Advanced Engineering Mathematics
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Lecture – 20
Meromorphic Functions

Hello friends welcome to my lecture on Meromorphic functions and analytic function whose only similarities in the finite complex plan or poles are called meromorphic function.

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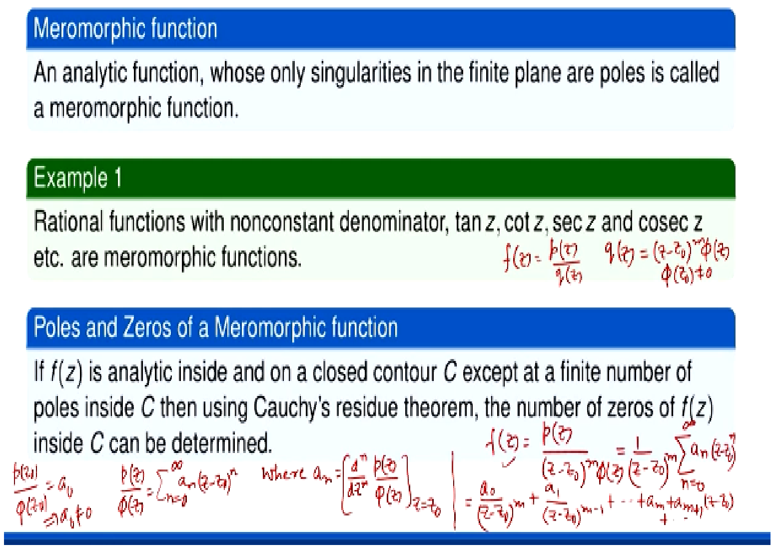

For example, we can consider rational functions with a non-constant denominator you know that the restaurant function is a quotient of 2 polynomials say suppose fz is a rational function then we can write fz=pz/qz and the 0s of qz are then the polynomials q0 of qz are then the poles of fz so we can easily prove that if qz has a 0 of order m at z=z0 then we can write qz as -z0 to the power m* some function phi z where phi z0 is not=0.

And phiz is analytic in some small neighbourhood of z0. So putting the value of qz we shall have fz=pz/z-z0 to the power m *phi z. Now pz/qz pz/phi z is an analytic function at z0. So, we can write ts as 1/z-zo power m* in the sense pz/phi z is analytic and that =z0 we can expand it but by the teller series sigma n= 0 to infinity say an z-z0 to the power n. Okay where you know that were n= nth derivative of pz/phi z at z=z0.

Okay now this is = if you write this expand this series then you can see a0 z-0 to the power 0 so we shall have 1/ a0/z-z0 to the power m then a1 z-z0 to the power m-1 and so on.. When you n=m then you will have am and then you will have am+1 z-z0 and so on. Okay now we can see here that a0 is not=0 because pz0/phi z0 is not=0 pz/phi z is =sigma n=0 to infinity am z-z0 to the power n.

So, pz/phi z0 is=a0 okay now pz0/phi z 0 is not =0 so this implies that a0 is not =0 okay so this means that now this is series of the function fz okay since a0is not =0 fz has a pole of order m fz=z0 o if qz has 0 of order m at z=z0 then fz has a pole of order m at z=z0. Similarly, tan z we know =sin z/cos z. SO, the 0s of cos z become the poles of tan z ad cot z is cos z/sin z the 0s of sinz at the 0s of sin z we have the poles of cot z.

Similarly, sec z sec z is 1/cos z the 0s of cos z gives us the poles of sec z and cosec z is 1/sin z the 0s of sin z give us the poles of fz. So, it gives us the poles cosec z so rational functions with a non-constant denominator tan z cot z sec z and cosec z are examples of meromorphic functions. Now let us look at a very useful result on meromorphic functions. It is regarding poles and 0s of meromorphic function.

If fz is analytic inside and on a closed contour C except a finite number of poles inside C and then using Cauchys residue theorem, we can determine the numbers of 0s of fz inside C let us see how we do that.

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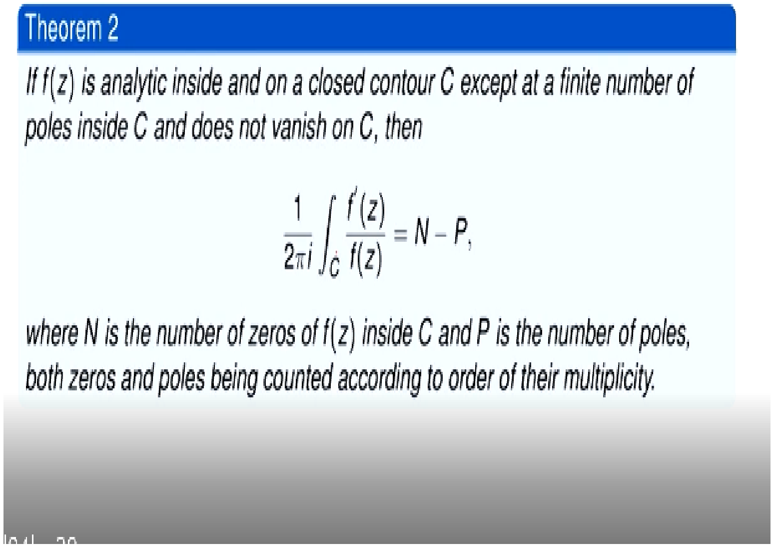

Suppose fz is analytic inside and on a closed contour C except at a finite number of poles inside C and does not vanish on C, Okay you can see here this formula 1/2 pi I integral /c f prime z/fz == N-P. So, here fz cannot be 0 on c, if fz takes value on C then integral/C prime fz/fz will not defined. So, except so then here N is the number of 0s of fz inside C and P is the number of also inside C. Both 0s and poles are being counted according to the order of their multiplicity.

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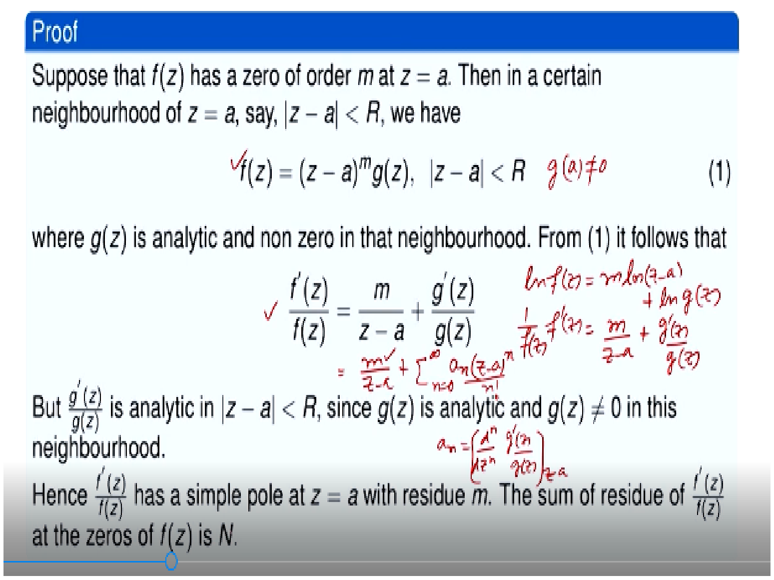

Now suppose that so let us prove this suppose that fz has a 0 of order m at z=a then in a certain neighbourhood of z=a then same order z- <= R we can write fz as z-a to the power m*gz mod of z-a<R where gz is an analytic function in the neighbourhood mod of z-a < R and gz gz does not become 0 at z=a that is ga is non-zero. And since gz is analytic in this neighbourhood it is continuous I the neighbourhood okay.

So, ga0=0 implies that we can get a sufficiently small neighbourhood of a z=a in which gs is not 0 throughout the neighbourhood. So, mod of z-a <r we can take the neighbourhood where gz is not =0 throughout the neighbourhood R can be taken sufficient small. Now from here it follows that if you take ln both sides this equation gives you ln fz =m times ln z-a+ln gz. Now differentiating with respect to z what we get 1/fz *f prime z =m/z-a+g prime z/gz that is.

We get this equation. So, now g prime z/ gz is analytic in model -a<r because gz is not 0 throughout this neighbourhood mod of z-a<r and gz is analytic. So g prime z/gz is analytic in this neighbourhood and hence f prime z/fz is such but we can do since g prime z/gz is analytic in mod of z-a<r okay we can expand g prime z/gz by a Taylor series. So, a actually this is nothing but m/z -a+sigma n= 0 to infinity an z-a to the power n.

Okay again were ans are dn/dzn g prime z/g z at z=a/n factorial. So, you can write n factorial here also here when I did this I wrote an=this so here also we can write an z-z0 to the power n/ n factorial. So, what we can do now f prime z/fz is now this is nothing but Laurent series of f prime z/fz. And the principle part of f prime z/fz contains only the term in negative powers of z-a m/z-a. Okay the co efficient of 1/z-a gives the residue of f prime z/fz.

So, we can say that residue of f prime z/fz at z=a which is a simple pole of f prime z/fz =m okay. So, f prime z/fz has a simple pole fz=a with residue m > now m bars the order of 0 of fz and we have seen that f prime z/fz has a pole at z=a with residue m. So, we can say that the sum of residues of f prime z at the 0s of fz is =N. If you recall N is the number of 0s of fz where the 0s are being counted in according to their odd of their multiplicity.

So, here o at z=a occurs m times so it will be so it will be counted as m 0s at z=a and at z=a we get a pole of f prime z/fz with residue m. So, we can say that the residue of f prime z sum of residue of f prime zfz at the 0s of fz is =n.

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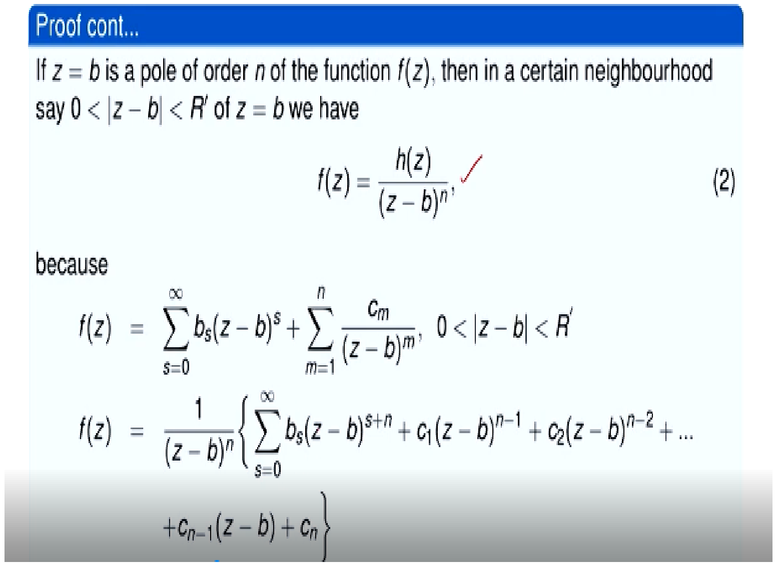

Now if z =b is a pole of order n of the function fz then in a certain neighbourhood say 0<mod of z-b <R dash of z=v we can write fz =hz/z-b to the power n here is the proof. You can say that z=b is a pole of order n of the function fz so fz can be written as sigma s=0 to infinity b as z-b to the power s m=sigma m=1to n cm/z-b power m o <mod of z-b<R dash Now this can be further written as fz= we can write 1/z-b to the power outside.

Then inside the bracket we shall have sigma s=0 to infinity bs z -b to the power s+n then we will have c1 times z-b to the power n-1 c2 times z-b to the power n-2 + and so on+cn-1 z-b +cn. Now at z=b this expression inside the bracket we have cn and cn is not =0 because fz has a pole of order n at z=b. So, we can write this expression inside the curve bracket as hz. Then we shall have fz= hz/z-b to the power n.

And this fz is a analytic function because it is represented by a power series. So, hz is represented by the power series therefore it is a analytic function and also that hb is not =0.

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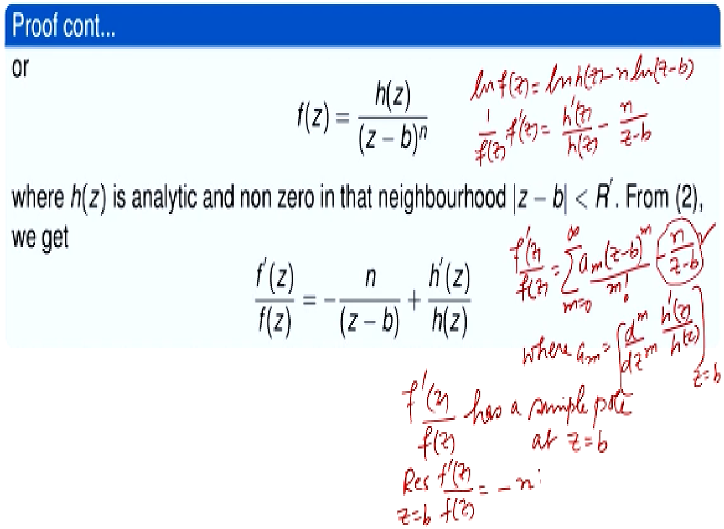

So, hz=fz fzhz/z-b power n where hz is analytic and non 0. So, we can take this R dash to show a small z inside the region mod z-b< R dash h ash is not =0 throughout. Because hz is analytic and hb is not =0 so we can find a small neighbourhood of z=b in which hz is not=0 throughout. So, let us take R dash to show a small that it is not 0 throughout this disc mod of z-b < R dash and then we take logarithm of this.

And logarithm arises in case of the 0 at z=a okay here also we can see that after taking logarithm both sides and differentiating with respect to z we shall arrive at this expression. So, you can see ln fz =ln hz-n times ln z-b then we differentiate with respect to z then 1/fz * f prime z =h dash z/hz -n/z-b okay. Now h prime z/hz is an analytic function in this neighbourhood mod of z-b <R dash n not=0 throughout this neighbourhood.

hz is not =0 throughout this neighbourhood. So, h prime z/hz can be expanded in a Taylor series and we shall write f prime z/fz= sigma n=0 to infinity okay some constants an *z-b to the power n/ I think we should write here m m=0 to infinity z-b power m /m factorial n- n/z-b so where am=dm/dzm h prime z /hz at z=b. Okay so this series is now the Laurent series of f prime z/ fz and this part is the principle part of f prime z/fz.

Okay so we can say that f prime z/fz has a pole simple pole at z=b and the residue of f prime z/fz at z=b is=-n okay and you can see that n is the order of the pole okay n is the order of the pole and z=b okay. So, f prime /fz has a pole at z=b with residue -b okay. So, we can say that this -P denote the so here P is the number of poles and therefore we can say that they are counted according to their multiplicity.

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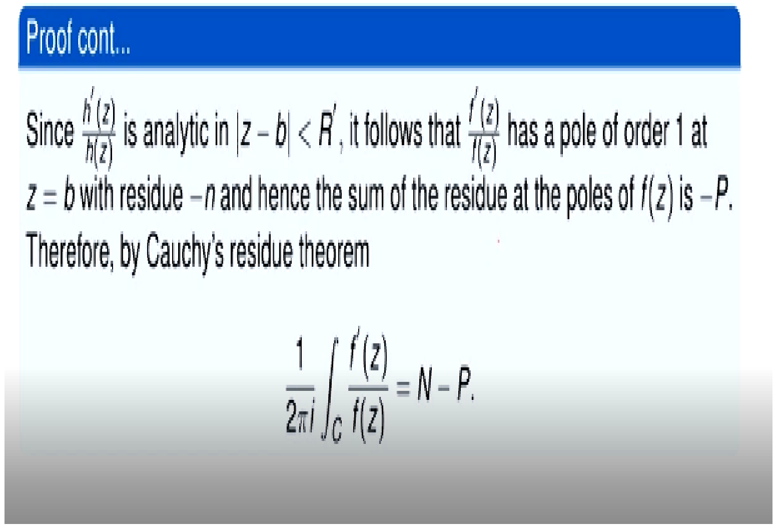

Okay so we can say that this -P is the sum of the residues of the sum of the residues at the poles of f prime z/fz sum of the residues at the poles of fz is -P. Okay because fz had a pole at z =b okay of order n so sum of the residues at the poles of fz is -P and therefore by Rouches residue theorem what do we say 1/2 pi I integral/C f prime z/fz is sum of the residues at the poles of f prime z/fz inside C okay. So, this is = N-P this proves the theorem. So, now let us now go to Argument principle.

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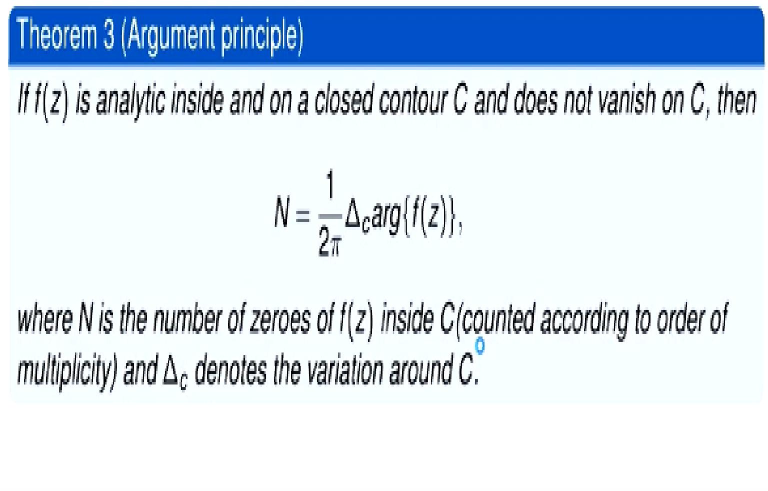

Which depends on this previous theorem if fz is analytic inside and on a closed contour C and fz does not vanish on the contour C, then n =1/2pi delta c argument of fz where n is the number of 0s of fz inside C counted according to order of multiplicity and delta c argument of fz denotes the variation of argument of fz around C.

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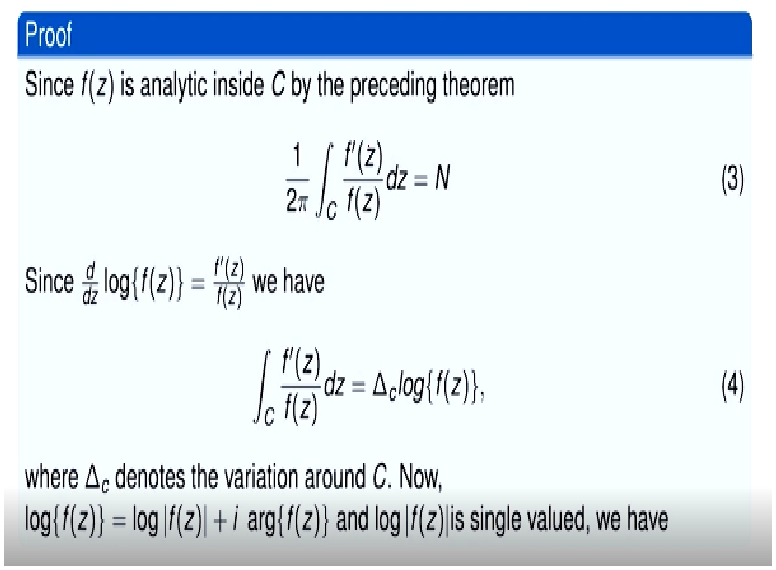

So, let us approve this is fz is analytic inside C okay we have this here we have given that fz is analytic inside and on C. So, since fz is analytic inside C by the preceding theorem 1/2 pi integral /C f prime z/fz dz will be =0 because f dash will not have similarities inside C. So, p will be 0 so this is =n so d/dz log of fz okay d/dz log of fz here you can see fz we are assuming that fz is not= on C okay.

We are assuming that fz is not =0 on C so we can take log of fz so let us cosier d/dz log of fz this is f prime z/fz and so what we can say here we integrate this equation if you integrate this equation then integral/C f prime z/fz will be =variation of log fz round the curves C. Let delta C denote the variation around the curve C. Now log of mod of fz log of fz=log of mod of fz +I times argument of fz.

Now we know that log of mod of fz is singe valued function and therefore from this equation from 3 and 4 what do we notice delta c log of fz= I times delta c of argument of fz. Because log mod of fz is single valued and does not change around C.

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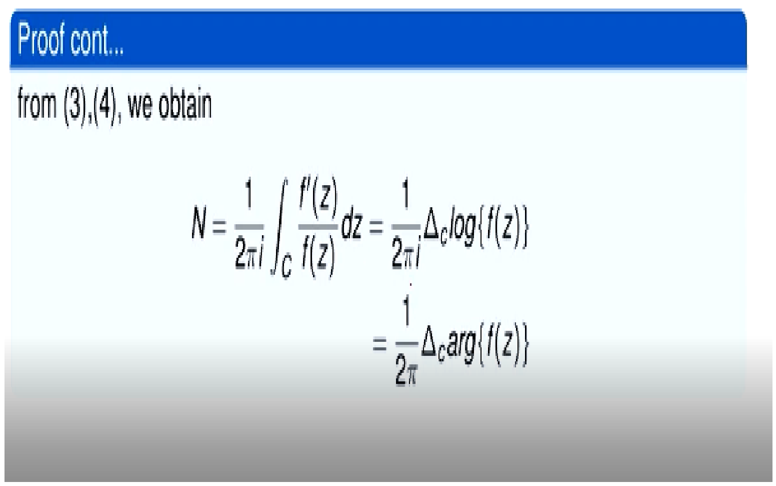

So, what we have and this will be =1/2 pi i integral/c f prime z/fz and this will be =1/2 pi i delta c log of fz and therefore delta c log of fz= I times delta C argument of fz o we shall have 1/2 pi delta c argument of fz. So, n=1/2 pi delta c argument of fz.

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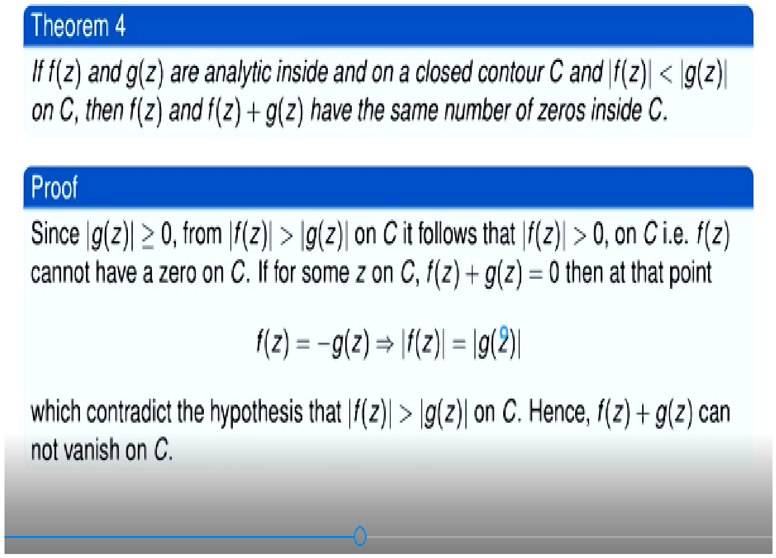

Now let us look at the Rouches theorem we can use to determine the number of 0s of fz the given function fz inside C. So, if f and gz are analytic inside and on a closed contour C and mod of fz < mod of gz on C then fz and fz+gz have the same number of 0s inside C. This is the theorem so here we can see here that mod of gz is the absolute value of gz so it is always > or =0 therefore from the condition mod of fz<gz it follow that mod of fz is always strictly >0.

And therefore fz cannot vanish on the curve C. Now let us also show that fz+gz also cannot be 0 on C. this we can easily show here suppose for some z on c fz+gz=0 then at that point kay fz=-gz and so mod of fz=mod of gz. But we have given that mod of fz is < mod of gz therefore fz+gz cannot be 0 at any point on the curve C. Now okay fz cannot be 0 on C and fz+gz cannot be 0 on C.

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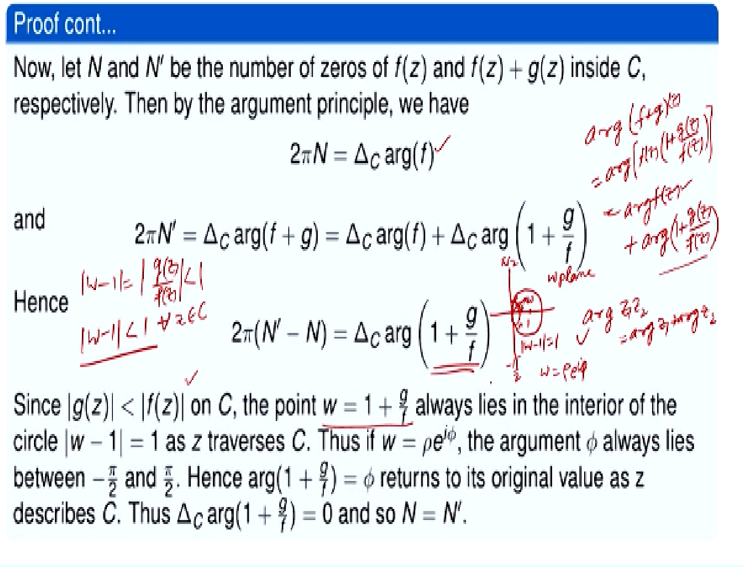

Proof cont...

Now, let N and N' be the number of zeros of $f(z)$ and $f(z)+g(z)$ inside C, respectively. Then by the argument principle, we have

$$2\pi N = \Delta_C \arg(f)$$

and

$$2\pi N' = \Delta_C \arg(f+g) = \Delta_C \arg(f) + \Delta_C \arg\left(1+\frac{g}{f}\right)$$

Hence

$$2\pi(N'-N) = \Delta_C \arg\left(1+\frac{g}{f}\right)$$

Since $|g(z)| < |f(z)|$ on C, the point $w = 1+\frac{g}{f}$ always lies in the interior of the circle $|w-1| = 1$ as z traverses C. Thus if $w = \rho e^{i\phi}$, the argument ϕ always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Hence $\arg(1+\frac{g}{f}) = \phi$ returns to its original value as z describes C. Thus $\Delta_C \arg(1+\frac{g}{f}) = 0$ and so $N = N'$.

Now let us consider that let us say that N and N dash denote the number of 0s of fz and fz+gz inside C respectively. Then by the argument principle by the theorem which we proved just now we will have 2 pi N= delta C argument of f okay because N denote the number of 0s of fz inside C. So, by the previous theorem 2 pi =delta C argument of fz and N dash denotes the number of 0s of fz+gz inside C.

So, 2 pi N dash =Deltas C argument to f+g now we know that argument of two complex numbers z1*z2 =argument of z1+argument of z2okay so by that argument of f+g can be written as argument of fz this fz+gz okay so argument of fz 1+gz/fz. So, using this result we have argument of fz +argument of 1+gz/fz. So, argument of delta c argument of f argument of f+g will be =deta c argument of fz+delta C argument of 1+gz/fz.

Now delta C argument of f=2 pi n so we can put the value here and then we shall have from this equation 2 pi n dash-n=delta C argument of 1+g/f. Now let us recall that we have given that mod of gz is < mod of fz on C. And therefore the point w=1+g/f always lies in the interior of the circle mod of w-1=1 s z moves on C. So, how does that happen we can consider this 1+g/f 1+gz/fz we are writing it as wz.

Okay so w= this gives you w-1 mod=mod of gz/fz okay mod f w-1 =mod of gz/mod of fz and mod of gz/mod of fzis < 1 so mod of gz /mod of fz is <1 we get mod of w-1<1 for all z belonging to C. So, as z moves on C okay w lies this w lies in the interior of the circle mod of w -1=1. Thus if you take w to be so let us draw this circle this is 1 in w plane and we have this circle this mod of w-1=1 so thus take w to be rho e i phi so the argument of w=phi.

And rho is the distance from the origin so thus w=rho e phi the argument phi now we can see that the argument of w varies between -pi/2 to pi/2 this is -pi/2 and here argument is pi/2 argument of wi is -pi/2 here it is -pi/2 so argument of pi lies between -pi/2 n+pi/2 and therefore argument of 1+g/f which is = pi because this we are denoting 1+gf we are denoting by w so argument if w is phi okay.

IT returns to its original value as z describes C okay you can see anywhere you take the argument if you go from any point this and okay come back to this point the argument will always be the same. So, the argument phi always lie between 1-i/2 and pi/2 and argument of 1+g/f that is argument of w which is =phi returns to its original value as z describes C okay you are moving from here to here.

Okay so when you come back to the original point the argument will return to its original value so thus delta C argument of 1+g/f there is no change in the argument of 1+g/f as z varies around C. So, it is =0 and therefore N=N dash.

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Example 5

Prove that every polynomial of degree n has n zeros.

Solution

Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n, \quad a_n \neq 0$$

be any polynomial of degree n. Let

$$f(z) = a_n z^n$$

and

$$g(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$$

Then, $f(z)$ has n zeros at the origin because $a_n \neq 0$.

Now let us also show that every polynomial of degree n has n 0s. Suppose if we take the polynomial pz=a0+a1z+a2z square+ so on anz to the power n where an is not=0 this be any polynomial of degree n. Okay so let us define fz to be=an z to the power n okay this is z we are taking and gz we take as a0+a1z+a2 z square +an-1 z to the power n-1 then you can see from the definition of z.

Fz= anz to the power n gives you if you out=0 n 0s of n 0s at z=0. Since an is non-zero so fz has n 0s at the origin because n is non 0.

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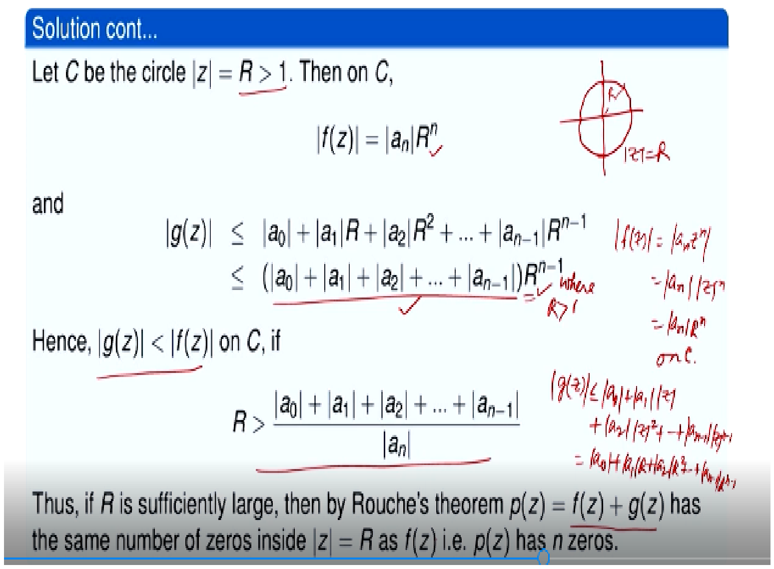

And let us take t to be the circle mod =z R where R is >1 let us take a circle mod z=R and let us take R to be >1 then on the circle C mod of fz= mod of an to the power n which is = mod of an *mod of z to the power n. So, mod of z=R on C so mod of an R to the power n on C. Okay mod of gz will be what mod of gz /triangularity<= mod of a0. Mod of gz is <= mod f a0+mod of a1 mod of z mod of a2 mod of z square and so on.

Mod of an-1 * mod of z to the power n-1 so this is mod of a0+ mod of a1*R+ mod of a2 R square and so on. Mod of an-1 R to the power n-1 okay so we will get this now mod of gz is < mod of fz okay so what we will get this quantity will be < this one. And therefore what we can do R will be > so using mod of gz < mod of fz and these estimates of mod of gz mod of fx here we arrive at r > mod of a0 mod of a1+mod of a2 mod of an-1/ mod of an okay.

So, if R is sufficiently large here you can see that while writing this inequality okay I have used the fact that R>1 when r >1 R square will be >R R cube will be >R okay, R to the power R square see when R is >1 then we can write R is <= Rn-1 R is <=Rn-1 and Rn- is <= Rn-1 so here R R square R to the power n-2 all can be replaced with R to the power n-1 so here we are using R >1.

Okay now with this estimate the estimate of od of fz we arrive at R> this okay. So, if you take R to be >1 and also greater than this value okay then by Rouche theorem pz=fz+gz. Okay as the same number of 0s inside mod of z=R as fz and fz has n zeros at z=0 so pz also has n 0s inside mod of z=R. You can see that z=0 z=0 lies inside mod of z=R it is the centre of mod of Z=R. So, mod of fz so fz and fz+gz have same number of 0s inside C.

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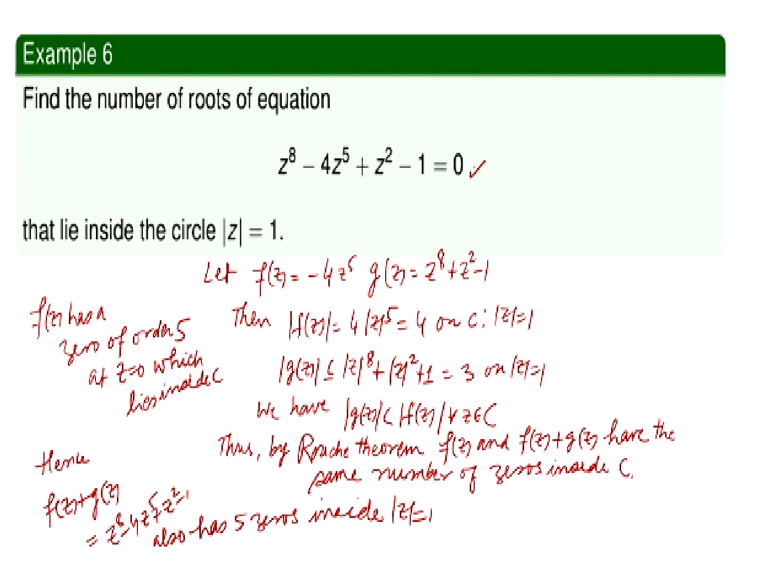

Now let us consider the equation z to the power 8-4 z to the power 5+z square-1=0 we will show that the 0s of this polynomial of degree 8 in z. There are 8 roots of this polynomial this 8 0s of this equation are the polynomial they lie inside mod of z=1. Okay so what we will do let us take let fz=-4 z to the power 5 and gz=z to the power 8+z square-1 okay then we can see that then mod of fz = 4 times mod of z to the power 5 okay which is =4 the curve mod of z=1.

Okay mod of gz mod of gz is <= mod of z to the power 8+mod of z square +1/ triangularity so this is =1+1 so 3 on mod of z=1. So we have mod of gz < mod of fz or all z belonging to the curve C okay and thus by Rouche theorem fz fz+gz have the same numbers of 0s okay inside C okay let us now notice that fz=-4 to the power 5 0s fz has 0 of order 5 at z=0 and therefore fz+gz also has 5 0s inside mod z=1.

So this z=0 lies inside which lies inside C and hence fz+gz which is = z to the power 8-4 z to the power r5 +z square – 1 also has 5 0s so thus fz+gz also has 5 0s inside mod z=1. So this means that there are 3 0s there are other 8 0s of this polynomial the other 3 0s lie outside mod z=1 okay.

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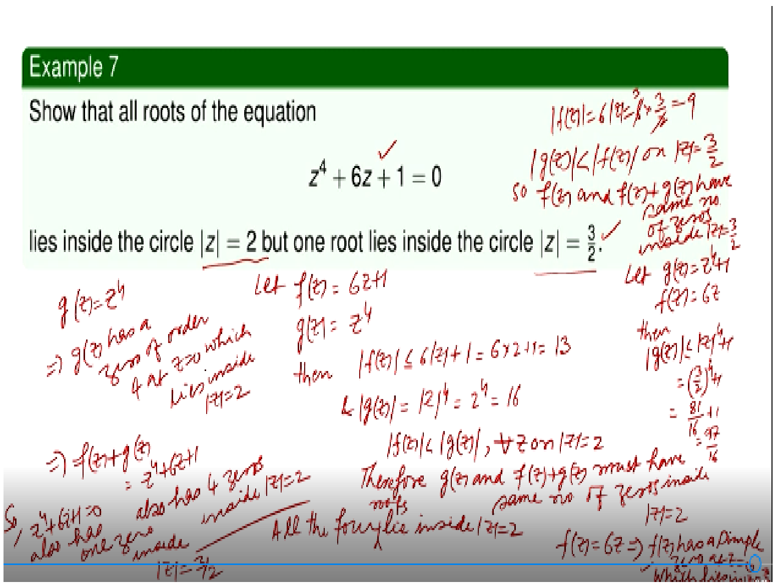

So, now let us go to next question suppose we have the equation z to the power 4+6Z+1=0 then let us find first where all the roots of this equation lie we see that all the four roots of this equation lie inside mod z=2 but if you consider this circle mod z=3/2 only 1 root lies inside mod z=3/2. So, let us take let fz=6z+1 okay let fz=6z+1 and gz=z to the power 4 okay then we see that mod of fz is <=6 times mod of z+1 okay.

So, this is =6 times 2+1 okay which is =13and od of gz =mod of z to the power 4=2 to the power 4 so we shall have 16 so mod of fz is <mod of gz. For all g on od gz= 2 and therefore gz and fz +gz must have same 0s same number of 0s inside mod z=2. Now let us look at the 0s of gz gz= z

to the power 4 implies that gz has a 0 of order 4 at z=0. Okay so fz+gz which lies inside mod z=2. So, gz has a 0 of order 4 and z=0 and z=0 lies inside mod z=2.

So, gz has 0s of order 4 inside mod z=2 and therefore fz +gz which is =fz+gz=z to the power 4+6Z+1 also has 4 0s inside modz=2. Now this is a fourth order equation this is a fourth order polynomial equation so it will have 4roots which means that all the 4 roots lie inside mod z=2. Okay let us take mod =3/2 so we will have to change now the choice of function fz gz in order to who that 1 root lies inside the circle mod z= 3/2.

Okay so let us take gz =z to the power 4+1 fz=6z okay what do we notice od of gz=<=mod of z to the power 4+1 which is =3/2 raised to the power 4+1 which is 81/16+1 which is =97/16 okay mod of gz an estimate of mod of gz is 97/16 on mod of z=3/2 and what about the function fz is 6z so od of fz= 6 times mod of z which is 6*3/2 so this means 9. Okay so we can see that mod of gz is <mod of fz on mod of z=3/2.

This means that gz and fz+gz have same number so 0 inside C. so, fz and fz+gz must have same number of 0sinside mod z=3/2. Now fz=6z okay fz=6z gives us fz has a simple 0 fz=0 okay which lies inside mod z=3/2 that is which lies inside mod z=3/2 okay so fz+gz which is z to the power +6z+1=0 which also has 1 0 inside mod z=3/2 okay. So, fz to the power 4+6z+1=0 has all the 4 roots inside mod z=2 but out of those 4 only 1 root lies inside mod z=3/2.

Because fz=6z fz has a simple 0 at z =0 and z=0 is a point inside mod z=3/2. So, fz has a simple 0 inside C and therefore fz+gz which is z to the power 4+z+1 also has 1 0 inside mod z=3/2. So, that is how we can decide the number of 0s inside the curve C using the Rouche's theorem. With that I would like to end my lecture. Thank you very much for your attention.