Advanced Engineering Mathematics
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Lecture – 23
Evaluation of Real Integrals Using Residues - III

Hello friends. Welcome to my lecture on evaluation of real integrals using residues. Now here we are going to consider some special kind of real integrals where we need to take a different kind of contour, different from the ones which we have studied so far. The contour will depend on the problem. It changes from problem to problem. So let us see. We will now illustrate some, show some examples and there we shall see how some certain types of real integrals can be evaluated by making a suitable choice of contours other than those considered earlier.

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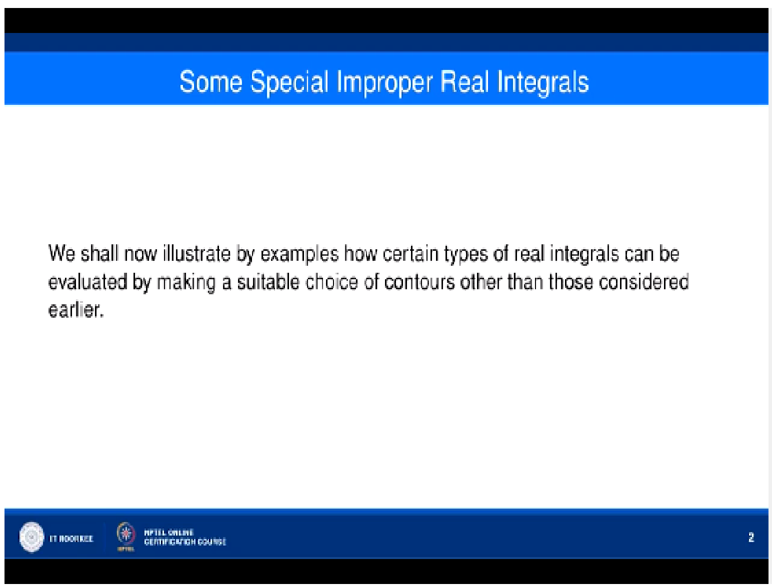

So let us consider this integral.

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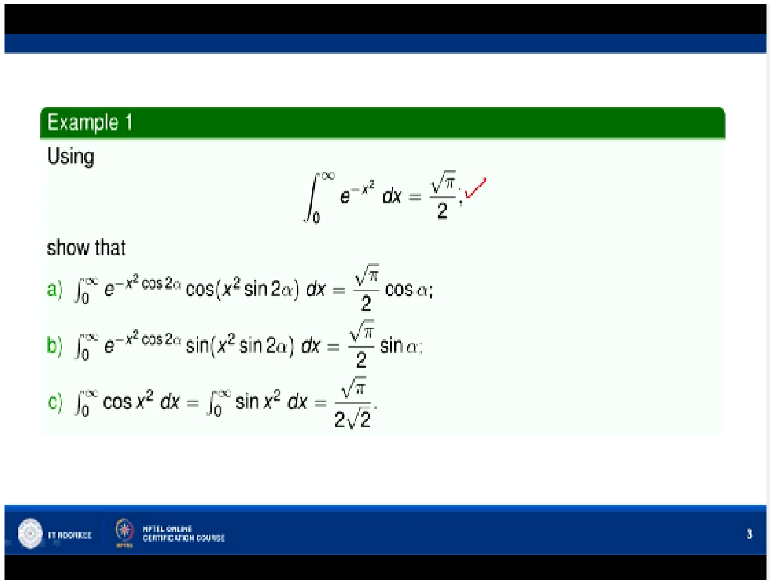

Integral 0 to infinity e to the power -x square dx where we know that by using gamma function, the value of this integral is root pi/2. So using this value of integral 0 to infinity e to the power -x square dx, let us find the value of the integral 0 to infinity e to the power -x square cos 2alpha cos x square sin 2alpha dx. And simultaneously, we will also get the value of integral 0 to infinity e to the power -x square cos 2alpha sin x square sin 2alpha dx.

So this is root pi/2 cos alpha and this is root pi/2 sin alpha. And here when you will take alpha=pi/4, you will get integral e to the power -x square cos pi/2. Cos pi/2 will be 0. So e to the power 0 will be 1 and we will get cos x square sin pi/2 is 1. So we will get integral 0 to infinity cos x square here. And here when you take alpha=pi/4, you will similarly get sin x square.

So integral 0 to infinity cos x square integral 0 to infinity and sin x square will be equal to root pi/2 cos pi/4 sin pi/4 which each of the cos pi/4 sin pi/4 have value 1/root 2. So we will get root pi/2 root 2. So by taking alpha=pi/4 in a and b, we get c. Now let us see how to get the value of a and b integrals by using residue calculus.

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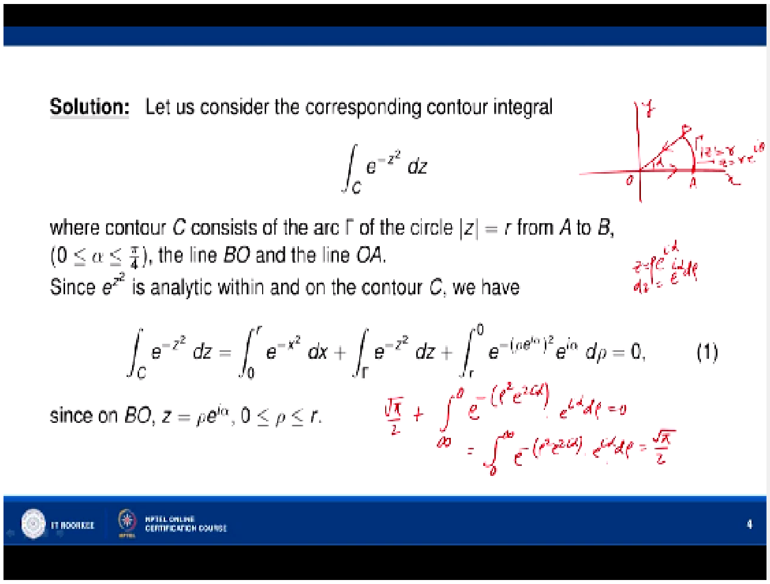

So let us consider the contour integral, integral over C e to the power -z square dz where the contour C consists of the arc of the circle mod z=r from A to B. So let us draw the contour. So this is origin, okay. Let me take this as A and let us say this is B, okay. So then where the contour C consists of the arc gamma of the circle mod z=r, this is gamma, okay, arc of the circle. mod z=r from A to B, 0 less than or equal to alpha less than or equal to pi/4.

So alpha lies between 0 and pi/4. The line BO and the line OA, this line OA and the line BO, okay. So this is 0, x, y. So our contour is integral, our contour is the line OA, the arc gamma and the line BO, okay. Now e to the power -z square is an analytic function within and on this contour C. So we have integral over C e to the power -z square dz=integral 0 to r. Now you see this AB is the arc of the circle with mod z=r. So OA=r.

So along OA, z=x and therefore, we have integral 0 to r e to the power -x square dx. Then integral along gamma e to the power -z square dz and then integral from r to 0, we are coming from B to 0, so integral from r to 0. Now z=rho e i alpha. Along BO, z=rho e to the power i alpha where rho varies from r to 0. So we get e to the power -rho e i alpha square e to the power i alpha.

Every point z on the segment OB has argument alpha, okay. And let us say magnitude rho, okay. So z=rho e i alpha we put here and then dz=e to the power i alpha*d rho because alpha is constant, okay. So since on BO, z=rho e i alpha and rho varies from 0 to r. We are coming from B to 0, so we are writing r to 0, okay.

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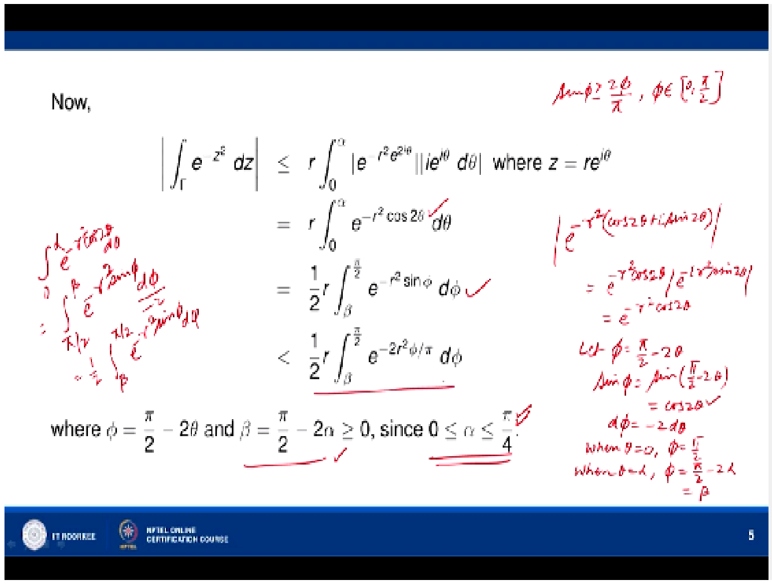

Now mod of, now let us find the value of the integral along gamma, the semi, the part of the circle, mod of z=r. So integral over gamma e to the power -z square dz is less than or equal to r, let us put z=rei theta. Any point here, on the arc, okay, I can write z=re to the power i theta, okay. So z=rei theta, when we put, we get e to the power -r square e to the power 2i theta and then dz=r3 to the power i theta*id theta, okay.

So r is constant, it comes here. Then 0 to alpha mod of e to the power -r square e to the power 2i theta, okay. e to the power -r square, e to the power 2i theta is cos 2 theta+i sin 2 theta. So mod of this=e to the power -r square cos 2 theta because mod of e to the power -ir square sin 2 theta=1, okay. So this is e to the power -r square cos 2 theta and then mod of i and mod of e to power i theta is 1, so we get this, okay.

Now here, we put first phi=pi/2-2 theta, okay. In this integral, let us take phi=pi/2-2 theta, okay. Then what do you get? Sin phi=sin pi/2-2 theta which gives you cos 2 theta. So for cos 2 theta, we write sin phi and you also notice that d phi=-2d theta, okay. Theta vary from 0 to alpha, okay. So when theta=0, phi=pi/2 and when theta=alpha, we get phi=pi/2-2 alpha, okay. Pi/2-2 alpha we denote by beta, okay.

So what do we get? Integral 0 to alpha e to the power -r square cos 2 theta d theta, will be replaced by; for 0, we get the limit pi/2; for alpha, we get the limit beta, e to the power -r square, cos 2 theta becomes sin phi and d theta becomes -d phi/-2. So this is 1/2 and the limits of integration can be changed because of this - sign. So we get beta pi/2 e to the power -r square sin phi d phi, okay.

And r was already there outside. So we get 1/2r integral over beta to pi/2 e to the power -r square sin phi d phi. Since let us look at this. Alpha lies between 0 and pi/4; therefore, 2 alpha lies between 0 and pi/2. So beta is greater than or equal to 0. Because of this condition, beta is greater than or equal to 0.

Beta is greater than or equal to 0, so integral over beta to pi/2 we have. And we know that as we have seen earlier sin phi is greater than or equal to 2phi/pi, okay when phi belongs to the interval 0 to pi/2, Jordan inequality. So we have put here, for sin phi, we put 2 phi/pi, so we get e to the power -2r square phi/pi, okay. So from here, we come to this, okay.

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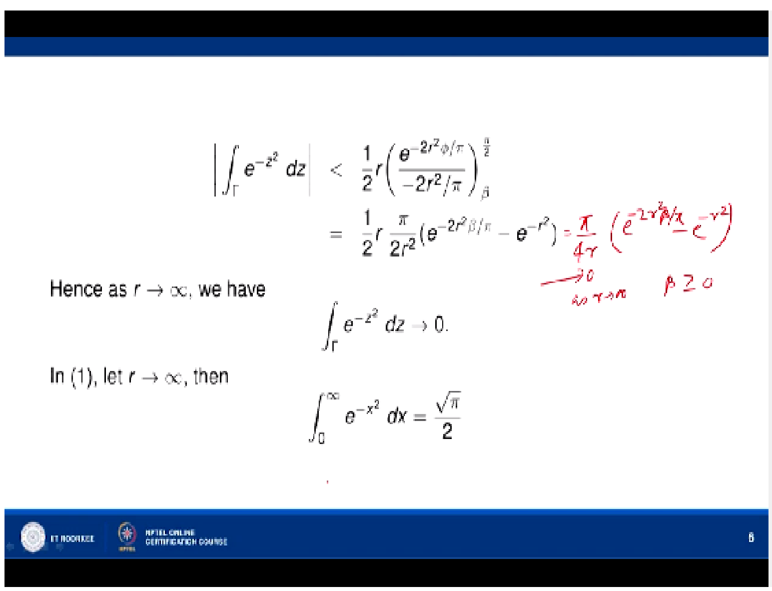

Now mod of integral over gamma e to the power -z square dz is less than, okay, so we have just integrated here e to the power -2r square phi/pi, when you integrate, you get e to the power -2r square phi/pi/-2r square/pi, okay. Because we are integrating over with respect to phi. And 1/2r is outside, so we get 1/2r and then we get pi/2r square from here and then we get, we put the upper and lower limits, we get e to the power -2r square beta/pi-e to the power -r square.

Now when r goes to infinity, what will happen? What will happen here when r goes to infinity? This is, when r will cancel, we get pi/4r, okay, pi/4re to the power -2r square beta/pi, okay, -e to the power -r square, okay. And beta as you have seen, beta is greater than or equal to 0, okay. So even if beta=0, when beta=0, this becomes 1. So 1-e to the power -r square, we have. And when r goes to infinity, this goes to 0, okay.

This is suppose 1. Then 1-0 is 1 and then 1/r here is there. So 1/r goes to 0 and therefore, this goes to 0 as r goes to infinity. If beta>0, then this goes to 0, this goes to 0 and this also goes to 0, okay. So r goes to infinity, integral over gamma e to the power -z square dz goes to 0. Now in the equation 1, let us go to the equation 1. So in this equation as r goes to infinity, what happens? This becomes integral 0 to infinity e to the power -x square dx and this goes to 0 and we get here infinity to 0, this quantity, okay.

Let us see what is this, okay. So when r goes to infinity, this is integral 0 to infinity e to the power -x square dx, okay. This quantity becomes integral 0 to infinity e to the power -x square dx. This is known to us. It is root pi/2. So what do we get? This we are using. So this becomes, this is root pi/2, okay. So root pi/2 and this goes to 0 and then we have infinity to 0 e to the power -rho square e to the power 2i alpha*e to the power i alpha d rho=0, okay.

So when it goes to the other side, the limits will change to 0 to infinity. So this becomes 0 to infinity e to the power -rho square e to the power 2i alpha*e to the power i alpha d rho=root pi/2. And then you put e to the power 2i alpha as cos 2 alpha+i sin 2 alpha and you put here for e to the power i alpha, you put cos alpha+i sin alpha. So we get this, okay.

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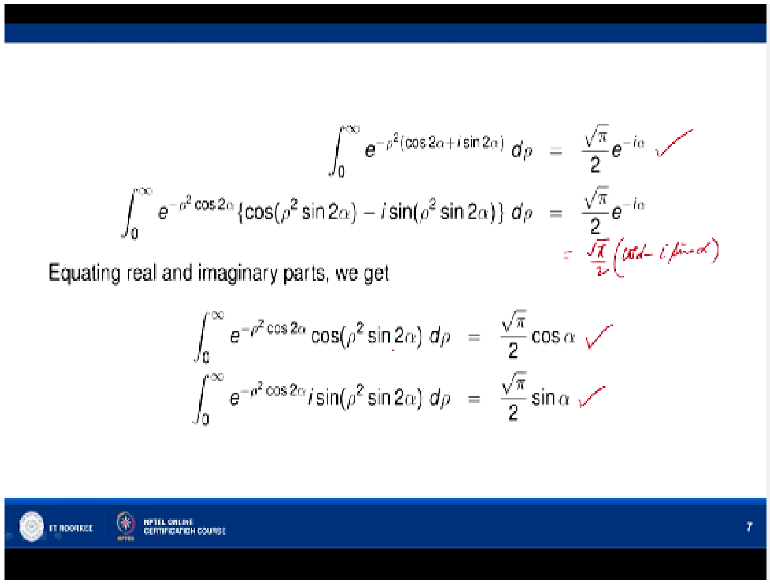

So e to the power i alpha we bring to the right side and then integral 0 to infinity e to the power -rho square cos 2alpha+i sin 2alpha d rho=root pi/2 e to the power -i alpha. This e to the power i alpha which was here, comes here because it is independent of rho. So then 0 to infinity e to the power -rho square cos 2 alpha*e to the power -i rho square sin 2alpha, that is cos rho square sin 2alpha-i sin rho square sin 2 alpha d rho=this.

Now let us equate here real and imaginary parts. This is root pi/2 cos alpha-i sin alpha. So then integral 0 to infinity e to the power -rho square cos 2alpha cos rho square sin 2alpha d rho=root pi/2 cos alpha and 0 to infinity e to the power -rho square cos 2 alpha sin rho square sin 2alpha d rho=root pi/2 sin alpha. So this proves part A and B and as I said, part C can be obtained from here by taking alpha=pi/4, okay.

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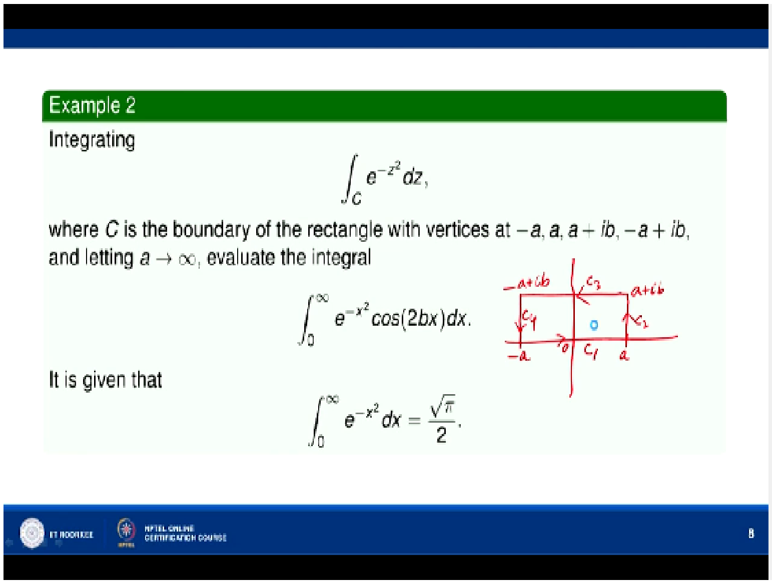

So we get part C. Now integral over C e to the power -z square dz, where C is the boundary of the rectangle with vertices at -a, a, a+ib, -a+ib and letting a tends to infinity, let us evaluate this integral. We are given that this is equal root pi/2. So you can see here we have this contour. This is -a, 0. Let us take a here and then this is a+ib and we have -a+ib here, okay. So let us take this segment -a to a as C1. This segment as a to a+ib as C2. From a+ib to -a+ib is C3 and -a+ib to -a as C4, okay.

So our contour C consists of 4 parts, okay. The 4 parts are C1, C2, C3, C4, okay. So we will evaluate the integral over C e to the power -z square dz along all these 4 parts, okay to evaluate the integral over C e to the power -z square dz which is equal to 0 because we see is a simple closed curve and e to the power -z square is analytic in the whole complex plane. So it is analytic inside and on this contour C, okay. So let us find this integral.

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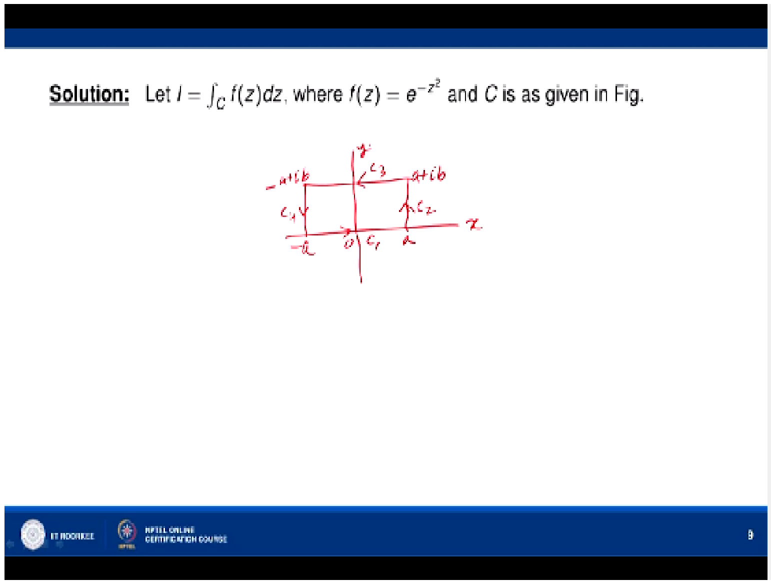

So integral over C fzdz where fz=e to the power -z square and C is given here, C as I draw there. Let us draw again here, okay. So what we do is.

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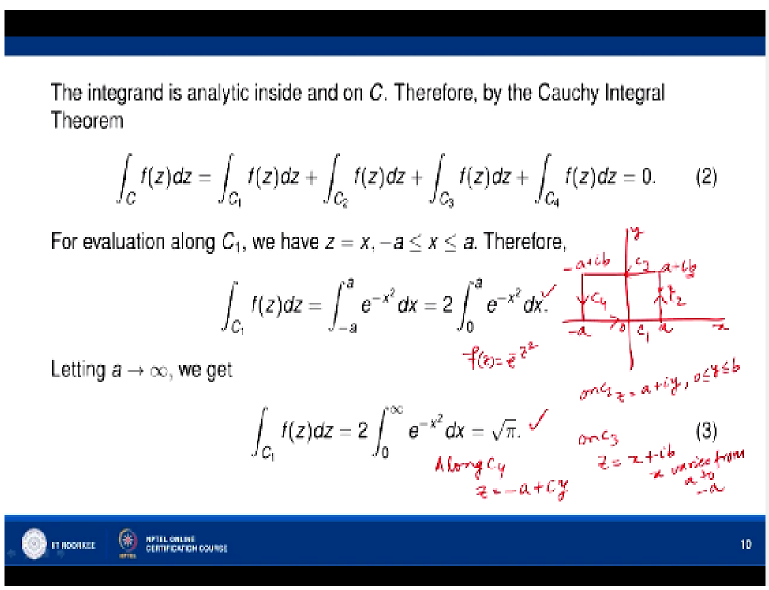

The integrand is analytic inside and on C. Therefore, by Cauchy integral theorem, integral over C fzdz=integral over C1+integral over C2+integral over C3+integral over C4=0, okay. Now for evaluation along C1, okay. Let us draw here now. Now you can see. When we want to integrate along C1, the line segment from -a to a lies on the real axis. Therefore, for z, we shall write x, okay and fz is e to the power -z square.

So we now write x for z. So e to the power -x square dx and x varies from -a to a. Now e to the power -x square is an even function of x. So it is 2 times 0 to a e to the power -x square dx. And when a goes to infinity, the limit of this is 2*integral 0 to infinity e to the power -x square dx. So when a goes to infinity, this becomes 2*root pi/2, that is root pi, okay. Now let us integrate along C2.

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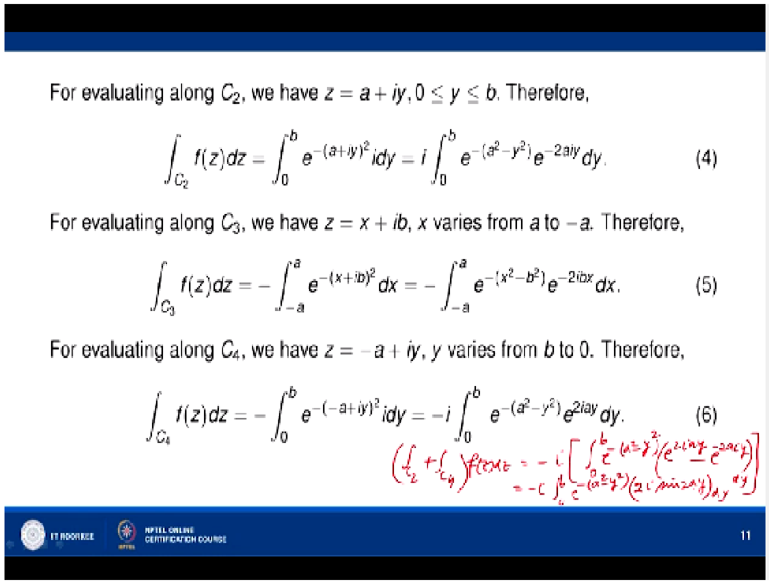

So for evaluating along C2, now let us look at this. Along C2, x remains constant. x=a, y varies from 0 to b. So we can take any point z here. The z can be written as a+iy, okay, where y varies from 0 to b along C2, okay. So on C2, we have this. So let us use this z=a+ib, y varies from 0 to b, then integral over C2 fzdz=integral over 0 to b, fz is e to the power -z square. So e to the power -a+iy whole square. dz is idy, okay.

So idy we put and then e to the power -a+iy whole square is a square-y square+2iay, so we get e to the power -a square-y square*e to the power -2aiy and then dy. This i we have kept outside, okay. For evaluating along C3, now let us look at C3. Along C3, you notice that this y is fixed, okay, y=b, okay. x varies, x varies from a to -a, okay. So on C3, we will write z as x+ib, okay, where x varies from a to -a, okay.

So you put x+ib, x varies from a to -a. So along C3, we have, now from a to -a, we can make it from -a to a, but we have to put negative sign here. So -integral over -a to a e to the power -z square. So e to the power -x+ib whole square and dz=dx. So we put dx here. So -integral over -a to a, you can now square this, so then x square-b square and then we get e to the power -2ibxdx. And when we integrate along C4, along C4 you notice again that y is fixed, okay, no y varies, x is fixed. x=-a, okay.

So along C4, z=-a+iy where y varies from b to 0, okay. So we get z=-a+iy, y varies from b to 0 and therefore, integral over C4 fzdz=-integral over 0 to b. It was from b to 0 when we want to make it 0 to b, we put negative sign. So -0 to b e to the power -z square, z is -a+iy whole square. dz=idy, okay. So this is -i and then integral 0 to b e to the power -a square-y square then e to the power 2iaydy, okay.

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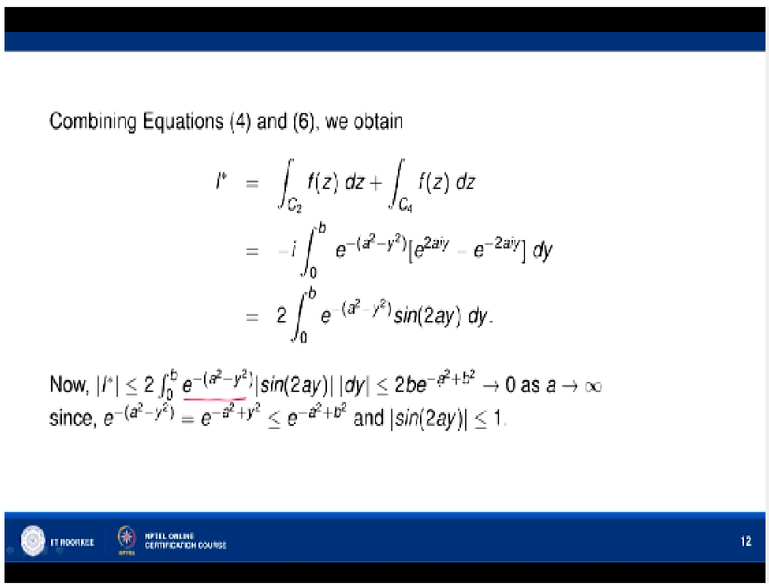

Now let us see combine equations 4 and 6, okay. 4 is this one, okay. 4 is this and 6 is this. When you combine C2 and C4, what do you get? You notice that e to the power -ia square-y square and this is same here, okay. So let me combine integral over C2+integral over C4 fzdz, this gives you -i*integral over 0 to b e to the power -a square-y square and e to the power 2iay and then here -e to the power -2aiydy, okay.

So this is -i*integral over 0 to b e to the power -a square-y square. And this 2i*sin 2ay, okay. So dy. So this is what we put here, okay. This is 2*, I square becomes -1, so 2* 0 to b e to the power -a square-y square sin 2aydy. Now let us estimate the value of i*. So mod of i* is less than or equal to 2*0 to b e to the power -a square-y square. This is positive quantity. Then mod of sin

2ay, mod of sin theta is less than or equal to 1.

So this is less than or equal to 2*b*e to the power -a square+b square, okay. Now how we get this? You can see here. Sin this quantity e to the power -a square-y square is e to the power -a square+y square, okay. So e to the power y y varies from 0 to b and e to the power y square is an increasing function. So e to the power y square will be less than or equal to e to the power b square. So this quantity is less than or equal to for -a square+b square.

And therefore, this is less than or equal to e power -a square+b square*2b, okay. So 2b e to the power -a square+b square. Now when a goes to infinity, okay, e to the power -a square goes to 0. So this quantity goes to 0 as a goes to infinity. So C2+C4, okay, when a goes to infinity go to 0. Now what is left? We are left with integral along C1. Integral along C1 we have already found. It is root pi, okay. So integral along C3, okay. C3 is this, okay. Let us concentrate on this.

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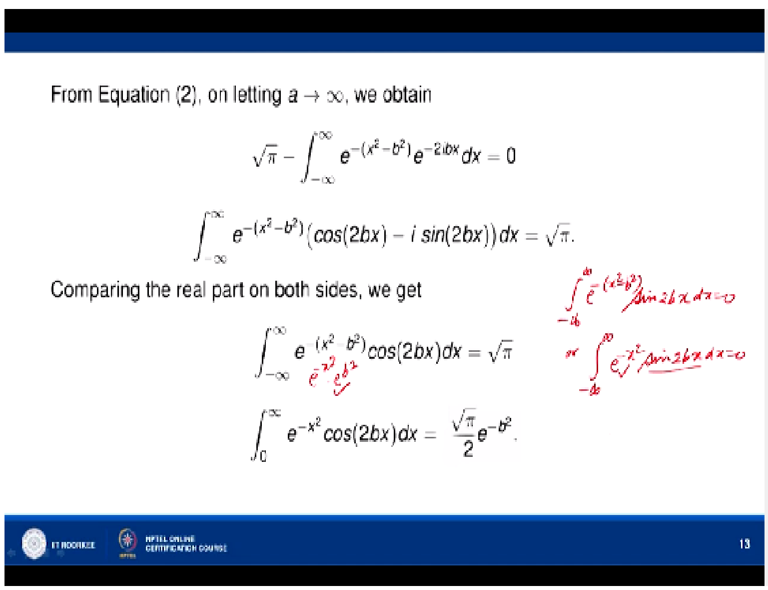

So we get now root pi-integral, okay. When a goes to infinity, this becomes -infinity to infinity, this quantity, okay. So root pi-this, okay. So root pi-integral over-infinity to infinity e to the power -x square-y square e to the power -2ibxdx=0 and we can put it as cos 2bx-i sin 2bx. So -infinity to infinity e to the power -x square-y square cos 2bx-I sin 2bxdx=root pi.

Now comparing real part on both sides because we want to evaluate the value of integral over

-infinity to infinity e to the power -x square-y square cos 2bx. We ultimately want to get this. So we just need to evaluate the, compare the real part. So -infinity to infinity e to the power -x square-b square cos 2bxdx=root pi, okay. And these are real quantities. So if you, imaginary part if you look at, the imaginary part will have value 0, integral over -infinity to infinity e to the power -x square-b square sin 2bxdx.

This will be equal to 0, okay. Or we can say integral over -infinity to infinity e to the power -x square sin 2bxdx=0. You can divide this equation by e to the power b square, okay. And this has to happen because this is an even function, e to the power -x square is an even function. Sin 2bx is an odd function. So its value over -infinity to infinity has to be 0. So that is verified. Now this is equal to root pi.

So we can now write it as e to the power -x square*e to the power b square. This is the power b square we can take to the other side and this is root pi*e to the power -y square now. e to the power -x square cos 2bx is an even function. So we can make it 2* 0 to infinity, okay. So 0 to infinity e to the power -x square cos 2bxdx is root pi*e to the power -b square/2, okay. So with that I come to the end of this lecture. Thank you very much for your attention.