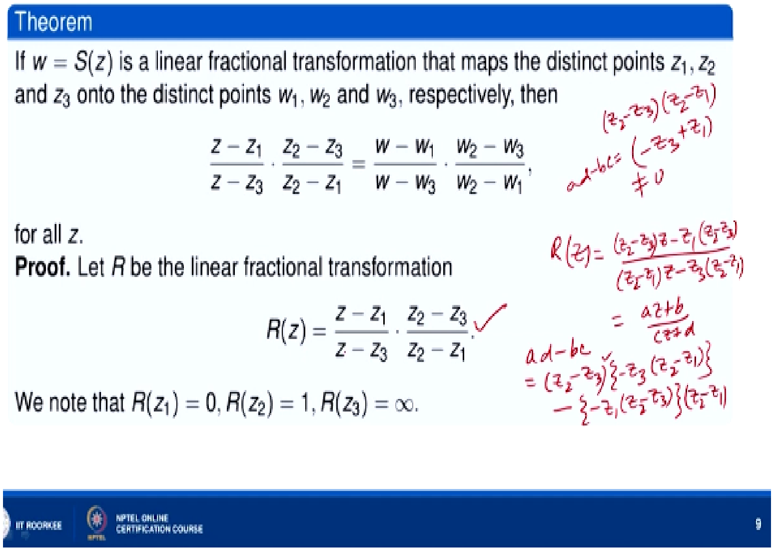

So this is the theorem, if w = Sz is a linear fractional transformation that maps the 3 distinct points z1, z2, z3 onto the 3 distinct points w1, w2, w3 respectively, then z – z1/z – z3 = z2 – z3/ z2 – z1, w – w1/w – w2, w2 – w3/w2 – w1. You can see the cross ratio remains invariant from here and this bilinear transformation also helps us in finding the required, actually this is the required bilinear transformation, which carries the 3 points z1, z2, z3 to w1, w2, w3.

You can see here when z = z1, this becomes 0, left side and so w = w1 okay and when z = z3 okay this becomes infinite and so w becomes w3 and when z = z2 okay then left side becomes 1 and here right side becomes 1 when w = w2 okay. Now let us in order to prove this, let us say R be the linear fractional transformation. Rz = z – z1/z – z3, z2 – z3/z2 – z1, then we can see that first of all this is the fractional transformation why because Rz = it can be written in this form z2-z3 * z – z1* z2-z3/z2-z1 * z – z3 times z2-z1.

So you can see it is of the form az + b/cz +d, but it will be called a linear fractional transformation provided we show that ad – bc is nonzero. So let us prove that here a is z2 – z3, b is –z1 * z2 – z3, c is z2 – z1 * z2 – z1 and d is –z3 * z2 – z1. They satisfy the condition that ad – bc is nonzero. So ad – bc is how much here, z2 – z3 * d, d is – z3 * z2 – z1, okay then –b = -z1 * z2 – z3, -bc okay.

So C is z2 – z1 okay, now from here what do we notice? z2 – z3 and z2 – z1 we can take common okay. So then taking these 2 factors common z2 – z3 * z2 – z1, what we get –z3 here and here what do we get z2 – z1, z2 –z3, so + z1 okay. So ad – bc is z2 – z3 * z2 – z1 * z1 – z3. Now z1, z2, z3 are your distinct points. So ad – bc is not = 0 and therefore this transformation is a linear fractional transformation.

Now further more you notice that Rz1 = 0, okay Rz2, z2 = 1 and Rz3, when z goes to z3 you see that Rz goes to infinity okay, so Rz3 = infinity.

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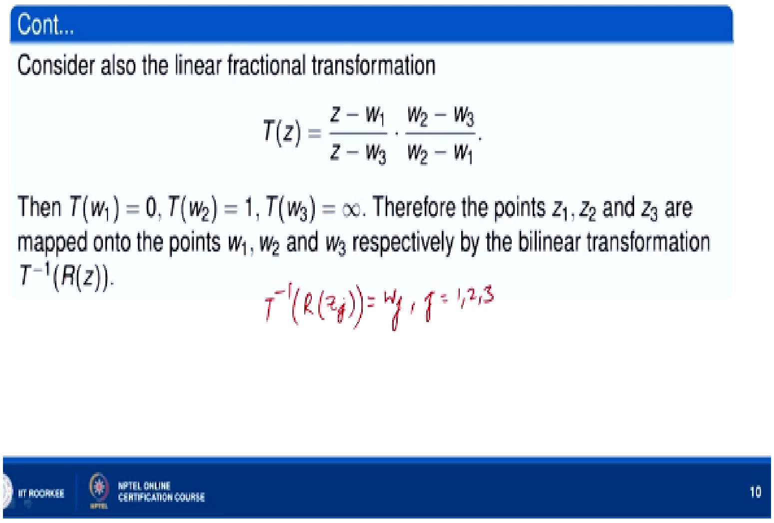

Now consider also this transformation Tz = z – w1, z – w3, w2-w3, w2-w1. Then here again when you replace z by w1, Tw1 = 0, Tw2 = 1, Tw3 = infinity. Therefore, the points z1, z2, z3 are mapped on to the points w1, w2, w3 respectively by the bilinear transformation T inverse Rz how, because T inverse Rzj okay, Rzj let us call them as z1, z2, z3, Rzj are 1, 0 infinity okay. So Rzj goes to 1, 0 infinity.

T inverse maps n okay and T inverse of, T maps w1, w2, w3 to 0, 1 infinity okay. So T inverse maps 0, 1, infinity * w1, w2, w3. So Rzj goes to 0, 1 infinity and 0, 1 infinity under T inverse go to wj okay. So therefore the points z1, z2, z3 are mapped to points w1, w2, w3 respectively by the bilinear transformation T inverse Rz. T inverse Rz is bilinear transformation because T is a bilinear transformation, this inverse is bilinear transformation and then T inverse and composition R is also a bilinear transformation.

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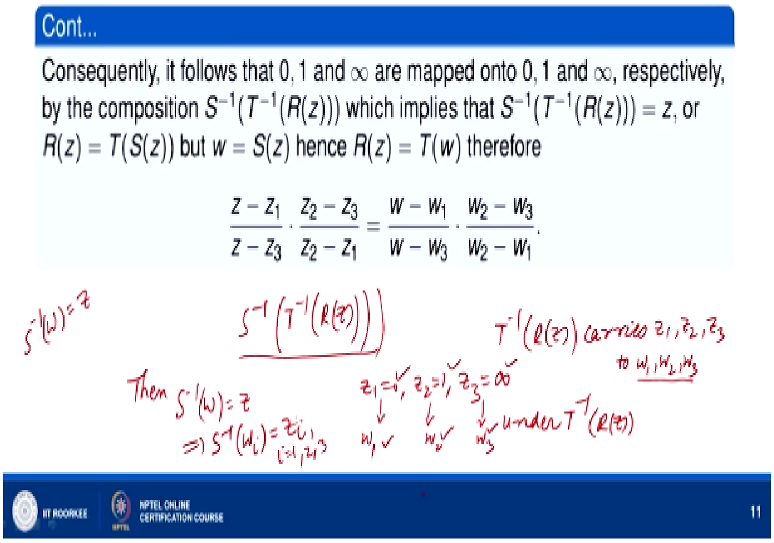

Now consequently it follows that 0, 1 infinity are mapped on to 0, 1 infinity respectively by this composition. So let us look at this composition, S inverse, T inverse Rz. So how we get this, 0, 1 infinity are mapped into 0, 1 infinity, okay what happened was here S inverse, T inverse Rz okay. S inverse, T inverse Rz did what from here let us look at this. T inverse Rz, T inverse Rz carries z1, z2, z3 to w1, w2, w3 okay.

So T inverse Rz carries z1, z2, z3 to w1, w2, w3, okay now we are applying S inverse on this mapping, S inverse this mapping okay. So let us look at this, w = Sz, we are given this information w = Sz which maps z1, z2, z3 on to w1, w2, w3 okay. So this means that z = S inverse W okay. So this means that w1, w2, w3 goes to z1, z2, z3 okay. Now you see let us take the points 0, 1.

We have seen that T inverse Rz carries z1, z2, z3 to w1, w2, w3 okay, let us take the points z1, z2, z3 to 0, 1 infinity okay, z1 = 0, z2 = 1 and z3 = infinity okay. Then z1, 0, 1 and infinity, okay, under this mapping T inverse Rz okay go to say w1, w2, w3 okay. Suppose this goes to w1, this goes to w2 and this goes to w3 okay under T inverse Rz okay. Then S inverse w = z implies that this w1, w2, w3 go to z1, z2, z3 that is 0, 1 infinity.

So then S inverse w = z okay, S inverse w = z implies that w1, w2, w3 go to z1, z2, z3, S inverse Wi go to Zi for i = 1, 2, 3, okay so let us take 3 points 0, 1, infinity in the z plane okay, then under T inverse Rz, we have seen T inverse Rz maps the 3 points z1, z2, z3 to w1, w2, w3, so these 3 points 0, 1 infinity okay, 0, 1 infinity they are supposing go to w1, w2, w3 okay. Then under the mapping T inverse Rz, you see because of this Tw1 = 0, Tw2 = 1, Tw3 = infinity okay.

So T inverse 0, 0, 1 infinity okay, 0, 1, infinity under T inverse, T inverse takes 0, 1 infinity to w1, w2, w3 okay, so what we get here. So T inverse, as inverse w = z, implies S inverse wi = zi, this means that the points 0, 1 and infinity are mapped into w1, w2, w3 and w1, w2, w3 under S inverse go to z1, z2, z3, this means that and z1, z2, z3 are 0, 1 infinity. So 0, 1 infinity are mapped into 0, 1 infinity and therefore because of the theorem on fixed points okay the transformation must be an identity because it fixes 3 distinct points 0, 1 and infinity.

So this S inverse, T inverse Rz fixes 3 distinct points 0, 1 and infinity, so it must be an identity map and therefore S inverse, T inverse Rz must be = z and therefore Rz = T of Sz okay, Rz = T of Sz, but Sz = w, so Rz must be = Tw okay and Rz = z – z1/z – z3, z2 – z3/z2 – z1, Tw = w – w1/w – w3 * w2 – w3/w2 – w1. So we get this Tw okay. So Rz = Tw and this gives us these transformation.

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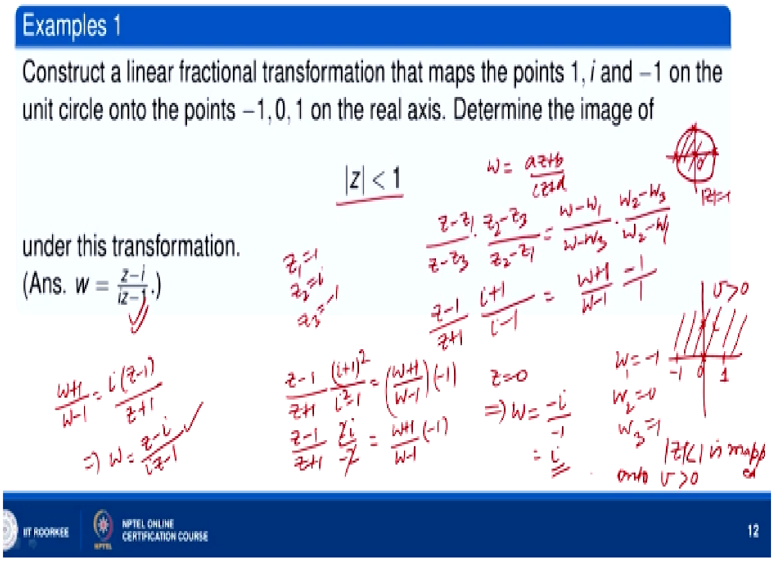

Now let us take an example on this, construct a linear fractional transformation that maps the points 1, i and -1 on the unit circle on to the points -1, 0, 1 on the real axis. So we have w = az + b/cz + d okay, this bilinear transformation and the transformation that maps 3 distinct points on to 3 distinct points from z plane to w plane we have z – z1/z – z3 * z2 – z3/z2 – z1 = w – w1/w – w3 * w2 – w3/w2 – w1.

So here we can put the value this z1 is 1, z2 is i and z3 is -1, okay, so z – z1, so z – 1/z – z3 that means z + 1 and z2 – z3, so i + 1/z2 – z1, so i -1, this is = w- w1, w1 = -1 and w2 =0, w3 = 1. So w – w1 means w + 1 okay and then w – w3 is w-1 and w2 – w3 is -1/w2 – w1 = +1 okay. So what we get? z-1/z+1 okay, this i can simplify i + 1/i-1 i can multiply by i + 1, so i + 1 whole square/i square – 1 this = w + 1/ w – 1 and what we get * -1.

w2 – w3 is -1, w2 – w1 is + 1 okay, so this is z-1/z+1 and here we have i square + 1 which is 0, because i square is -1, so we get 2 i/-2 and this is w+1/w-1 * -1 okay. So, this means that w+1/w-1 = now this cancels with this –i and -1 here, so this gives you i times, i times z – 1/z + 1 okay and when we simplify this okay, we get w = this gives you w = iz – i/iz – 1 okay. So this transformation can be obtained by simplifying this equation.

Now let us find the; so this is the required transformation which maps 1i – 12 -101 and we have to find the image of this interior of this circle mod z = 1 here under this transformation. So let us take the test point z = 0, z = 0 goes to w = -i/-1, this means i okay. So i lies here okay, i lies here okay. Now 1i – 1, they are the points on the circle mod z = 1, you see 1 is here, i is here and -1 is here.

We have taken 3 points on the circle mod z = 1 and we are looking for and these 3 points are being mapped on to -1, 0, 1 on the real axis okay. So the boundary of the circle is going into the real axis of the w plane okay and we are now looking for the interior of mod z = 1 where does it go. So we have taken a test point here z = 0 and we found the image of z = 0 under the bilinear transformation w = z-i/iz – 1 as i and we see that i lies in the upper half of the w plane.

So this mean that upper half of the w plane is the image of mod z < 1, okay upper half of the w plane means v > 0. So mod z < 1 goes to mapped into v > 0 okay, rather we could write mod z < 1 is mapped on to v> 0 okay.

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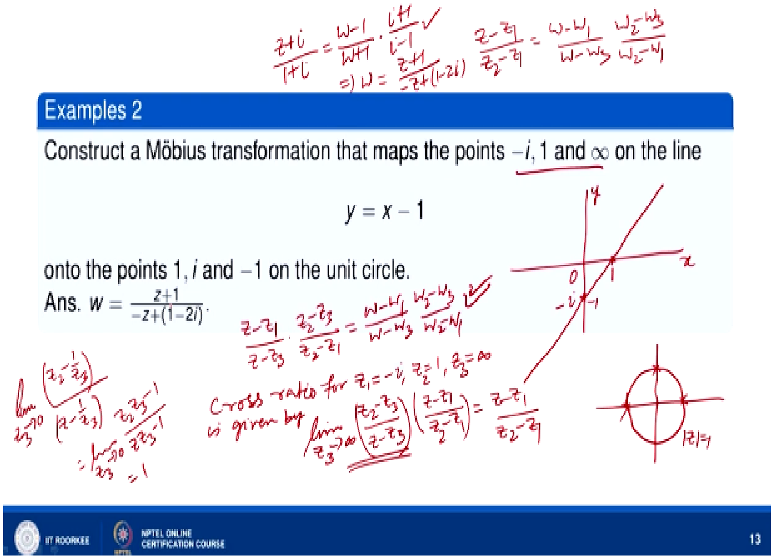

Let us take one more question, construct a Mobius transformation that maps the point –i, 1 and infinity on the line y = x – 1 you can draw the line y = x – 1. So this is your x = 1 and when your, this is when x, 0 till -1. So this is the line y = x – 1 in the z plane and we have, so construct a Mobius transformation that maps the point –i, this is –pi okay. This point is –i, this point is 1 and we have a point infinity on the line okay.

So they are mapped on to the points 1, i and -1 on the unit circle, that means under the transformation straight line is going into circle. This is mod z = 1. The points are 1, i and -1 okay, let us find the transformation. So we have z-z1/z-z3 z2-z3/z2-z1 okay, this cross ratio okay, we have this equation w – w1, w – w3, w2 – w3, w2 – w1 okay, the cross ratio for the points –i, 1 and infinity okay will be this following.

For z1 = -i, z2 = 1, z3 = infinity, okay will be limit z3 tends to infinity, z2 – z3/z – z3 * z – z1/z – z3 or z2 – z1 okay. So we can write it as limit this limit we can find by taking z3 goes to 0 and z3/1/z3. So z2 – 1/z3/z – 1/z3. So this will be limit z3 goes to 0 okay, z2, z3 – 1/z z3-1, so this is 1 okay. So this will be replaced by 1 okay and this gets reduced to z- z1/z2 –z1. So we have this equation, this equation becomes z – z1/z2 – z1 = w – w1/w – w3 * w2 – w3/w2 – w1.

So now let us put the values here, z1 is –i, so z + i okay/at z2 – z1, so 1 + i, = w – w1, so w – 1/w – w3, so w + 1 and then w2 – w3. So i + 1 and then w2 – w1, so i – 1 okay. So this is the equation, this can be simplified and we get w =, this gives you w = after simplification z + 1/-z + 1 – 2i. So this is the required transformation which maps the points –i 1 and infinity to 1, i and -1 on the unit circle. So with this I would like to end my lecture. Thank you very much for your attention.