**Advanced Engineering Mathematics**
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**Lecture – 02**
**Cauchy-Riemann Equations**

Hello friends. Welcome to my lecture on Cache-Riemann equations. Suppose we have a function, complex function fz=uxy+ivxy which is differentiable at a point z=x+iy. Then this theorem says that at the point z, that is xy point, the first order partial derivatives of u and v exist, u is a function of xy, v is a function of xy.

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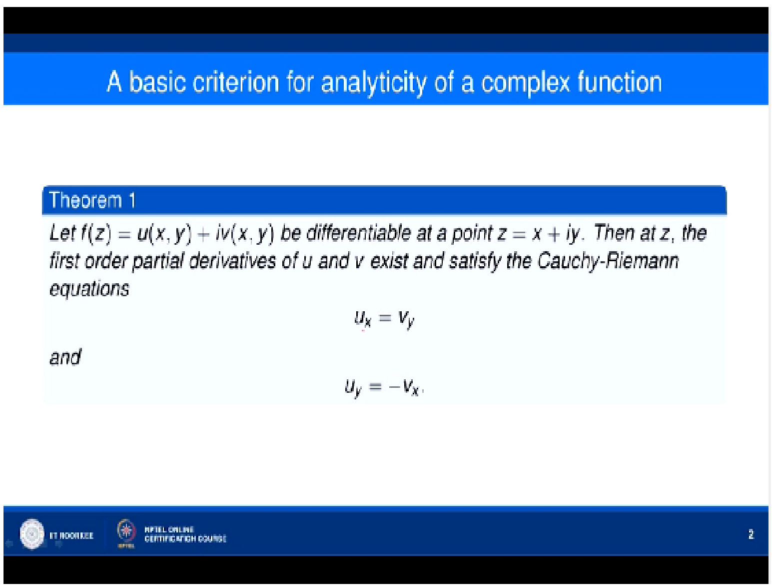


So their first order partial derivatives are uxuy vxvy. So they exist and they satisfy Cache-Riemann equations, that is ux=vy or we can say del u/del x=del v/del y and del u/del y=-del v/del x. So at each point z where the function fz is differentiable, the partial derivative uxuy vxvy exist and they are related by these equations. They are known as Cache-Riemann equations.

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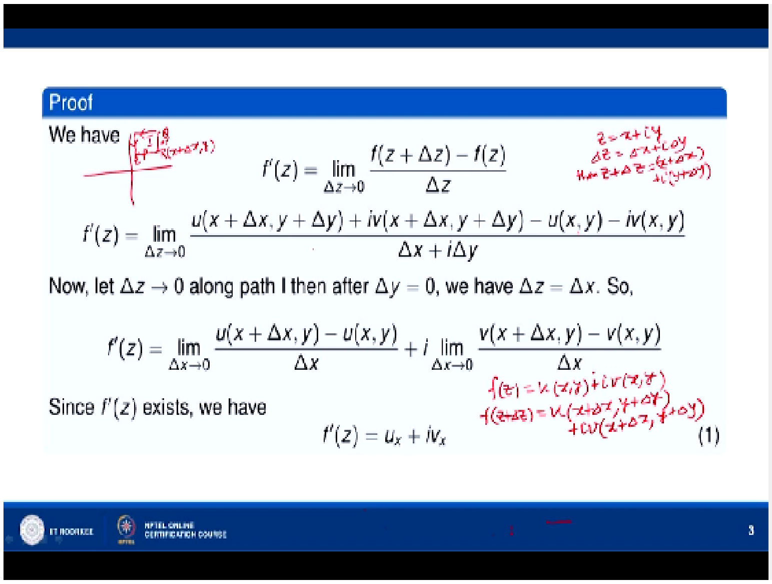


So we can say that whenever the function fz is differentiable at a point z, the Cache-Riemann equations are bound to be true, okay. That means they are necessary for the function to be differentiable at the point z. So we can let us prove this theorem. Since the function fz is differentiable at the point z, f prime z=limit z delta z tends to 0, fz+delta z-fz/delta z, okay. Now if z=x+iy and if z=x+iy, delta z=delta x+i delta y, then z+delta z=x+delta x+iy+delta y.

And so fz being uxy+ivxy will give us fz+delta z=ux+delta x, y+delta y+iv x+delta x, y+delta y, okay. So fz+delta z=ux+delta x, y+delta y+iv xdelta x, y+delta y-uxy-ivxy. And delta z is delta x+i delta y. Now let us take delta z to go to 0 along the path 1. As we had earlier discussed, here is P, here is Q, okay. We are moving parallel to y axis. After this delta y has become 0, this is your point R, this is x+delta x, y.

So after delta y has become 0, when we move towards P, delta x goes to 0. So from Q to P, we move along the path 1, okay, then when delta y has become 0, delta z becomes equal to delta x. So f prime z=limit delta x goes to 0 ux+delta xy-uxy/delta x, okay. We can rearrange the terms uxy we can subtract here. So this is minus this ux+delta xy-uxy/delta x+i*delta limit delta x goes to 0, vx+delta xy-vxy/delta x.

Now when delta x goes to 0, you see y has not changed. y remains fixed. There is only increment in x. So this gives us partial derivative of u with respect to x. And here same, this expression when delta x goes to 0, goes to partial derivative of v with respect to x. And therefore, f prime z becomes equal to ux+ivx. So along path 1 when we move, we see that f prime z=ux+ivx. We have assumed that function is differentiable at the point z.

So along whatever path we move to the point z, that is the point P, okay, the limit will have to be same, okay. So now let us go to the point P along path 2, okay. So when we do that, what happens?

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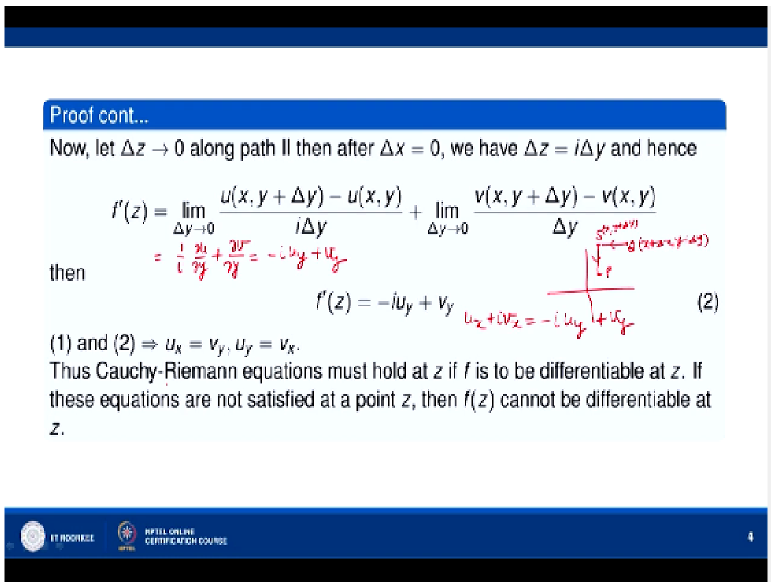


So when delta z goes to 0 along the path 2, this is x+delta x, y+delta y and this is your x, y+delta y point, okay. So we are moving in this, like this, okay. So after delta x has become 0, okay, when we are moving towards S from Q, okay, when we reach to the point S, what happens? Delta x has become 0. So when we, from S we move towards P, delta y goes to 0, okay. So after delta x has become 0, delta z becomes i delta y.

So when delta z goes to 0, delta y will go to 0. So f prime z will be the limit, delta y goes to 0, uxy+delta y, okay, because delta x has already become 0. So uxy+delta y-uxy/i delta y+limit delta y goes to 0 vxy+delta y-vxy/delta y. Now you can see here x has not changed, x remains fixed. Only there is a change in the value of y. So this gives you partial derivative of u with respect to y.

So 1/i*, this is equal to 1/i*partial derivative of u with respect to y and what we get there? Partial derivative of v with respect to y. But 1/i is -i. So -iuy+vy, okay. So along path 2, f prime z=-iuy+vy. Since f prime z exist, both the values must be same, okay. So we have ux+ivx, that is the value of f prime z along path 1. This must be equal to -iuy+vy, okay. Now equating real and imaginary parts.

Here real part is ux, here real part is vy. So ux=vy and imaginary part here is vx, here imaginary part is -uy. So uy must be equal to, vx=-uy or uy must be equal to -vx. So thus Cache-Riemann equations must hold at the point z if f is to be differentiable at z. If these equations are not satisfied at a point z, then fz cannot be differentiable at z, okay.

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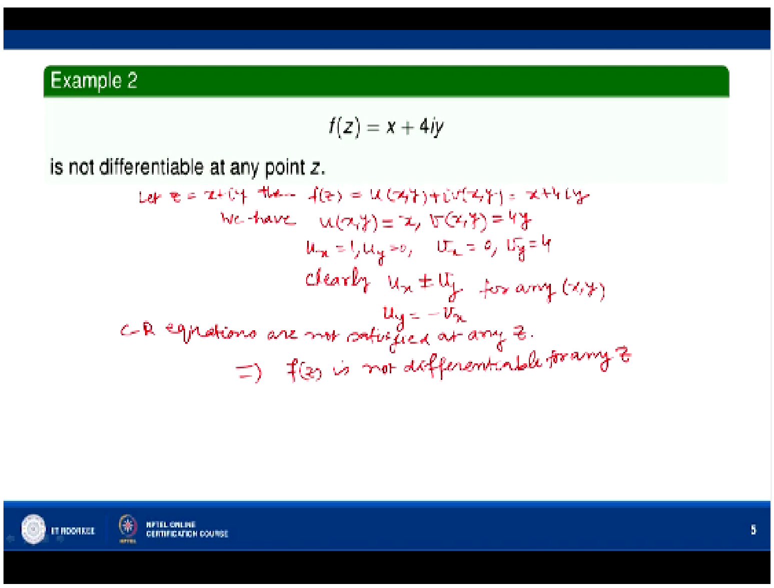


So let us take an example here, fz=x+4iy and so that it is not differentiable at any z, okay. So here you can see, let z=x+iy. Then fz=uxy+ivxy, u and v are real and imaginary parts of fz. Now we are given fz=x+4iy. So what we have? Equating real and imaginary parts, we have uxy=x, vxy=4y, okay. Now let us find the partial derivatives here. So partial derivative of u with respect to x that is ux=1 and partial derivative of u with respect to y is 0, okay.

Partial derivative of v with respect to x is 0 and partial derivative of v with respect to y is 4, okay. Now you can see here clearly ux is not equal to vy, okay. Because ux is 1, vy is 4. Of course, uy=-vx, okay. Because uy is 0, vx is 0. So ux is not equal to vy for any xy, okay. And uy=-vx for any xy. So at any point xy, both the equations do not hold, okay. Both the equations must hold at any point xy and therefore, the CR equations, CR equations means Cache-Riemann equations, are not satisfied at any point z, okay.

So fz is not differentiable for any z, okay. Now we shall see that CR equations, in some cases, CR equations will hold at a point, okay, but the function is not differentiable there, okay. So it can happen. CR equations, the satisfaction of CR equations is necessary condition for differentiability. It is not sufficient.

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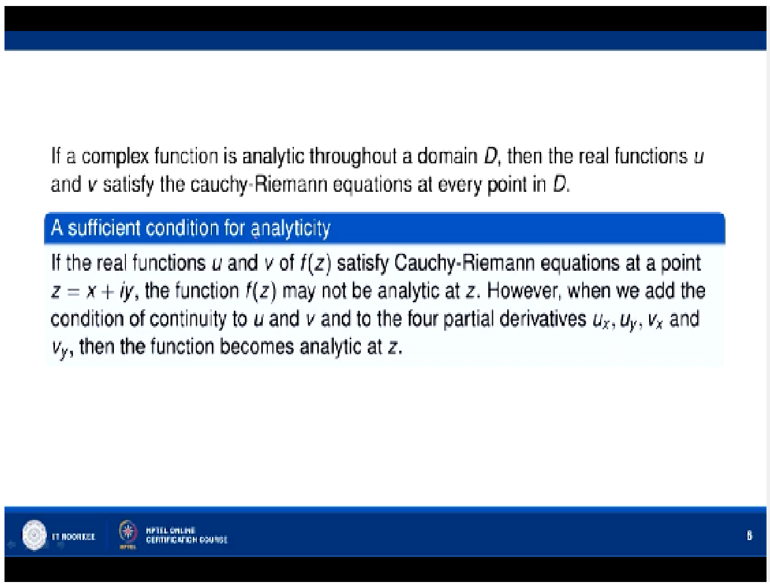


If a complex function is analytic throughout a domain D, then the real functions u and v satisfy Cache-Riemann equations at every point in D, okay. So this we have seen. Now a sufficient condition, now suppose the Cache-Riemann equations hold at a point xy, then if we want at a point z, then what do we need for the function to be differentiable at the point z, okay. So this theorem gives us the sufficient condition for that.

If the real functions u and v of fz satisfy Cache-Riemann equations at a point z=x+iy, the function fz may not be analytic at z. However, when we add the condition of continuity to u and v and to the 4 partial derivatives, ux, uy, vx and vy, then the function becomes analytic at z. So at the point z, suppose the 4 partial derivatives, ux, uy, vx and vy are continuous together with the continuity of u and v, then the function fz will become analytic at z. So let us prove this.

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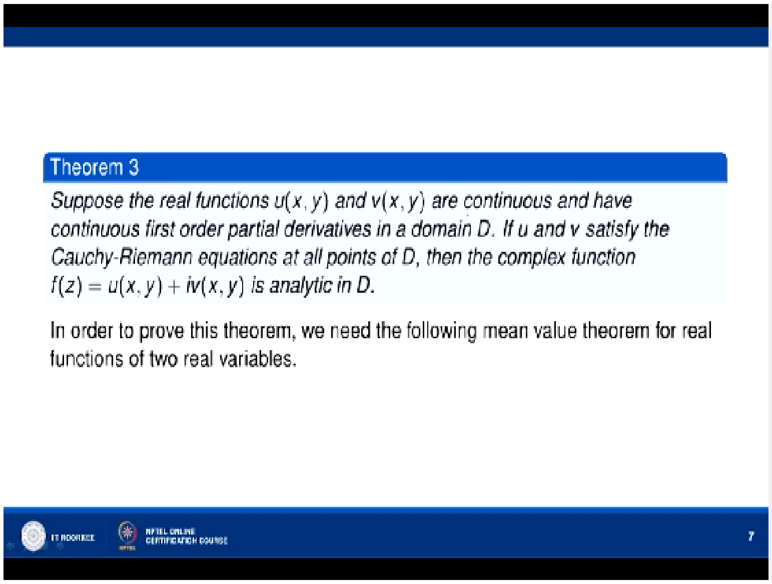


So this theorem we have, this result we have formularized in this theorem. Suppose the real functions uxy and vxy are continuous and have continuous first order partial derivatives in a domain D. If u and v satisfy Cache-Riemann equations at all points of D, then the complex function is fz=uxy+ivxy is analytic in D. Now in order to prove this theorem, we need the following mean value theorem for real functions of 2 real variables.

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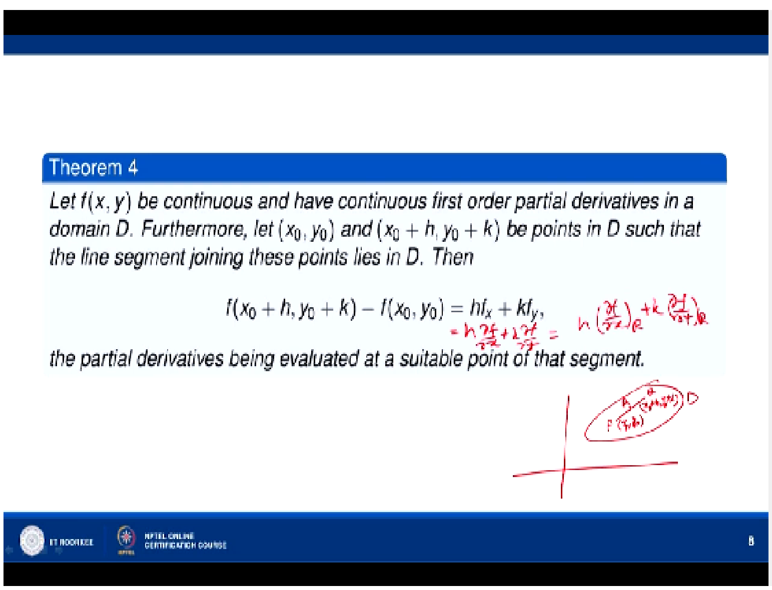


So let us look at this real mean value theorem for real value functions of 2 real variables. Let fxy be continuous and have continuous first order partial derivatives in a domain D. Furthermore, let x0y0 and x0+h, y0+k be points in D. So suppose you take any domain in the xy plane. Let us take the domain D, okay. Say 1 point xy is here, another point is x0+h, y0+k in D, okay. So x0y0 is here.

This is x0, this is y0, okay. And x0+h, y0+k is here, such that the line segment joining these points, this is line segment joining these points, that also lies in D. Then fx0+h, y0+k-fx0y0=h*fx, fx means partial derivative of f with respect to x. And similarly, fy is partial derivative of f with respect to y. So h*fx+k*fy, the partial derivatives fx and fy are being evaluated at a suitable point of that segment.

So some point is there, say suppose this x0y0 be denoted by P and x0+h y0+k be denoted by Q, then there is some point, let us say R, okay, in between x0y0 and x0+h y0+k, such that fx0+h y0+k-fx0y0=h*fx+kfy where fx and fy are evaluated at R, okay. So I can write like this. This is equal to h*fx at R+k*fy at R. There is some point R in between, x0y0 and x0+h y0+k where fx and fy are being evaluated, okay. So this is the mean value theorem for functions of 2 variables. We are going to use this in order to prove the sufficient condition for analyticity.

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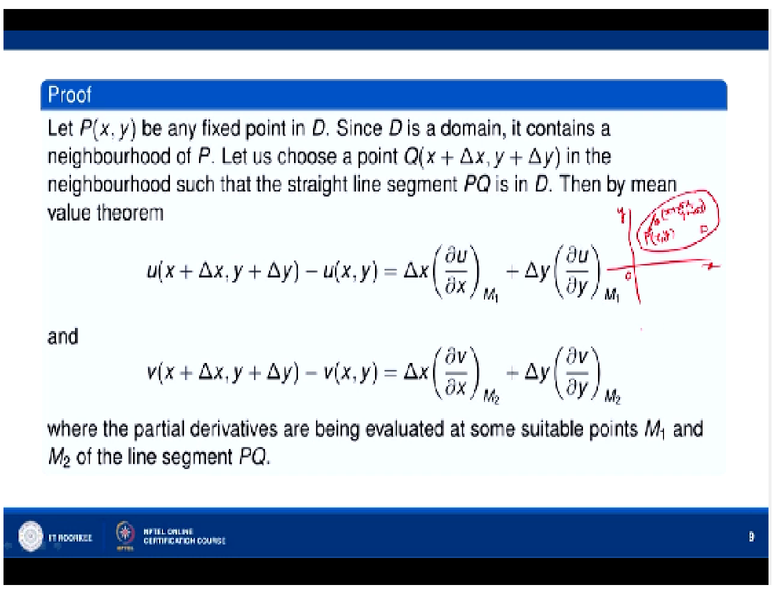


So let us take a P to be any fixed point in D. Suppose you take any fixed point, let us take a domain D, okay. Let us take a fixed point P here. Its coordinates are xy. Since D is a domain, okay, it contains a neighbourhood of the point P, okay. So let us choose a point Q in that neighbourhood. You take a neighbourhood of this. So let us take a point Q in this neighbourhood, okay.

What we have? It is x+delta x y+delta y. So in the neighbourhood of P, let us take a point Q here, okay. Let us choose a point Q, x+delta x y+delta y in the neighbourhood such that the straight line segment PQ, this PQ, okay, is in the D. Then by mean value theorem, ux+delta x y+delta y-uxy=delta x*ux at m1 delta y uy at m1. m1 is some point in between P and Q, okay.

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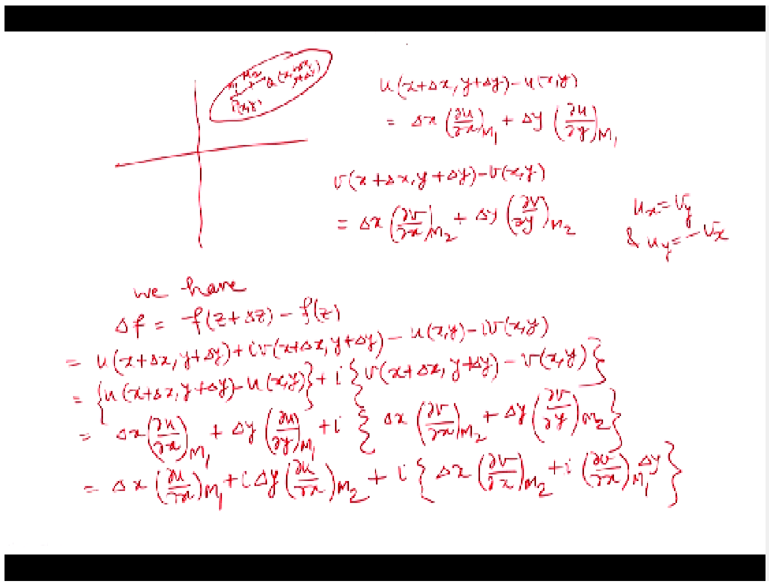


So let us draw this figure. This is your point P, that is your point Q. This is xy, this is x+delta x, y+delta y. In between P and Q, there is a point, let us say m1, okay. There is a point m1 at which we have ux+delta x y+delta y-uxy=delta x, ux at m1 delta y uy at m1. And similarly, for the function vxy, we have vx+delta x y+delta y=vxy=delta x. The partial derivative of v with respect to x at m2+delta y partial derivative of v with respect to y at the point m2.

So that we have, so there is another point m2 here, okay at which. So see what we have to mean, I mean that ux+delta x y+delta y-uxy, there exists some point m1 in between P and Q at which we have this, okay. For the real function uxy, we have this and similarly for the real function vxy, we have, okay, for some point m2 in between P and Q, okay. So we have these 2 equations where the partial derivatives are being evaluated at some suitable points m1 and m2 of the line segment PQ.

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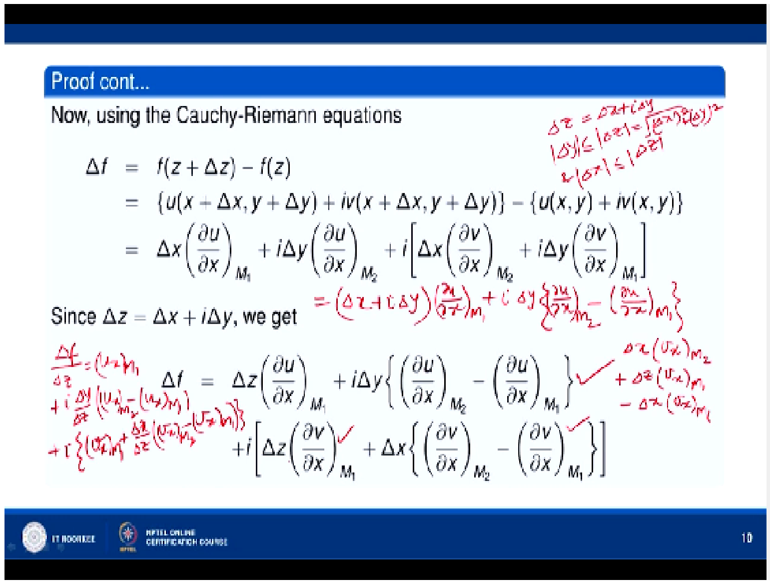


Now using the Cache-Riemann equations, now delta f, the increment in f, delta f is the difference fz+delta z-fz. fz+delta z is ux+delta x y+delta y+ivx+delta x y+delta y. fz is uxy+ivxy. So we have written the value of fz+delta z-fz here. Now we put the value of ux+delta x y+delta y-uxy here from the previous slide, from here, okay. And similarly, from here, we put the value of the other difference, okay.

So we have delta x delta u/delta x m1+delta y delta u, okay. So what we have? ux+delta x, let me write it, okay. So let me clarify this. See we have delta f=, what we have? fz+delta z-fz. So this we can write as ux+delta x y+delta y, okay, +ivx+delta x y+delta y-uxy-ivxy, okay. Putting the value of fz and fz+delta z and fz we have this. Now let us write ux+delta x y+delta y-uxy+i*vx+delta x y+delta y-vxy, okay.

So this is delta x ux at m1+delta y uy at m1+i*, here we have delta x, vx at m2+delta y vy at m2, okay. So what we do now? Let us look at this. So here we have written delta x delta u/delta x m1, okay, we are using actually Cache-Riemann equations here. So what we do is, in this term, if we are making use of ux=vy and uy=-vx. All the derivatives which were there with respect to y are being changed to partial derivatives with respect to x.

So let us do that. So what do we do? Partial derivative of v with respect to y, using ux=vy and uy=-vx, okay. What we will get? This will be equal to delta x delta u/delta x m1 and here we will get i delta y vy at m2. vy at m2 will be ux at m2, okay. And then what we will get? i*, here let us see. Delta x delta v/delta x m2, okay. Now here we have delta y uy at m1. uy at m1 will be equal to -vx at m1, okay.

So -vx at m1 I can write as i*; *delta y. Because iota*iota is -1. iota square is -1. So uy at m1 is -vx at m1. So this term will become -delta y vx at m1 and that -delta y vx at m1, I have put here because this is i square, so i square is -1, so this is -vx m1*delta y, okay. So this is how we come to this term. This is what we get, okay, using Cache-Riemann equations. Now let us see delta z=delta x+i delta y.

So I can write it as delta x ux at m1, this you see here what I do? I add iota delta y ux at m1 and iota delta y ux at m1 I subtract, okay. So if I do that, actually what I do? Delta x+i delta y ux at m1 and I subtract that. This is what I do, okay. So here I add iota delta y ux at m1 and I subtract iota delta y ux at m1 and I get this term, okay. Then add iota delta z. So here also I add iota delta y, okay. And I subtract iota delta y.

So delta x+iota delta y will become delta z, partial derivative of v with respect to x, okay. What I do is, this term delta z, okay. Here I add, let me write this term, okay. So delta x vx, okay. This is delta x vx m2 and then I add delta x here. So I make it delta z. Delta z vx at m1 and I subtract delta x vx at m1, okay. So delta z vx at m1 is here, okay. And then delta x*vx at m2-vx at m1 is here, okay. So this is what we do.

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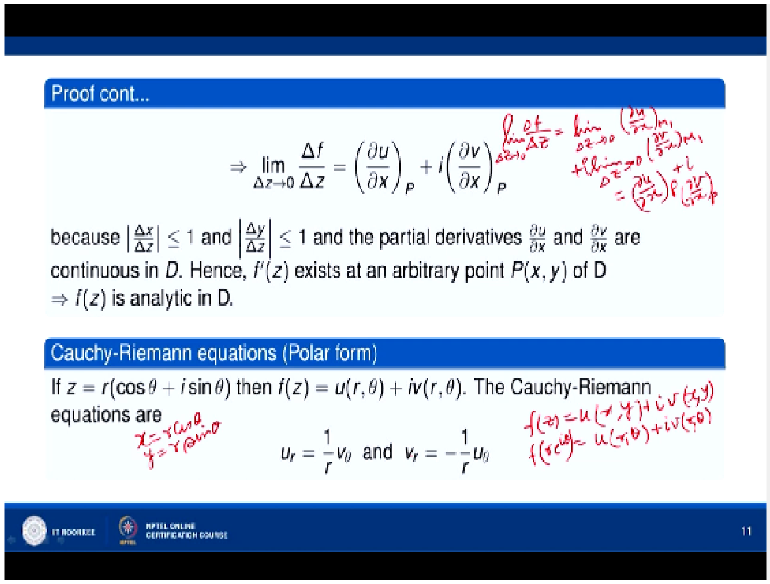


And then let us take the limit of; so delta f/delta z as delta z goes to 0, for that what do you do? Divide both sides by delta z. When you divide by delta z, delta f/delta z, okay, what you get? This is equal to ux at m1+iota delta y/delta z ux at m2-ux at m1+iota, this is delta z. When we divide, this becomes 1. So vx at m1, okay, and then delta x/delta z vx at m2-vx at m1, okay. This is what we get, okay.

So when delta z will go to 0, what will happen? Mod of delta y, see delta z is delta x+i delta y. So delta y is always less than or equal to mod of delta y, is always less than or equal to mod of delta z which is under root delta x whole square+delta y whole square. And also, and similarly, mod of delta x is always less than or equal to mod of delta z, okay. So what will happen? Mod of delta y/mod of delta z is less than or equal to 1.

So delta, when we divide by delta z, okay, delta y/delta z is bounded by 1 and here delta x/delta z is bounded by 1. And when delta z goes to 0, the m1 and m2 points will tend to the point P. So the partial derivatives by the continuity of the partial derivatives, ux at m2 ux at m1, they will tend to ux at P and this will also tend to ux at P. So this will be 0 and here we will have partial derivative of v with respect to x at P.

Here we will have partial derivative of u with respect to x at P. And this will be partial derivative of v with respect to x at P. This will be partial derivative of v with respect to x at P. So this will also be 0 and delta x/delta z is bounded by 1. So what I do is, when we divide by delta z, we get the following actually. So delta f/delta z, okay, limit delta z tends to 0 when we do. This is what we get.

Limit delta z tends to 0, okay. Partial derivative of u with respect to x at m1 we have. Then limit delta z tends to 0 iota*delta v/delta x at m1. So they will tend to partial derivatives of u at the point P+partial derivative of v with respect to x at the point P. The other derivatives will tend to the respective derivatives at the point P and will cancel, okay.

So this happens because of the fact that mod of delta x/delta z is less than or equal to 1, mod of delta y/delta z less than or equal to 1 and the continuity of the partial derivatives of ux and vx. So what has happened then? f prime z exists at an arbitrary point P of D and therefore, fz is analytic in D. Now let us consider the Cache-Riemann equations in polar form. So if z=r cos theta+i sin theta, then we have fz=uxy+ivxy.

So if we use the polar coordinates r theta, then we know that the partial derivatives, the Cartesian coordinates are xy and the polar coordinates r theta are related by x=r cos theta and y=r sin theta, okay. So xy depend on r and theta. So I can say that u is a function of r theta, okay when we replace z by r e to the power i theta, okay. So u is the function of r theta and v is the function of r theta.

So in this article, we shall be finding the corresponding Cache-Riemann equations in the polar coordinates r and theta. And we see that they are, ur=1/rv theta. ur is the partial derivative of u with respect to r. v theta is the partial derivative of v with respect to theta. Similarly, vr is the partial derivative of v with respect to r-1/r, partial derivative of u with respect to theta. So let us now see how we derive them, okay.

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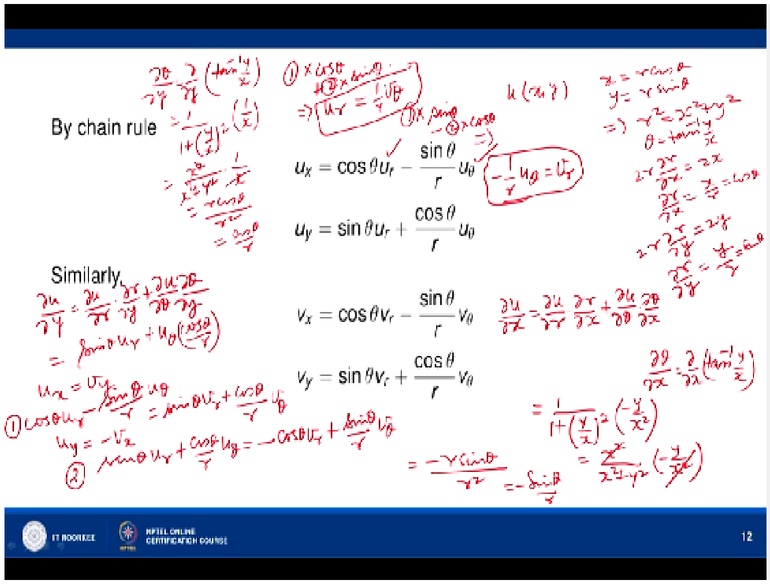


So let us see we have u is the function of xy, x is a function of r and theta, y is a function of r and theta and these give you r square=x square+y square and theta=tan inverse y/x. Now r square=x square+y square gives you 2 r rx=2x, okay. So rx which is partial derivative of r with respect to x is x/r and this is equal to cos theta. Similarly, 2r delta r/delta y=2y, okay. So ry=y/r which is equal to sin theta, okay.

So ux by chain rule, partial derivative of u with respect to x is given by partial derivative of u with respect to r*rx+partial derivative of u with respect to theta*theta x, okay. So we need to find theta x and theta y also. So partial derivative of theta with respect to x will be partial derivative of tan inverse y/x with respect to x. And this is 1/1+y/x whole square*-y/x square. So this will be equal to x square/x square+y square and -y/x square.

So this will give you -r sin theta, y is r sin theta/r square. So this is equal to -sin theta/r, okay. So partial derivative of u with respect to x is ur*rx, rx is cos theta. So we get this and then partial derivative of theta with respect to x is -sin theta/r. So -sin theta/r u theta. So we get this first equation. Similarly, let us find uy. uy is; so what we have? ry. ry=sin theta. So we have sin theta*ur, okay.

And then u theta, let us find theta y. Theta y we can find in similarly like we have found theta x. So theta y=partial derivative of tan inverse y/x with respect to y which is 1/1+y/x whole square*1/x. So this is x square/x square+y square*1/x. So this will give you x/x square+y square. So r cos theta/r square. So we get cos theta/r, okay. So we get this, okay. So uy=sin theta*ur+u theta*cos theta yr.

Now similarly, we can write the expressions for vx and vy. Only we have to replace u by v. So vx=cos theta vr-sin theta/rv theta and vy is sin theta vr and +cost theta/rv theta. Now what we do? So ux=vy. ux=vy gives cos theta ur-sin theta/r u theta=sin theta vr+cos theta/r v theta, okay. This is ux=vy. uy=-vx gives what? uy is sin theta ur, okay, +cos theta/r u theta=-, uy=-vx. So -vx means -cos theta vr+sin theta/r v theta, okay.

Call it equation number 1 and this as equation number 2. Now what you do? Multiply equation 1 by cos theta and 2 by sin theta, okay. So 1*cos theta and 2*sin theta, okay. Then give you what? You see cos square theta ur+sin square theta ur, which will be equal to ur. So ur. And then here what we will have? -sin theta cos theta/r u theta. Here we will have sin theta cost theta/ru theta. So they will cancel.

We are multiplying equation 1 by cos theta, equation 2 by sin theta and adding. So this will become 0. The right side will be equal cos theta sin theta vr. Here we will have -sin theta cos theta vr. So that will also cancel. Here we will have a cos square theta/r v theta. Here we will have sin square/r v theta. So when we add, we get 1/r v theta. So this is one Cache-Riemann equation in polar form.

The other Cache-Riemann equation if we want, then what we do? Now we multiply equation 1 by sin theta, okay. So equation 1 by sin theta. So we get sin theta cos theta ur here and here we multiply by cos theta and subtract, okay. So -2*cos theta, okay. So we multiply 1 by sin theta, 2 by cos theta and subtract.

Then what will happen? Sin sin theta cos theta ur, sin theta cos theta ur will cancel. Here we have -sin square theta/r, okay. Here we have -cos square theta/r, okay. Sin square theta by r, cos square theta/r, will give us -1/r u theta. So this will give you -1/r u theta. And here right side, what will happen?

Sin square theta vr and then +cos square theta vr because we are subtracting. So that will give you vr. The other term will be sin theta cos theta/r v theta-sin theta cos theta/r v theta, that will cancel. So we will get this equation, okay. This equation and this equation, okay. They are the Cache-Riemann equations in the polar form, okay.

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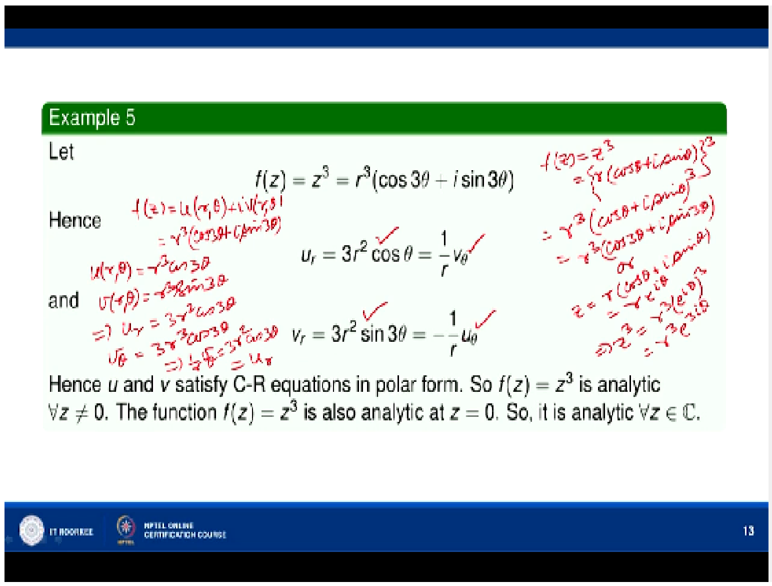


Now let us consider, use this Cache-Riemann equations in polar form to see whether they are satisfied for a complex function. So let us say fz=z cube, okay. Taking z=r cos theta+ir sin theta, what we will get? fz=z cube will give us r cos theta+i sin theta whole cube, okay. This is r cube and cos theta+i sin theta whole cube. So by de Moivre's theorem, I can write cos 3 theta+i sin 3 theta or I can say using the Euler's formula, z=r cos theta+i sin theta, I can write as r e to the power i theta.

So this will give you z cube=r cube e raise to the power i theta to the power 3 which is r cube e raise to the power 3 i theta. So again r cube*e to the power 3 i theta is cos 3 theta+i sin 3 theta. So fz=r cube cos 3 theta+i sin 3 theta. So here fz=u r theta+iv r theta, okay. So this is equal to r cube cos 3 theta+i sin 3 theta. So equating real and imaginary parts, ur theta=r cube cos 3 theta and vr theta=r cube sin 3 theta.

Now let us find ur. ur=3r square cos 3 theta, okay. So this is ur. And v theta is what? If you find v theta, sin 3 theta when we differentiate, we get 3 cos 3 theta. So 3r cube cos 3 theta. So v theta/r, okay, this gives 1/r v theta=3r square cos 3 theta which is equal to ur, okay. So ur=1/r v theta, okay. Similarly, vr=3r square sin 3 theta. This is vr and we find u theta. u theta=-3r cube sin 3 theta.

When we divide by r, we get -3r square sin 3 theta. So when we multiply by -1, we get 3r square sin 3 theta. So vr is equal to this. Hence u and v satisfies CR equations in polar form and so fz=z cube, these are valid at any point z of the complex, so fz=z cube is analytic for all z not equal to 0. See why we are saying this, because the partial derivatives, the functions ur theta vr theta and their partial derivatives ur vr, they are all continuous functions of r and theta.

So by the sufficient conditions for analyticity theorem, okay. By that theorem, fz=z cube is analytic for all z. Because not just the thing that u and v satisfy Cache-Riemann equations, fz=z cube is analytic, it follows because of the continuity of u and v and the partial derivatives of u and v with respect to r and theta. So fz=z cube is also analytic, okay. Now here when you use the Cache-Riemann equations, you see that we are dividing by r, that means r cannot be 0.

So Cache-Riemann equations here can be used only for all z which are not equal to 0 because z=0 means the origin point, okay. So from these validity of Cache-Riemann equations, we can simply say that fz=z cube is analytic for all z not equal to 0. For the z=0 point, we have 2 separate (()) (42:13), okay.

So fz=z cube is also analytic at z=0, that we can see by definition of derivative, okay. We can easily show that fz=z cube is differentiable fz=0. So it is differentiable in any neighbourhood fz=0 because at all non=0 z, okay, fz=z cube satisfies CR equations in polar form and its first order partial derivatives together with u and v are continuous. So it is analytic for all z belonging to c.

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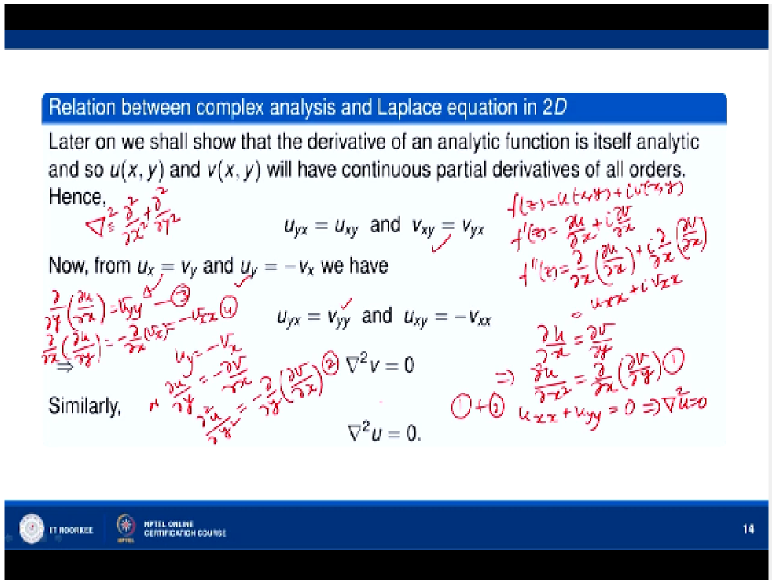


Now relation between complex analysis and Laplace equation in 2 dimensions, okay. 2D, this D means dimension, 2 dimensional case. So first later on we shall show that derivative of an analytic function is itself analytic, okay. If fz is an analytic function in a domain D, then f prime z is analytic in D. And when f prime z is analytic in D, f double prime z is analytic in D. So all order derivatives of fz exist and they are analytic functions in D.

This we do not have. This kind of result we do not have in real calculus. There we know that if the function fx is differentiable at a point x=x0, then f prime, f double prime, x need not exist, okay at x=x0. But here, if the function is analytic at z0, then its all order derivatives are also analytic at z0. So because of that, if fz=uxy+ivxy, okay. We have seen that if fz is analytic and fz=uxy+ivxy, then f prime z is ux+ivx, okay.

So if fz is analytic at any point z, then what will happen? f prime z, f double prime z, they are also analytic. So that means they are all differentiable functions, differentiable functions are continuous functions. So uxvx will be continuous, okay. uyvy will be continuous because uxvx are related to uyvy by Cache-Riemann equations. So uxvx are continuous, so uyvy will be continuous for all derivatives, first order derivatives will be continuous.

f prime z is analytic means f double prime z is analytic. So second order partial derivatives f double prime z we can write as, because this is real part of u, f prime z. So we can write like this and so on. So second order partial derivatives are also continuous. So continuing like this, all ordered partial derivatives, okay of u and v exist. They are continuous functions. So in particular uxy and vxy will have partial derivatives of second order partial derivatives which are continuous.

And when second order partial derivatives of a real value function of 2 real variables is continuous, then the order of differentiation can be interchanged, this we know. So uyx=uxy. And similarly, vxy=vyx. Now we have, when the function is analytic, we have ux=vy, uy=-vx. So when you pick up the equation ux=vy, that is this, differentiating with respect to x, because u is a function of xy, so its partial derivative with respect to x is also a function of xy.

So I can again differentiate it with respect to x. So when we differentiate this with respect to x, I get this. Similarly, the other equation is uy=this one, uy=-vx. If I, this is or I can say this. So if I differentiate it with respect to y, what I get? Second derivative of u with respect to y, okay, is equal to. Now let us call this as equation 1, this as equation 2. So adding 1 and 2, okay, what we will get? uxx+uyy, okay=partial derivative of v with respect to y first then with respect to x.

And here partial derivative of v with respect to x first, then with respect to y, their sum is equal to 0. This minus this is equal to 0 because of this, okay. So this gives you the Laplacian of u=0, Laplacian del square is the defined like this. Del square we define as, in 2 dimensions. So del square u=0 and here we have shown that del square v=0. We can similarly do. Here what I did? I differentiated this with respect to y. So I get vyy, okay. I get this. And then this I differentiated with respect to x. So I get this. So minus, okay. And this I can call as 3, this as 4.

Then I subtract 4 from 3, okay. So 3-4, 3-4 will give us vyy+vxx. And here this minus this, this minus this is 0 because of this, okay. So del square v=0 and similarly del square u=0, this we have already seen. So u and v have second order partial derivatives which are continuous and they satisfy Laplace equation in 2 dimensions, okay. They are solutions of Laplace equation in 2 dimensions.

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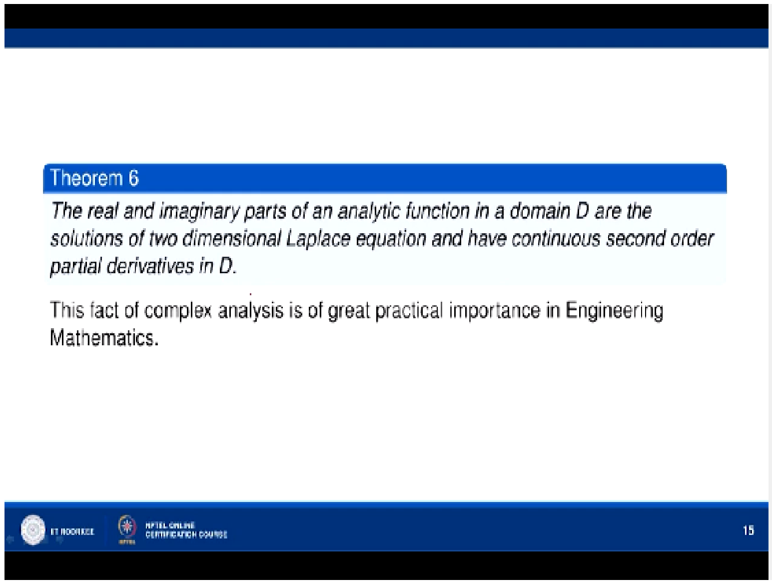


So they are harmonic functions. The real and imaginary parts of an analytic function in a domain D are the solutions of 2 dimensional Laplace equation and have continuous second order partial derivatives. We shall see that when we have the third lecture, in the next lecture when we define the harmonic functions, we shall see that the real and imaginary parts of this analytic function are harmonic functions in D, okay.

And this property of analytic functions is very important, has a predicted practical importance in engineering mathematics as we shall see in our next lecture. Thank you very much for your attention.