Advanced Engineering Mathematics

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Lecture – 24
Evaluation of Real Integrals Using Residues - IV

Hello friends. Welcome to my lecture on evaluation of real integrals using residues. This is the fourth lecture on this topic. In this lecture, we shall consider certain real integrals for which some special techniques and some special contours are needed. So let us consider the integral, integral over -infinity to infinity x cos pi x/s square+2x+5*dx.

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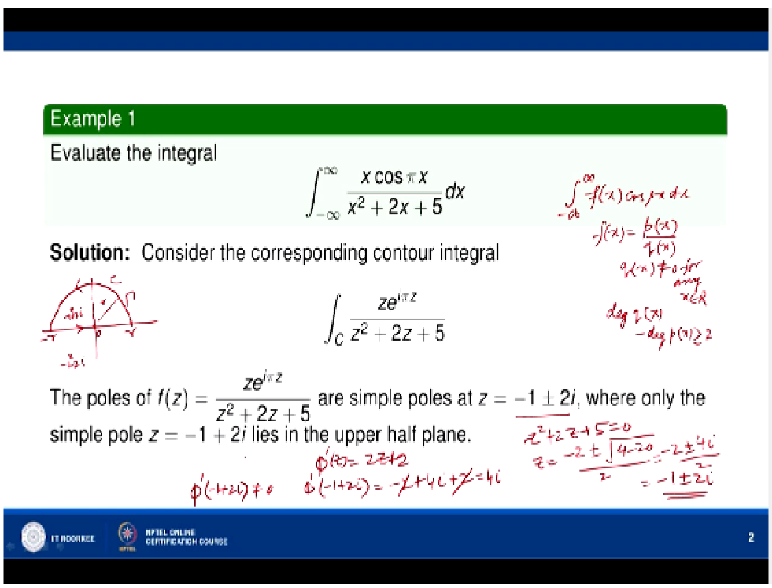

Now here you can see we have considered such type of integrals in the previous lecture. But there what we had was that we considered integral over -infinity to infinity fx cos sxdx and we assumed that fx is a rational function say px/qx where qx is not 0 for any real x and degree of qx-degree of px is greater than or equal to 2. But here we notice that although fx=x/x square+2x+5, that is it is a rational function, okay.

But here the degree of the denominator is 2 while the degree of the numerator is 1. So the difference in the degrees of the denominator and the numerator is not greater than or equal to 2. And therefore, the technique which we applied to show that the integral along gamma, that is integral along the semicircle goes to 0 as r goes to infinity, that technique will have to be modified in order to show that the integral along gamma of the function fz*e to the power i pi z.

Here we will have, we are taking cos pi x. So we will be considering e to the power i pi z. So integral over gamma fz e to the power i pi zdz goes to 0. So we will have to make a certain modification in that technique to show that the integral along gamma goes to 0. So what we do is, let us consider the contour integral z e to the power i pi z/z square+2z+5dz. Now you can see that fz=z e to the power i pi z/z square+2z+5.

So it will have poles wherever the denominator z square+2z+5=0. Now z square+2z+5=0 gives you z=-2+-under root 4-20/2 which will be equal to -2 +-4i/2 or we can say -1+-2i. Now here again we shall be considering the contour which consist of the semicircle with center at origin of radius r and center at the origin and the line segment from -r to +r. So you can see that here there are 2 poles.

One pole lies at -1-2i. -1-2i will be in the third quadrant, here, -1-2i. And -1+2i will be here in the second quadrant, okay. So and moreover you notice that the denominator, suppose if denominator is 5z, then 5 prime z. 5 prime z=2z+2, okay. So at z=-1+2i, what we get? 5 prime -1+2i=-2+4i+2, okay. So this is equal to 4i. Therefore, 5 prime-1+2i is not equal to 0 and so we can say that at -1+2i, the function fz has a simple pole, okay.

Actually it has simple poles at both of these points -1+-2i, but out of these 2 poles, the pole at -1+2i lies in the upper half plane. And our contour which we have taken, contour C consist of this semicircle gamma and the line segment from -r to r contains the simple pole at z=-1+2i when you take r to be sufficiently large. So we shall be only considering the residue at the simple pole which lies in the upper half plane.

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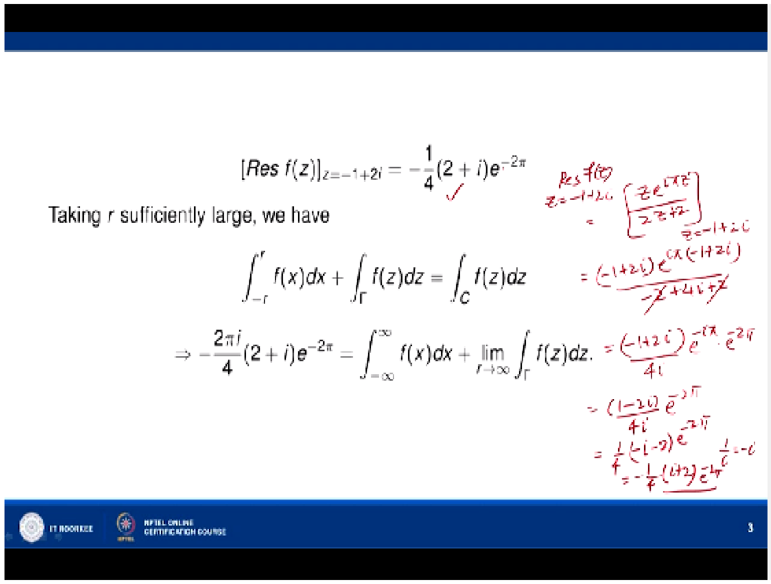

So let us find the residue of fz at z=-1+2i. So we have, in order to find this, we consider residue of fz at z=-1+2i will be equal to, we have, z e to the power i pi z/z square+2z+5. So we will differentiate the denominator and put the value of z. So we have z e to the power i pi z/2z+2 and you put the value of z=-1+2i. So what we will get? -1+2i*e to the power i pi-1+2i/-2+4i+2. So this will cancel with this and what we will get here?

We have -1+2i/4i and then we have e to the power -i pi*e to the power -2pi, e to the power -i pi*2i is e to the power -2pi. So e to the power -i pi is cos pi-i sin pi; and therefore, it is -1. So we get 2i-1/4i*e to the power -2pi. We can simplify it further and write it as, 1/i is -1. So we will have, 1/i is -1. So when you multiply -i to 2i-1, what you will get? -2i square, means 2 and -i when you multiply to -1, you get +i, okay.

So -1 when you multiply here, we get 1-2i actually. Now 1/i is -i, okay. So 1/4 and we multiply -i here, so we get -i and then +2i square. +2i square means -2, okay. So we get -1/4 i+2*e to the power -2pi, okay. So we get this. Residue of fz at z=-1+2i comes out to be -1/4 2+i e to the power -2pi.

Now taking r to be sufficiently large, okay, the integral over C. Integral over C can be written as integral over gamma + integral over -r to r fzdz. So integral over -r to r, now fz becomes fx because we are moving along the x axis. So fz becomes fx and dz becomes dx. So integral over

-r to r fxdx+integral over gamma fzdz=integral over C fzdz.

Now by residue theorem, integral over C fzdz=2pi i*sum of residues at the poles in the upper half plane. So we get 2pi i, we multiply to that value and we get -2pi i/4 2+i*e to the power -2pi. And then here we have integral over -infinity to infinity fxdx + limit r tends to infinity integral over gamma fzdz, okay.

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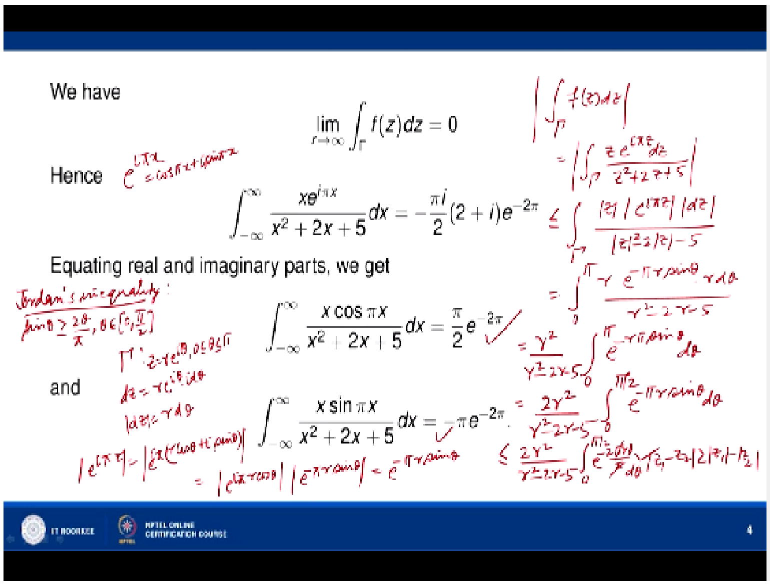

Now we will show that limit of integral over gamma fzdz as r goes to infinity is 0, okay. So let us prove this. So we have fz=z e to the power iz. So let us consider integral over gamma fzdz, fz we have taken as z e to the power i pi z/z square+2z+5, okay. So we put that. So dz=modulus of integral over gamma z e to the power i pi zdz/z squre+2z+5, okay.

So this is less than or equal to integral over modulus of z mod of e to the power i pi z, okay, and mod of dz/modulus of z square-2*mod of z-5 using the fact that mod of z1-z2 is greater than or equal to mod of z1-mod of z2. This follows from the triangle inequality. So using this triangle inequality, this result from the triangle inequality, we get the modulus of z square+2z+5 is greater than or equal to mod of z square-2*mod of z-5, okay.

Now gamma is the semicircle with center at origin, radius r. So mod of z=r, okay. And 0 less than or equal to theta less than or equal to pi, we can write z=, so we can write z=r e to the power i theta, small r we should put. So z=rei theta 0 less than or equal to theta less than or equal to pi. So mod of z we shall replace by r, mod of dz=r e to the power i theta*id theta. So mod of dz will be equal to rd theta, okay.

And what is e to the power i pi z? Mod of e to the power i pi z=mod of e to the power i pi and then z is r cos theta + ir sin theta. So this is equal to mod of e to the power i pi r cos theta*modulus of e to the power -pi r sin theta, okay. Modulus of e to the power i pi r cos theta=1 and e to the power -pi r sin theta is greater than or equal to 0. So we have put greater than 0. So we can write it as e to the power -pi r sin theta, okay.

So this can be written equal to integral 0 to pi r e to the power -pi r sin theta*rd theta/r square-2r-5, okay. Now this is equal to r square/r square-2r-5 integral 0 to pi e to the power -pi r sin theta d theta. Now let us notice the difference between this expression and the expression that we used to get when we had this, in the coefficient of cos sx, we had the rational function where degree of the denominator was at least 2 units higher than the degree of the numerator.

Here the denominator has only 1 degree higher than the degree of the numerator, okay. So when the degree of the denominator was at least 2 units higher than the degree of the denominator, what we did was? This e to the power -r pi sin theta, we replaced by 1. This is always less than or equal to 1. So we replaced it by 1. And here, we had the difference in the degree of the numerator denominator was 1, okay.

And so when r tends to infinity, this expression tended to 0 there. But here what is happening is that because the difference in the degrees of the denominator and numerator is only 1, you can see that the degree of the numerator is 2 and the degree of the denominator is also 2 and therefore, we cannot replace this by 1.

If we replace this by 1, then will not go to 0 as r goes to infinity due to the fact that the degrees in the numerator and denominator are equal, okay. So now what we do is? We use the Jordan's inequality, okay. And Jordan's inequality says that, okay, let me first use a property of the definite integral that when you replace the integrand by this, the barrier of integration is theta.

When you replace theta by pi-theta, there is no change in the value of the integral. So it is 2r square/r square-2r-5 0 to pi/2. We can use this property of the definite integral, okay. So this follows because sin pi-theta is sin theta. So there is no change when we replace theta by pi-theta. And now what you do is, use Jordan's inequality.

And Jordan's inequality says that sin theta is greater than or equal to 2 theta/pi when theta belongs to 0pi/2, okay. So here, we can write this is less than or equal to 2r square/2 square-2r-5 and then integral 0 to pi/2 e to the power -pi r*2 theta/pi. So -2pi r theta/pi d theta, okay. So we replace sin theta by 2 theta/pi. So 1 pi will cancel and we get e to the power -2r theta d theta, okay. So let us go further.

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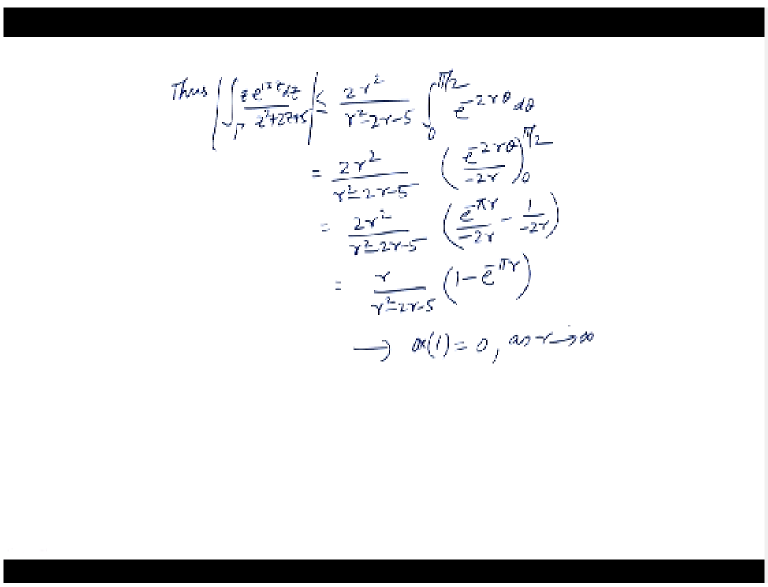

We have thus integral over gamma z e to the power i pi zdz/z square+2z+5 modulus of this is less than or equal to 2r square/r square-2r-5 integral 0 to pi/2. And we have e to the power -2r theta d theta, right. So this is equal to 2r square/r square-2r-5 and now we can easily integrate this. This is e to the power -2r theta/-2r. And what do we get here? 2r square/r square-2r-5 and we will get now, when you put pi/2 here, e to the power -pi/r, okay, /-2r and then you get here -1/-2r, okay.

So this is, 2r we can cancel and we get r/r square-2r-5 1-e raise to the power -pi r. Now you can see when r goes to infinity, e to the power -pi r goes to 0. So the expression inside the brackets goes to 1 as r goes to infinity and this is a rational function where the degree of the denominator is 1 unit more than the degree of the numerator. So this also goes to 0. So this goes to 0 and this goes to 1.

So this goes to 0*1, okay. 0*1, so this is 0 as r goes to infinity. So this is how we prove that integral along gamma goes to 0 as r goes to infinity, okay. So when we have proved this, then what do we get? Now let us go back. Now integral over -infinity to infinity fxdx=this quantity. This is 0 now, okay. So we get, and this is how much? -pi i/2 2+i*e to the power -2pi. So this we have written here, -pi i/2 2+i e to the power -2pi.

Now e to the power i pi x, we can write as cos pi x+i sin pi x, okay. So when we write cos pi x as e to the power i pi x as cos pi x+i sin pi x and equate real and imaginary parts, then what we will get? Left side we will have, on equating real parts, integral over -infinity to infinity x cos pi xdx/x square+2x+5 and what is real part here? When you multiply -pi i/2 to i, i square is -1. So you will get pi/2. Pi/2 e to the power -2pi.

And if you compare the imaginary parts, then left side imaginary part will be integral over -infinity to infinity x sin pi x/x square+2x+5dx. And here what we will get? -pi/2*2, that is -pi e to the power -2pi. So this is how we evaluate the value of the integral over -infinity to infinity x cos pi x/x square+2x+5dx.

We remember that when the degree of the denominator is not 2 units higher than the degree of the numerator, then e to the power -r pi sin theta will not be replaced by 1. We will be using the Jordan's inequality. First we will convert the integral from 0 to pi/2 and then we shall replace sin theta by 2 theta/pi so that we can integrate this expression.

As such e to the power -pi r sin theta d theta we cannot integrate, okay. So we replace sin theta by 2 theta/pi and then we are able to integrate this and then take the limit as r goes to infinity, okay. So this is how we evaluate this integral. Now let us go to next integral where we shall be seeing that the integrand has a branch point.

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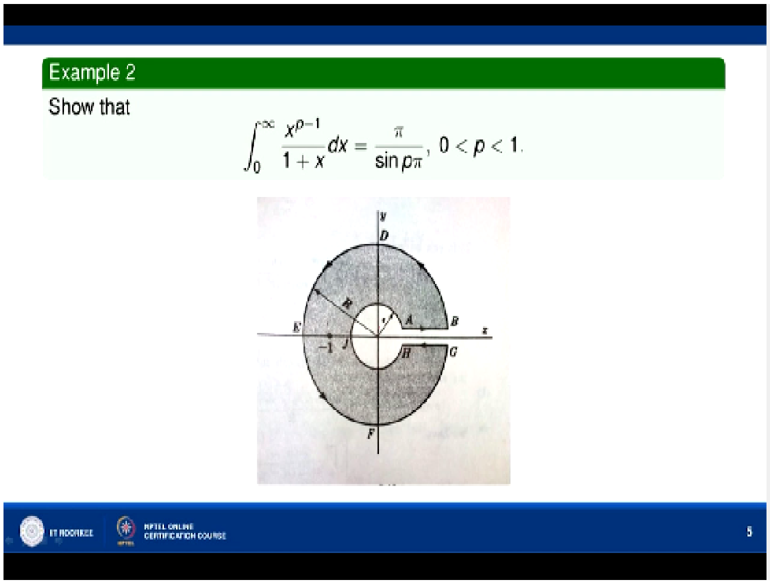

So let us consider this integral. Integral over 0 to infinity x to the power p-1/1+x dx where we have to show that the value of the integral is pi/sin p pi 0<p<1, okay. Now p is a value lying between 0 and 1, so x to the power p-1/1+x is a multi-valued function, okay. And therefore, we have a branch point at x=0. So it is a similarity of the function and therefore, we have to remove it in order to use the Cauchy residue theorem.

So we consider this kind of a contour, okay. It consists of; the contour C consist of the circle with it radius R, okay BDEFG, okay and then AB and GH. AB and GH coincide with x axis. But they are shown separate because so that for visual purposes, we have shown them separately, okay. And then this small circle, around the origin of radius is epsilon which is very small, okay. This is just to exclude the similarity at x=0, okay. So AB and GH, they coincide with x axis but we have shown them separately, okay for visual purpose. So let us see.

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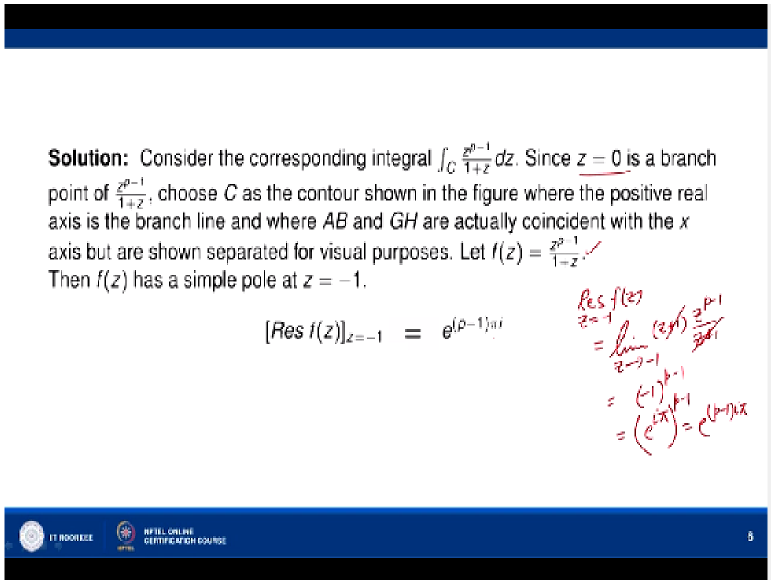

We consider the corresponding contour integral, you see. Corresponding to the integral 0 to infinity x to the power p-1/1+xdx, we consider the corresponding contour integral, integral over C z to the power p-1/1+zdz. Now this z to the power p-1/1+zdz is a multi-valued function. So that is why it has a similarity at z=0. Now we choose the contour C as shown in the figure, as shown in this figure, okay.

So here, the positive real axis is the branch line, this positive real axis is the branch line. And AB and GH are actually coincident with the x axis. But we are showing them separately for visual purpose. Now let us consider fz=z to the power p-1/1+z. Then fz has a simple pole at z=-1. The branch point at z=0 we have removed by taking this small circle, okay. So fz has only 1 similarity which is a simple pole at z=-1, okay, this -1.

Now residue of fz at z=-1 is then, okay, so when you consider this contour, okay, you move along AB, then you move along this outer circle, you move along GH, then HJRA, okay. And then you come back to B, okay. So we have this contour simple closed curve C which contains the simple pole at z=-1. So we have to find the residue of fz at z=-1. And in order to find the residue of fz at z=-1, what we do?

Residue of fz at z=-1, okay. We write limit z tends to -1 -z+1*z to the power p-1/z+1. So this will cancel with this and we will get -1 to the power p-1, okay. So -1 to the power p-1 will be, this is

-1 is e to the power i pi, okay. So e to the power i pi*to the power p-1. So this is e to the power p-1*i pi. So residue of fz at z=-1 is e to the power p-1*pi i.

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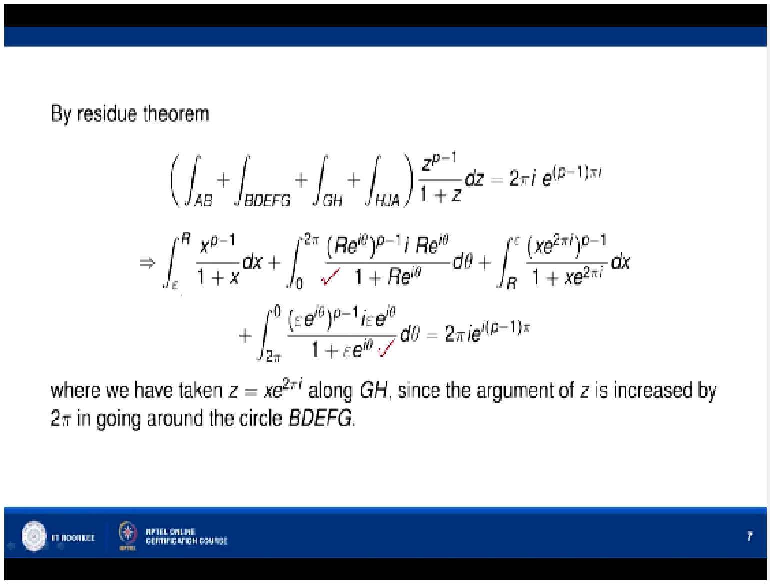

Now by residue theorem, integral over AB + integral over BDEFG + integral over GH + integral over HJRA, okay, is equal to 2pi i*e to the power p-1*pi i. 2pi i*the residue at z=-1. So this follows by the residue theorem. Now integral over AB, let us write the integral over AB. This radius is epsilon. So this is A, is at a distance epsilon from origin, okay. So epsilon to R and AB is coincident with the x axis, okay.

So you can say along AB x varies from epsilon to R and we write z as x here, okay. So epsilon to R x to the power p-1/1+xdx and then we move along the outer circle BDEFG, okay. So we write integral over 0 to 2pi. Along the outer circle, radius is R. So z will be replaced by Rei theta raise to the power p-1, dz is iR e to the power i theta/1+R e to the power i theta. And then we move along GH, okay.

When we move along GH, GH is also coincident with x axis. But at the point G, x=R and at the point H, x=epsilon. So x varies from R to epsilon. But while starting from G to H, we have taken one complete round about the origin and therefore, theta now changes by 2pi and so we will put z as x e to the power 2pi i, okay, not z as x but z=x*e to the power 2pi I because we have taken one round about the circle BD, around we have gone once around the circle BDEFG.

So x e to the power 2pi i, then raise to the power p-1/1+x e to the power 2pi i and we have dx, dz is dx, okay. Now we have to move along HJA, this HJA, okay. So along HJA, z=epsilon e to the power I theta. So we have z=, and theta varies from 2pi to 0 because we are moving now clockwise, so theta varies from 2pi to 0 and z=epsilon e to the power i theta to the power p-1 and then i.

For dz, we will have epsilon e to the power i theta*id theta/1+epsilon e to the power i theta which is equal to the right side, 2pi i e to the power p-1*I pi, okay. Now what we will do? Now let us take the limit a epsilon tends to 0 and R goes to infinity. We note that the second and fourth integral, this integral and this integral, they tend to 0 which we will be showing later, okay.

So the second and fourth integrals tend to 0 and these integral, first and third integral, they approach to 0 to infinity, okay. This goes to, epsilon goes to 0, R goes to infinity. So integral 0 to infinity x to the power p-1/1+xdx and this will become infinity to 0 and this quantity, okay. So we have 0 to infinity x to the power p-1/1+xdx and then infinity to 0 and here we will have e to the power 2pi i*p-1 x to the power p-1.

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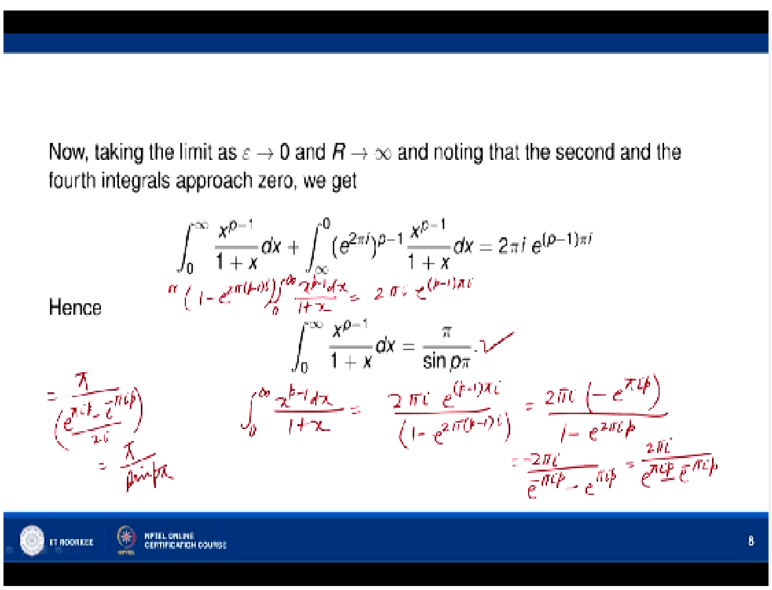

Now in the denominator, e to the power 2pi i=1. So we will have 1+x. So what we have is this, e to the power 2pi i*p-1 x to the power p-1/1+x dx, okay. Now what we will do here? We can make the limit from -infinity to 0 to 0 to infinity by writing a negative sign. So we can write this left hand side as or we can say 1-e raise to the power 2pi*p-1i, okay. So 1-e raise to the power 2pi i p-1 and then integral 0 to infinity x to the power p-1 dx/1+x=2pi i e to the power p-1*pi i.

Now we can divide by this small coefficient of this integral. So integral over 0 to infinity x to the power p-1 dx/1+x, this is equal to 2pi i e to the power p-1*pi i/1-e to the power 2pi*p-1 i, okay. Now this is 2pi I, this is e to the power p pi i*e to the power -pi i. So e to the power -pi i is -1. So we will get -e to the power pi ip, okay. Because e to the power -pi i is -1. And here what we will get, 1-e raise to the power 2pi ip*e to the power -2pi i. e to the power -2pi i=1.

So we will have 1-e to the power 2pi ip. Now e to the power pi ip, we will divide in the denominator and numerator. So we will get -2pi i/e to the power -pi ip-e to the power pi ip. When we divide e to the power pi ip in the numerator and denominator. Now this is equal to 2pi i/, because of this -1, it is e to the power pi ip-e to the power -pi ip, okay. Now this is equal to i/e to the power pi ip-e to the power -pi ip/2i, okay. And this is pi/sin p pi, okay.

So this is how we come to this result, okay. Now let us show that the second and fourth integrals go to 0 as epsilon goes to 0 and R goes to infinity.

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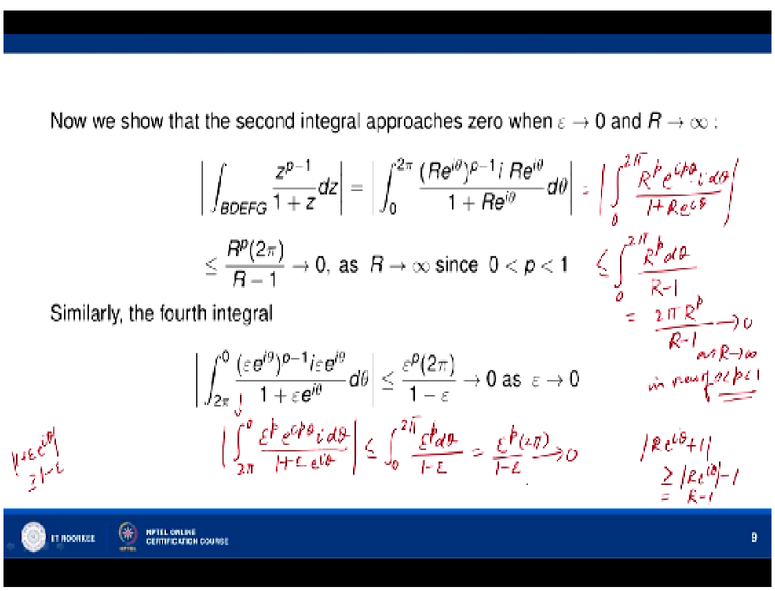

So first we show that the second integral approach is 0 when epsilon goes to 0 and R goes to infinity. The second integral is integral over BDEFG z to the power p-1/1+zdz. Now here z=Rei theta. So we have Rei theta to the power p-1 iR e to the power i theta/1+Rei theta d theta and mod, okay. Now what do we get? This is equal to mod of integral 0 to 2pi and here what we will get?

e to the power i theta*p-1*e to the power i theta. So this is e to the power ip theta, okay. So and we have R to the power p e to the power ip theta*id theta/1+Ri theta, okay. This is less than or equal to integral 0 to 2pi R to the power p modulus of e to the power ip theta is 1, okay. Because e to the power ip theta is cos p theta+i sin p theta and modulus of that is always 1. So this is, and mod of this is 1, mod of i is 1.

So we get R to the power p d theta and R to the power Rei theta+1 is greater than or equal to mod of Rei theta-1, okay. This is R-1. So this is R-1 here, okay. So this is equal to 2pi R to the power p/R-1. Now this quantity goes to 0 as R goes to infinity because 0<p<1, okay. In view of 0<p<1. This power of R is less than 1. Here the power of R is 1. Here the power of R is less than 1.

So this goes to 0 as R goes to infinity. So this is how we show that this goes to 0. Now let us come to this integral. Here what do we notice in the fourth integral? Modulus of integral 2pi to 0. Now this quantity, okay. This is equal to modulus of integral 2pi to 0, this is equal to integral 2pi to 0, then epsilon to the power p-1*epsilon to the power p e to the power i*p-1 theta*e to the power i theta.

So e to the power ip theta, same thing as we have learnt earlier. And then we have id theta. So *id theta/1+epsilon e to the power i theta. Now this is less than or equal to, you can say now this is integral over 2pi to 0, we can write as -integral 0 to 2pi and -1 mod will be equal to 1. So we can write 0 to 2pi, okay. And then epsilon to the power p mod of ei theta and mod of i is 1. So this is d theta/, here we will have 1+epsilon ei theta.

When we take the modulus of this epsilon is very small. Let us, this is greater than or equal to 1-epsilon, okay. Instead of writing epsilon -1, we will write 1-epsilon because epsilon is tending to

0. So 1-epsilon. So epsilon to the power p/1-epsilon*2pi. Now epsilon is going to 0, so denominator goes to 1 and numerator goes to 0 because p lies between 0 and 1.

So this goes to 0, okay. So second and fourth integral goes to 0. First and third integral we have combined. And we have shown that as epsilon goes to 0, R goes to infinity. We get the value of the integral 0 to infinity x to the power p-1/1+x dx=pi/sin p pi. With that I would like to end my lecture. Thank you very much for your attention.