Advanced Engineering Mathematics
Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology- Roorkee

Lecture – 22
Evaluation of Real Integrals Using Residues - II

Hello friends, welcome to my lecture on evaluation of real integrals using residues. We will consider some more cases where the real integrals we evaluated using the residue method let us first consider the real integrals in improper real integrals of the form integral over –infinity to infinity fx cos sxdx. and integral over –infinity to infinity.

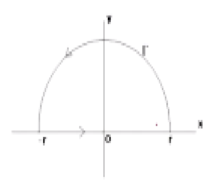
(Refer Slide Time: 00:53)

Improper Real Integrals of the Form $\int_{-\infty}^{\infty} f(x)\cos sx\,dx$ and $\int_{-\infty}^{\infty} f(x)\cos sx\,dx$

Such type of real integrals occur in the connection with the fourier integral. If $f(x)$ is a rational function satisfying the conditions of the previous article then we may consider the corresponding counter integral

$$\int_C f(z)e^{isz}\,dz \quad (s>0)$$

over the contour C.



Fx cos sxdx you might have seen such type of real integrals occurring connection with the Fourier integral if fx is the rational function satisfying the conditions of the previous article in the previous article when we evaluated the integral of fx/- infinity to infinity interval we had considered fx to be a rational function of x that is fx= px/qx where qx does not vanish for any x and the degree of qx is at least 2 units are when the degree of px.

So, we assume the same conditions here on the function fx so if fx is a rational function satisfying those conditions then we may consider the corresponding contour integral, so we are considering the contour integral /c fz to the power i fzdz where s is positive now contour c consist of the semi-circle with the center of the origin and radius r which we denote by gamma and then it consist of the line segment from – r to r along the x axis.

So, c consists of 2 parts 1 is the semi-circle with the center at d origin and radius r which we denote by gamma and the other part is integral other part is the line segment from –r to r along the real axis so let us consider the integral/c fz e 3 power isz dz and c to be is contour.

(Refer Slide Time: 02:36)

Then

$$\int_C f(z)e^{isz}\,dz = \int_\Gamma e^{isz} f(z)\,dz + \int_{-r}^{r} e^{isx} f(x)\,dx$$

where r is taken to be so large that all the poles of $f(z)$ in the upper half of the z-plane lie inside C.

By residue theorem

$$\int_C f(z)e^{isz}\,dz = 2\pi i \sum Res\,\{f(z)e^{isz}\}$$

Now,

$$\int_{-r}^{r} e^{isx} f(x)\,dx = 2\pi i \sum Res\,\{f(z)e^{isz}\} - \int_\Gamma e^{isz} f(z)\,dz$$

And what we notice is that we can write integral/c fz e to the power isz dz = integral/gamma e to the fz e to the power isz dz and integral/ -r to r but when we go from –r to r we are moving along the x axis so the imaginary part of z is 0 that is by 0 and we can then take z=x and so while moving along the line segment from –r to r we put x for z so integral/ -r to r e to the power isz fxdx.

Now let us take because r is at our choice the radius of the semi-circle at our choice so let us choose r to be so large that all the poles of fz which lie in the upper half plane come within the contour c so c encloses all the poles of fz that lie in the upper half plane. Let us take r to be sole r by the residue theorem the integral/c fz e to the power isz dz will be 2Pi i times sigma residue of fz e to the power isz where sigma residue fz e power isz means.

We find the residue of fz e to the power isz at all the poles that lie in the upper half plane and take their sum and multiply by 2Pi i to get the value of integral / c fz e to the power isz dz now integral/-r to r e to the power isx fxdx and then can be written as 2Pi i sigma residue fz e to the power isz – integral/ gamma e to the power isz fz dz.

(Refer Slide Time: 04:23)

Let $r \to \infty$, then we shall show that

$$\int_{\Gamma} e^{isz} f(z)\, dz \to 0$$

Hence,

$$\int_{-\infty}^{\infty} f(x) \cos sx\, dx = -2\pi \sum \Im Res\left\{f(z)e^{isz}\right\}$$

and

$$\int_{-\infty}^{\infty} f(x) \sin sx\, dx = 2\pi \sum \Re Res\left\{f(z)e^{isz}\right\}$$

$$\int_{-\infty}^{\infty} f(x) e^{isx} dx = 2\pi i \sum Res\left(e^{isz} f(z)\right)$$

$$\int_{-\infty}^{\infty} f(x)(\cos sx + i \sin sx)\, dx = 2\pi i\,(\alpha + i\beta)$$

where $\alpha = Re \sum Res\left(e^{isz} f(z)\right)$, $\beta = Im \sum Res\left(e^{isz} f(z)\right)$

$$\int_{-\infty}^{\infty} f(x) \cos sx\, dx = -2\pi\beta$$

$$\int_{-\infty}^{\infty} f(x) \sin sx\, dx = 2\pi\alpha$$

Now let us take r to go to infinity then we shall show that integral/ gamma e to the power isz fz dz goes to 0 and once we get this what will happen then integral/- infinity to infinity e to the power is x fxdx will be = 2Pi i times sigma residue fz e to the power isz dz so we will have integral /- infinity to infinity fx e to the power is x dx= 2Pi i times sigma residue of e to the power isz*fz.

So, what we will do now let us we may write this further integral /-infinity to infinity fx e to the power is x we can write it as cos as x + i sin sx dx = 2Pi i times okay now let us say sigma residue is r to the power isz fz is some complex number say alpha + i beta where alpha is real part of sigma residue of e to the power isz*fz and beta = imaginary part of sigma residue e to the power isz*fz then what will happen when we get real imaginary parts.

We will have integral/ - infinity to infinity fx cos sxdx = now this is 2Pi i* alpha i then 2Pi i* I beta will be – 2 Pi beta so this will be -2Pi beta because on the right side real part – 2Pi beta and integral/- infinity to infinity fx sin sxdx = 2Pi alpha so what we will have integral/- infinity to infinity fx cos sxdx= - 2Pi times imaginary part of sigma residue of fz e to the power isz whether we write imaginary part here or we write imaginary part here the same thing.

Because we are taking summation later on so sigma residue imaginary part of residue of fz e power isz that is beta. So that is multiplied by -2Pi n here we have integral/ - infinity to infinity fx sinsxdx 2Pi times real part of this summation r summation of the real parts sigma is the real part of residue of fz e to the power isz for every z which lies which is a pole and lies in the upper half plane. Now let us show that integral/ gamma e to the power isz fzdz goes to 0 or goes to infinity.

(Refer Slide Time: 07:54)

For sufficiently large r_0 such that $|z| = r > r_0$,

$$\begin{aligned} |f(z)e^{isz}| &= |f(z)||e^{is(x+iy)}| \\ &= |f(z)|e^{-sy} \leq |f(z)| < \frac{M}{r^d}, \end{aligned}$$

where $d \geq 2$ and M is any number greater than $|\frac{a_n}{b_m}|$. Thus

$$\begin{aligned} \left| \int_\Gamma f(z)e^{isz}\, dz \right| &< \frac{M}{r^d}.\pi r \\ &= \frac{\pi M}{r^{d-1}} \to 0, \end{aligned}$$

as $r \to \infty$.

$$\begin{aligned} &|e^{is(x+iy)}| \\ &= |e^{isx}.e^{-sy}| \\ &= |e^{isx}||e^{-sy}| \\ &= e^{-sy} \end{aligned}$$

$f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ & degree $q(x)$ - degree $p(x)$ $= d \geq 2$

Since $s > 0$ and $y > 0$ we have $e^{-sy} \leq 1$, on Γ

$$f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}$$

So, mod of fz e to the power isz = mod of fz and to the mod of e to the power isz = x+iy so x+iy we put here and then when you multiply mod of e to the power is*x+iy = mod of e to the power isx* e to the power –sy which = mod of e to the power isx* mod of e to the power –sy now mod of cos i mod of e to the power is x =1 okay as s is real and positive so this = e to the power –s/ because e to the power –sy which always >0.

So, we will have mod of fz *e to the power isz = mod of fz* e to the power –sy now our fx we have assumed to be =px/qx where qx is != 0 and degree of qx- degree of px let us take = d so than this d is >=2 now in our previous lecture we have seen that when fx satisfies this conditions than mod of fz is<m/r to the power d for sufficiently r 0 that mod of z = r r is >r0 so and e to the power -sy what happened to this e to the power –sy.

Since s >0 and y lies in the upper half plane and we are in the upper half plane estimating this mod of fz e to the power isz for this computing this integral estimating this integral and gamma lies in the upper half plane so if you take any point in the gamma then y > than 0 so what we will have since s> 0 and y>:0 we have e to the power – sy<=1 on gamma this is valid on gamma so we can place it for –sy/y 1 so this we have M/r to the power d where d>=2.

And m is any number we have seen> mod of an/bm where we have taken fx we have taken fz we have taken to the anz to the power n+an-1z to the power n -1 and so on a1z+a0 and we had taken denominator polynomial to be bm z to the power m+bm-1z to the power m-1 and so on =b1z+b0 then we had seen that if m-n=d and d >=2 then mod of fz<=m/r to the power d where m is any number > mod of an/bm.

So, mod of by Cauchy inequality let us apply Cauchy inequality so integral we have estimated integrant </ r to the power d * the length of gamma, so length of gamma is Pi r because gamma is semicircle of radius r so m/r to the power d Pi r so this Pi M/r to the power d-1 now d >= so d – 1>=1 and therefore it goes to 0 as r goes to infinity so this is how we prove that this integral around gamma goes to 0 when r goes to infinity.

(Refer Slide Time: 12:10)

Example 1

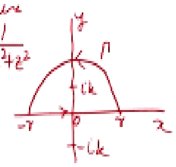
Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2}\, dx = \frac{\pi}{k} e^{-ks},$$

$$\int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2}\, dx = 0, \ (s > 0, \ k > 0).$$

Let us consider the contour integral $\int_C f(z)\, e^{isz} dz$, where $f(z) = \frac{1}{k^2+z^2}$

Then $\int_C f(z) e^{isz} dz = \int_\Gamma f(z) e^{isz} dz + \int_{-r}^{r} f(x)\, e^{isx} dx$



Let us take r to be so large that all the poles of f(z) in the upper half plane lie inside C.

$z^2 + k^2 = 0 \Rightarrow z = \pm ik$

The simple pole at $z = ik$ lies inside C.

$q(z) = k^2 + z^2$
$q'(z) = 2z$

$\text{Res}_{z=ik}\, f(z) e^{isz} = \text{Res}_{z=ik} \frac{1}{k^2+z^2} e^{isz} = \left[\frac{e^{isz}}{2z}\right]_{z=ik}$

Now let us evaluate integral / - infinity to infinity cosxsx/ ksquare+x square dx and simultaneously we will also be able to get value of the integral / - infinity to infinity sinsx/ksquare + x square dx now you can notice here that sinsx is an odd function ksquare + x square is an even function so sinsx/ ksquare + x square is an odd function of x and therefore it is value /- infinity to infinity interval must be 0.

So, we have to verify that by using residue calculus and here this is an even function cosx is an even function ksquare + x square is an even function so cos sx/k square+x square is an even function when we use the counter integral / cfz e power isz dz then you notice that we get the value of both the integrals simultaneously value of integral / -infinity to infinity gx cosxdx and integral/ - infinity to infinity fx sinxdx by equating real imaginary parts.

So, let us consider the corresponding contour integral here so let us consider the contour integral / c fz e to the power isz dz now let us notice that here fx is1/ksquare+x square so fx is a rational function of x where the denominator of the polynomial is up to degree 2 denominator polynomial is k square + x square so it is degree is 2nx and the numerator is constant 1 so it is of degree 0 the difference of the degree in the numerator and the denominator is 2.

Numerator and denominator exceeds by 2 units then the degree of the denominator the degree of the denominator is 2 units higher than the degree of the numerator so those conditions and moreover that the denominator polynomial k square + x square which we have denoted by qx is not 0 for any real value of x so we can apply the article which we have first now studied so let us consider the integral /cfz e power is zdz where your c is this.

Integral we are taking over this path c consist of semicircle gamma center at 0 radius r and the real x is the line segment on the real axis from –r to r so this is our c then we can split integral /cfz e to the power is zdz= the integral/ gamma fz e to the power isz dz+integral/- r to r fx e to the power is xdx here fz = replace x by z in the expression for fx is 1/ksquare + x square so this 1/ksquare + z square alright let us take r has to be so large.

Let us take r to be so large that all the poles of fz in the upper half plane lies inside c okay what are the poles of fz the fz has poles at the points where z square + k square = 0 so z square + ksquare = 0 gives you z = + - i*k we are given k to be positive so ik lies here –ik lies here so and moreover that denominator if you write qz=k square + z square then q dash z = 2z so q dash z is not 0 and z =ik and therefore qz has a simple pole at z = ik.

So, the simple pole at z = ik lies inside c let us find the residue of fz e to the power fz at z = ik so this is residue of fz is 1/k square + z square * e power isz at z = ik so we have e to the power isz denominator polynomial differentiate we get 2z and then you put z =ik so we get e to the power – sk/2ik so then integral over c.

(Refer Slide Time: 18:21)

By residue theorem $\int_C \frac{1}{k^2+z^2} e^{isz} dz = 2\pi i \frac{e^{-sk}}{2ik} = \frac{\pi e^{-sk}}{k}$

Thus taking $r \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{1}{k^2+x^2} e^{isx} dx = \frac{\pi e^{-sk}}{k} - \lim_{r\to\infty} \int_\Gamma \frac{e^{isz}dz}{k^2+z^2} \quad \text{— (2)}$$

Let us prove that $\lim_{r\to\infty}\int_\Gamma \frac{e^{isz}dz}{k^2+z^2} = 0$

$$\left|\frac{e^{isz}}{k^2+z^2}\right| = \left|\frac{e^{is(x+iy)}}{k^2+z^2}\right| = \left|\frac{e^{isx}e^{-sy}}{k^2+z^2}\right| \le \frac{1}{|z|^2-k^2} e^{-sy} = \frac{1}{r^2-k^2}e^{-sy}$$

hence by Cauchy's inequality $\le \frac{1}{r^2-k^2}$ since $e^{-sy} < 1$

$$\left|\int_\Gamma \frac{e^{isz}dz}{k^2+z^2}\right| \le \frac{\pi r}{r^2-k^2} \to 0, \text{ as } r\to\infty$$

Therefore (2) $\Rightarrow \int_{-\infty}^{\infty}\frac{1}{k^2+x^2}e^{isx}dx = \frac{\pi e^{-sk}}{k}$

Equating real and imaginary parts by putting $e^{isx} = \cos sx + i\sin sx$

$\int^{\infty} \cos sx\, dx = \pi e^{-sk}$ $\quad \int^{\infty} \sin sx\, dx = 0$

By residue theorem integral /c1/ksquare + z square e to the power is zdz this will be = 2 Pi i times e to the power – sk/2i k so this cancels and this and we get Pi times e to the power –sk/k so we have found the value of the integral /c fz dz. Now let us go back and see what we have to do now next so let us take the limit let me call it as equation number 1 let us take r to go to infinity then this will be this we have already evaluated.

So, this will come integral /-infinity to infinity fx e to the power is xdx so that will be = to the value of the residue here multiplied by 2Pi i we get the value of this integral – integral /gamma so we have thus taking r to go to infinity we have integral / - infinity to infinity 1/k square + x square e to the power is x dx = Pi times e to the power –sk/k-limit r tends to infinity integral /gamma e to the power isz/k square + z square dz.

So, we can so show in the article let us prove that limit r tends to infinity integral / gamma e to the power is zdz/k square +z square = 0 so mod of e to the power isz/k square +z square let us first calculate this so this is mod of e to the power is*x +iy/ksquare+z square so this = mod of e to the power is x* e to the power –sy/ksquare+ z square e to the power is x modules of e power sx = 1 e to the power –sy>0 so it will not be affected by mod.

And then this is <=2//mod of z square- ksquare * e to the power –sy so this is 1/r square –k square * e to the power –sy so e to the power –sy let us replace /1 this <=1 so we can write 1/r square – k square since e to the power –sy<1 gamma lies in the upper half plane now hence by Cauchys inequality modules of integral / gamma e to the power is zdz/ k square + z square <=1/ r square – k square * length of gamma which is Pi r, so this goes to 0 as r goes to infinity.

Thus from this goes to 0 this becomes 0 so therefore let me call it 2 so therefore 2 implies integral/- infinity to infinity 1/ksquare + x square e to the power is xdx=Pi e to the power –sk/k now let us put e to the power is x cossx+ is sin x and then equator imaginary parts both sides on the right side you see is the real value so imaginary part is 0 so equating real and imaginary parts by putting e to the power is x = cos sx+ i sinx.

So, we get integral/- infinity to infinity cos sx/k square + x square dx=Pie to the power-sk/k while integral /-infinity to infinity sinsxdx/ksquare+ x square= 0 so this is how we do this problem, so this is the solution to this problem.

(Refer Slide Time: 24:41)

Improper Integrals with Singular Points on the Real Axis

Another kind of improper integral is a definite integral

$$I = \int_a^b f(x)\,dx$$

whose integrand becomes infinite at a point c in the integral of integration i.e.

$$\lim_{x \to c} |f(x)| = \infty.$$

Then we express I as

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\eta \to 0} \int_{c+\eta}^b f(x)\,dx.$$

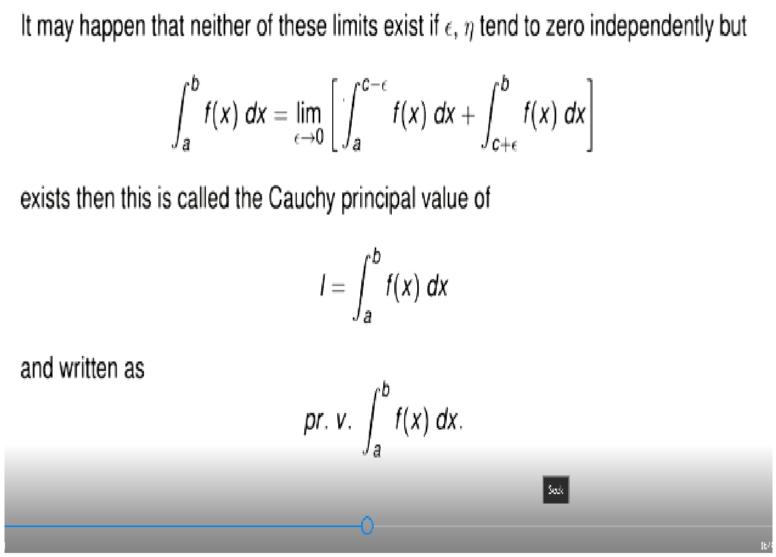
where both ϵ and η tend to zero independently through positive values.

And now let us go to next article so here we consider improper integrals where we have similar points on the real axis in the previous article, we consider those improper integrals where the integral does not have any similar point on the real axis. It does not become 0 the denominator we had taken fx=px/qx and we had assumed that qx is not 0 for any real axis. So, we considered all those integrals where similarities of the integrant do not lie on the real axis.

Here we consider an improper integral where similarities will lie on the real axis. So, another kind of improper integral is a definite integral I- integral a to b fx dx whose integrand. Becomes infinite at a point c in the integral of the integration. Here we will see those kind of integrals where there is a point in the integral of integration at which the integrand becomes infinite. That is limit of mod of fx and it extends to C infinite and C lies between a and b.

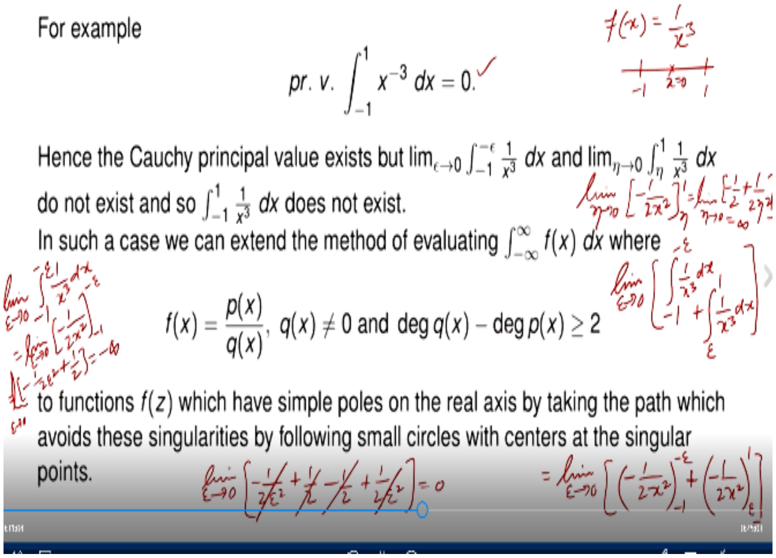
So, when we express i as integral over a to b affect dx limit epsilon tend to 0 a to integral over. So, here you see this is your integral a and b okay and c lies in between a and b. So, we integrate over a to c -epsilon this is c-epsilon integral over a to c- epsilon limit epsilon goes to 0 and then c+eta to b okay integral/c+beta to b fx dx where eta goes to 0, both epsilon and eta both tend to 0 through positive values independently.

(Refer Slide Time: 26:31)



Okay now it may happen that neither of these limits adjust this limit or this limit or both of them may not exist. But limit of integral/a to c-epsilon fx dx +integral/c+eplsion b fx dx as epsilon tends to 0 exists. So, then this limit is called the Cauchy principal value of integral a to b fx dx and we write it as principal value of a to b fx dx. So, for example let us consider this example.

(Refer Slide Time: 27:06)



You see if we take fx=1/x cube and the integral of integration we take as -1 to 1 then at the point x=0 you see fx becomes infinite. So, now let us see that the Cauchy principal value exits here so Cauchy principal exits means integral over limit epsilon tends to 0 integral/-1 to c-epsilon c is 0 here, so -epsilon 1/x cube dx+we have then integral/c+eplison so c+c 0 so we get epsilon 2 1 here 1/x cube dx okay.

Let us see whether this limit exists so this is =limit epsilon tends to 0 we have here -1/x to the power -3 means when you integrate you get x to the power-2/-2 so -/2x square and then you put the limits -1 -epsilon + we have here same thing -1/2 x square and you put the limits epsilon n1 and what do we notice this is limit epsilon tends to 0 let us put the upper limit so we get -1/2 epsilon square and then we get +1/2 okay when you put x=-1 and here you get what.

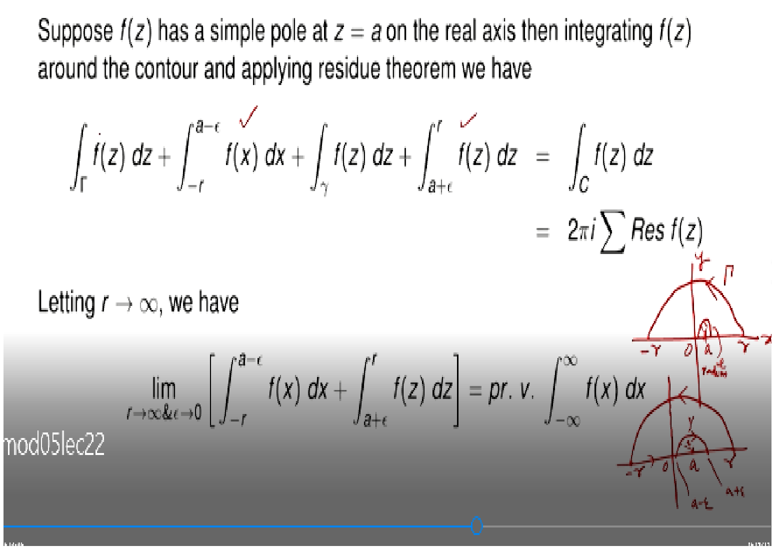
-1/2 and then you get 1/2 epsilon square so this cancels with this this this cancels with this and what we get is 0 okay. So, the Cauchy principal value of this integral exits and this is =0. But you see the limits epsilon tends to 0-12 -epsilon /x cube dx this limit okay this limit and tis limit separately you find they do not exists. Let us see that limit epsilon tends to 0-1 to – epsilon 1/x cube dx this will be = when you put the limits you will get -1/2 epsilon square+1/2 okay.

And you take the limit as epsilon tends to 0 okay you see that the limit when epsilon tends to 0 through positive values this gives you -infinity okay similarly when you find limit eta tends to 0 eta to 1 what do you get limit eta tends to 0 you get -1/2 x square eta to 1 okay so limit eta tends to 0 and you get -1/2+1/2 eta square and this becomes +infinity okay so both these limits separately do not exist.

While the Cauchy principal value exists and is=0 so we will be considering Cauchy principal value here in such a case we can extend now we will extend the method of evaluating integral /-infinity to infinity fxdx where fx=px/qx where qx is not to =0 and degree qx-degree of px>=0 this we have already seen we will extend the method of evaluating such kind of integrals to functions fz which have simple poles on the real axis.

By taking the path which avoids these similarities by following small circles with centers at the singular points. Now let us see an example of this.

(Refer Slide Time: 31:27)



We will first consider the general case suppose fz has a simple pole at z=n okay so let us consider the contour this is gamma suppose I have a simple pole here at z=a suppose fz has a simple pole at z=a and the real axis then integrating fz around the contour C is now this this we have this is semi-circle at center a and radius is epsilon okay radius of this semicircle this radius is=epsilon okay integral/gamma.

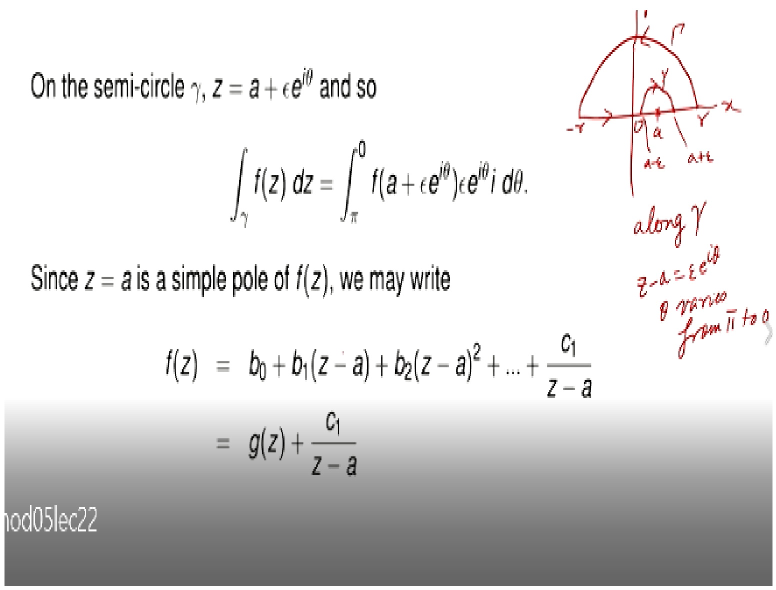
When you consider integral/c fz dz, it will be =integral/gamma fz dz+ integral from -r to now this is what you see here. Let me write it separately -r r this is your point a so I have made a small circle here of radius epsilon okay and here is your origin so -r to a -epsilon this point is a-epsilon this point is a+epsilon okay because epsilon is the radius. So, what do we do we have this figure okay from here we are moving like this so r to a-epsilon.

We integrate along the x axis, so we write fxdx then integral/gamma this is integral/gamma this semicircle. So, integral/gamma fdz+integral from a+epsilon to r again along x axis we write integral/a+epsilon to r fx dx this is =integral/c fzdz. So, you can see by drawing a small circle with sufficiently small radius with center at the point a we can avoid the similarly at point a okay.

Now letting r go to infinity let us take r to go to infinity then n epsilon go to 0 we will have this will become this goes to 0. Integral/gamma fz dz goes to 0 integral/-r to a-epsilon fxdx+integral/a+epslion to r fxdx that gives us the Cauchy principal value of integral/-infinity fxdx. This value is 2 pi i * sum of residues of fz that lie in the upper half plane and integral/gamma fzdz we shall see how to evaluate separately okay.

So, when taking r to go to infinity and epsilon to go to 0 this becomes Cauchy principal value. So, this part and this part tends to Cauchy principal value of integral/-infinity to infinity fxdx this we have evaluated by using residue theorem. This will go to 0 and this we will see integral/gamma fzdz how do we evaluate okay so everything will be known and then we will get the Cauchy principal value of integral/-infinity to infinity fxdx.

(Refer Slide Time: 34:58)

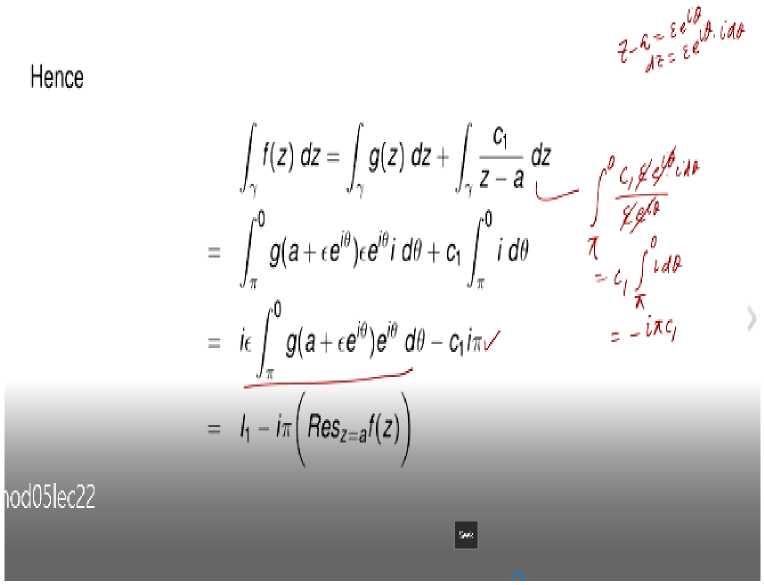


Now let us see here how we evaluate the integral along gamma okay so this is your figure this is integral/gamma this is r this is r this is x axis this is y axis this is a-epsilon point, and this is a+epslion and gamma because center is at a and radius is epsilon along gamma z=epsilon e e to the power I theta okay and since we are moving clockwise along gamma theta varies from pi to 0 okay so along gamma, we can write z=because center is at a.

So it will be not z it will be z-a okay z-a will be = epsilon e to the power I theta or we can say z= a+eplsion I theta. Si, integral/gamma fz dz will be integral/ pi to 0 because theta varies from pi to 0fz f a+epslion ei theta and dz is z= a+epslion e I theta gives dz= epsilon e to the power t theta * I d theta so this is what you get. Now z=a is a simple pole of fz so by Laurent series expansion fz can be written as b0+b1 z-a +b2 z-a square.

And so on and the principal part of fz contains only one term that is c1/z-a where c1 is non-zero. Okay c1 is the residue of fz at z=a. Now this part which contains the positive integral powers of z-a non-negative integral powers of z-a that constitutes a power series which center at z=a so that we can denote by gz where gz is analytic in some neighborhood of z=a so fz can be written as gz+c1/z-a

(Refer Slide Time: 37:17)



And then integral/gamma let us put integral/gamma let us put in place of f a+e i theta f a+epslion e i theta let us put this gz+c1/z-a so we will get integral/gamma gz this integral/gamma fz dz can be written as integral/gamma dz gz +integral/gamma c1/z-a dz now z is a+eplison e i theta dz is epsilon e i theta i d theta so pi 2 0 we get this then c 1 times dz is now let me write this integral/gamma integral/pi to 0 c1.

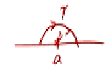
We have z-a=epsilon e to the power i theta. So, we get epsilon e to the power i theta dz becomes epsilon e to the power i theta * id theta so epsilon e to the power theta*i d theta and we cancel this out and we get c1 times integral/pi to 0 I d theta which is -i pi*c1 okay so -i pi c1 this is I epsilon we can write outside integral pi to 0 g a+epslion ei theta ei theta d theta or I can write it as let us call this as i1 this I call as i1 i1-I pi *residue of fz at z=a.

This we have seen here c1 is residue of fz =a so I can write i pi times residue of fz=a and then let us evaluate the value of i1.

(Refer Slide Time: 39:10)

Now,

$|g(a+\epsilon e^{i\theta})| \leq M$



$$|I_1| = \left| i\epsilon \int_{\pi}^{0} g(a+\epsilon e^{i\theta}) e^{i\theta}\, d\theta \right| \leq \epsilon M \pi,$$

where $|g(a+\epsilon e^{i\theta})| \leq M$ on γ. It follows that $I_1 \to 0$, as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon \to 0} \int_{\gamma} f(z)\, dz = -\pi i \, Res_{z=a} f(z)$$

mod05lec22

Now mod of i1 =mod of i epsilon*integral/ pi to 0 g a+eplsion e i theta d theta okay so mod of this quantity. Now gz is an analytic function in some neighborhood of a okay therefore integral/pi to 0 g a+epslion e i theta d theta this can be evaluated by Cauchy mod of g a+epslion e i theta this is >=m okay you see we have a semi-circle okay whose radius is epsilon is very small okay and dz is analytic at z=a.

So, some neighborhood of z=a this function is continuous and therefore this gamma we can write that mod of g a+epslion e i theta<=m it is bounded in that neighborhood and now so this is epsilon this epsilon we have here and then m for this and then e to the power i theta mod of integral and then integral/ pi 0 d theta will be -pi mod of that will be pi. So, epsilon M pi we have and.

So, when epsilon goes to 0 i1 goes to 0 and therefore limit epsilon goes to 0 f gamma limit epsilon goes to 0 integral/gamma fz dz that is =-i epsilon *-i pi *residue of fz at z=a. so, this is what we get this is how we evaluate the integral along the semi-circle gamma of small radius so this value we shall put here for this expressions we this is =0 when r tends to infinity this integral.

And this integral gives us the Cauchy principal value of integral/-infinity to infinity fxdx this we know by residue theorem and this we have calculated okay. So, we have put the value here okay and then calculate the Cauchy principal value.

(Refer Slide Time: 41:25)

Further $\int_\Gamma f(z)\,dz \to 0$, as $r \to \infty$ because $f(z) = \frac{p(z)}{q(z)}$ is a rational function and $\deg q(z) - \deg p(z) \geq 2$. Thus

$$pr.\,v. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum Res\,(f(z)) + \pi i \left(Res_{z=a}\, f(z) \right).$$

If $f(z)$ has finite number of simple poles on the real axis then repeating the above argument for each pole we have

$$pr.\,v. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum Res\,(f(z)) + \pi i \sum Res^*$$

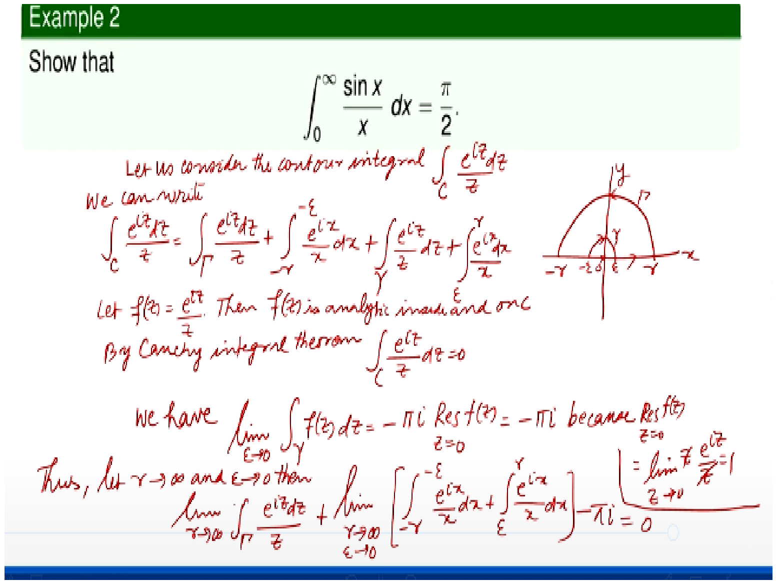
where $\sum Res^*$ is the sum of residues of $f(z)$ at its poles on the real axis.

And further we will explain this so since this goes to 0 as r goes to infinity because fz =pz/qz is a rational function and degree qz-degree of pz is >=2 we will get the same thing degree of qz-degree of pz>=2. Okay thus principal value of integral/-infinity to infinity fx dx =2 pi i sigma residue of fz +pi i because on the left integral/gamma was -pi i c1 so +pi i * residue of fz at z=a/.

Now if fz has a finite number of poles we have taken the case where fz has a simple pole at z=a if there are more than 1 pole okay then we will find the residues at all the poles. Then we if fz has a finite number of simple poles on the real axis then repeating the above argument for each pole we will have here this expression this one will change or replace by pi I times sigma residue star sigma residue star is the term of the residues of fz at its poles on the real axis.

We showed this formula for a case of a simple pole at z=a but there can be one such similarity on the real axis. So, we collect the we find out the residue at all the similarities on the real axis take their sum and multiply by pi i. So this is what we do to evaluate the Cauchy principal value of the integral/-infinity to infinity fxdx.

(Refer Slide Time: 43:07)



Now let us consider the case of this integral integral/0 to infinity sin x/x dx =pi/2 now this integral you can see is not of the type which we have discussed now okay because here the difference is not that of 2 units. Of course sin if you when you write the corresponding contour integral here e to the power I z/z then z=0 which lies on the real axis the integrand will have a pole a simple pole.

But the other condition regarding integral/gamma of fz dz to go to 0 that is not satisfied here. So, it is a another case another example where we are going to find the Cauchy principal value, we are going to find the integral of sin x /x over the interval 0 to infinity. So, what we do let us consider the corresponding contour integral and contour is this okay C is the contour we can write integral/C e to the power iz dz/z=integral/this is gamma.

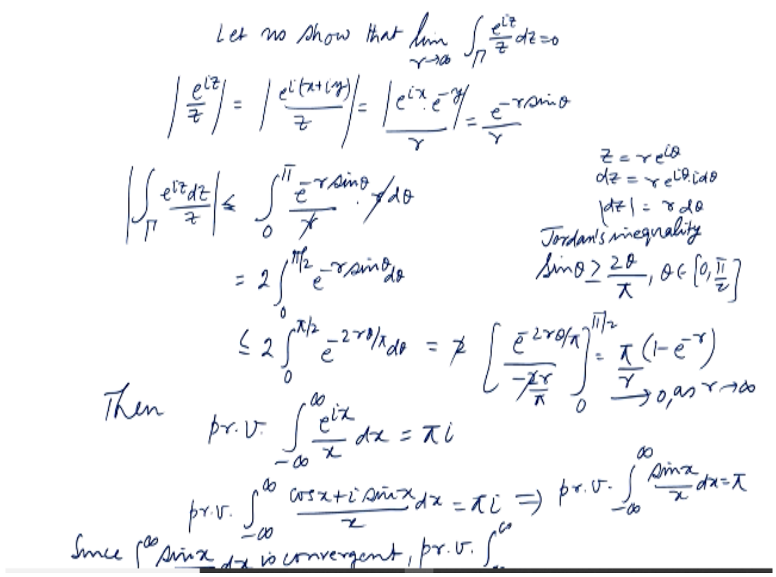
This is small gamma integral/gamma e to the power izdz/z+integral /-r to -epsilon because e to the power iz/ has a similarity z=0 so -r to -epsilon e to the power ix/x dx+integral/gamma e to the power iz e to the power iz/z dz +integral/epsilon to r. Okay now let fz=e to the power iz/z okay e to the power iz/z does not have any similarly in the contour okay c. So, by Cauchy integral theorem if you assume fz to be iz/z and fz is analytic inside and on C.

So, by Cauchy integral theorem integral/c e to the power iz/z=0 let us recall the article from the article let us evaluate the integral/gamma limit epsilon tends to 0 integral /gamma fz dz is – pi i residue at z=a fz so let us follow that. So, we have limit epsilon tends to 0 integral/gamma fzdz =-pi*i residue of fz at z=0 okay and that is =-pi i because residue of fz at z=0is limit z tends to 0 z times fz fz is e to the power iz/z.

So, this is =1 okay this is – pi I so thus let r go to infinity and epsilon tends to 0 okay when we will get, we can show that integral/gamma e to the power iz dz/z goes to 0 as r goes to infinity okay so we will have limit tends to infinity integral/gamma okay e to the power iz dz/z +limit epsilon tends to 0, r tends to infinity we have integral/-r to -epsilon e to eth power ix/x dx +integral /epsilon to r e to the power ix/ dx.

This should be here because we are moving along the real axis. So it should be x so this and integral/gamma we got -pi I this is =0 okay this is what we get okay first we shall show that this value=0.

(Refer Slide Time: 49:30)



So, let us show this is =0 okay so mod e to the power iz/z along gamma we have to find so e to the power i x+iy/z and this is mod of e to the power ix* e to the power -y/r because along gamma mod z=r. So, mod of e for i=1 this is e to the power -r sin theta /r okay now integral/gamma e to the power iz dz /z=integral 0 to pi mod of this is <=e to the power -r sin theta/r and the mod of dz dz=r e to the power i theta so dz is r e to the power i theta* id theta.

So, dz mod =rd theta. So, we get r d theta here, so this r cancels now let us notice that by the property definite integral sin pi-theta=sin theta so this becomes 0 to pi/2 times 0 to pi.2 e to the power r-r sin theta d theta. Now there is an important result sin theta >=2 theta /pi when theta belongs to 0 to pi/2 this is called as Jordons inequality it is very easy to prove we can prove it by calculus jordons inequality.

So, this is now <=2 times 0 to pi/2 e to the power -2r theta/pi d theta and now we can integrate easily. So, e to the power -2 r theta /pi-2r/pi so this is = pi times this 2 will cancel with this 2 and pi /r we get 1- limits are 0 to pi/2 so we get 1- e to the power -r. Now if r goes to infinity e to the power -r goes to 0 and pi /r goes to 0. So, 0*1-0 so which goes to 0 s r goes to infinity. Okay this is how we prove this =0.

Now what we have this becomes 0 so we will get this =pi i okay this is nothing but Cauchy principal value Cauchy principal value = pi i so we get the following. So, then principal value of integral/-infinity to infinity e to the power i x/x dx=pi I okay now you can put here cos x +i sin x so integral /-infinity to infinity cos x+i sin x /xdx. Okay so the principal value of this =pi i. This means that if you get real imaginary parts.

Then this is principal value of integral / -infinity to infinity sin x/x dx=pi okay now we know that integral/-infinity to infinity is sin x/x okay is a convergent integral it is a convergent integral therefore Cauchy principal value of integral/-infinity to infinity sin x/x dx is same as integral/-infinity to infinity sin x dx. So, since okay this is convergent okay, we have cos principal value=integral/-infinity to infinity sin x/x dx=pi.

Now sin x/x dx is an even function integral/-infinity to infinity sin x/x dx =2 times okay. So, since sin x/x is an even function we get integral 0 to infinity sin x/x dx= pi/2 this is how we evaluate the integral. With this we come to the end of this lecture thank you very much for your attention.