Advanced Engineering Mathematics
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Lecture - 19
Residue Theorem

Hello friends. Welcome to my lecture on Residue Theorem. We will first determine the residue in the case of a simple pole. There are methods which we can use to determine the residue in the case of a simple pole, okay. So let us first discuss a method where we will be able to find the residue in the case of a simple pole.

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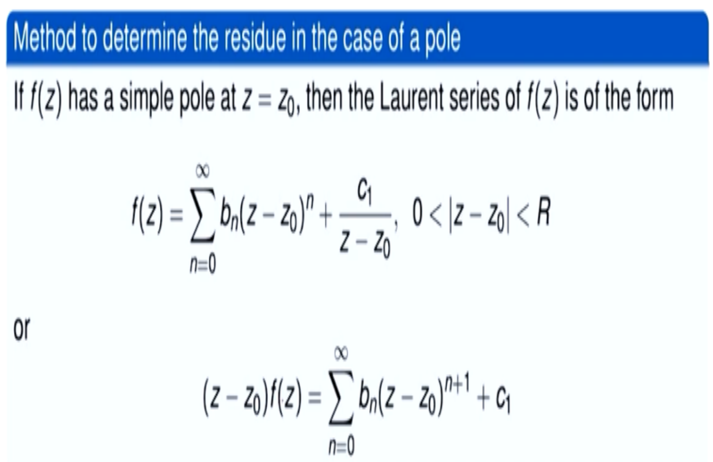

If f(z) has a simple pole at z=z0, then the Laurent series of f(z) is of the form f(z)=sigma n=0 to infinity be an z-z0 to the power n+c1/z-z0 and 0<mod of z-z0<R. Because the function f(z) has an isolated singularity at the point z=z0, so there exist a neighbourhood, deleted neighbourhood of z=z0, in which the Laurent series converges. Now this Laurent series can be put in the form z-z0*f(z)=sigma n=0 to infinity bn(z-z0) to the power n+1+c1.

We can multiply this equation by z-z0. Now then, if we take the limit of z tends to z0, then limit z tends to z0, z-z0*f(z) will be equal to limit z tends to z0, sigma n=0 to infinity bn(z-z0) to the power n+1+c1, but here we see that each term contains z-z0 as a factor, even the first term when you put n=0 is b0(z-z0), okay. So each term here contains z-z0 as a factor. Therefore, when we take the limit as z tends to z0, this part on the right side will tend to 0.

And so right side will tend to c1, okay and c1 will be the limit of z-z0*f(z) as z tends to z0, okay.
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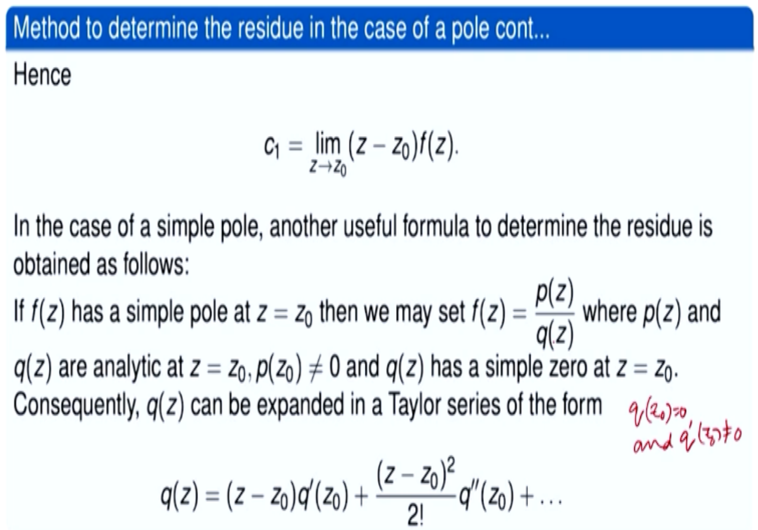

So c1 we know. The c1 is the coefficient of 1/z-z0 and therefore c1 is the residue of f(z) at z=z0. So when we want to find the residue c1 of a function f(z) in the case of a simple pole at z-z0, we do not need to write the Laurent series for the function f(z) with center at z=z0. We can directly use the formula to evaluate c1. The formula is c1=limit z tends to z0, z-z0*f(z). In the case of a simple pole, there is another useful formula to determine the residue at the point z=z0.

Let us see the other formula. If f(z) has a simple pole at z=z0, then we can write f(z) as pz/qz where pz and qz are analytic at z=z0. Pz0 is not equal to 0 and qz has a simple 0 at z=z0, okay. So qz has a simple 0 at z=z0 means qz0=0 and q prime z0 is not equal to 0, okay. So if qz has a simple 0 at z=z0, then qz0=0 and q prime z0 is not equal to 0. Now qz is analytic and has a simple 0 at z=z0, so qz can be expanded in a Taylor series of the form qz=this, okay.

Here the first term qz0 is not written because qz0=0. So qz=z-z0 q prime z0 (z-z0) whole square/2 factorial q double prime z0 and so on. Let us put this value of qz in the denominator here for qz.

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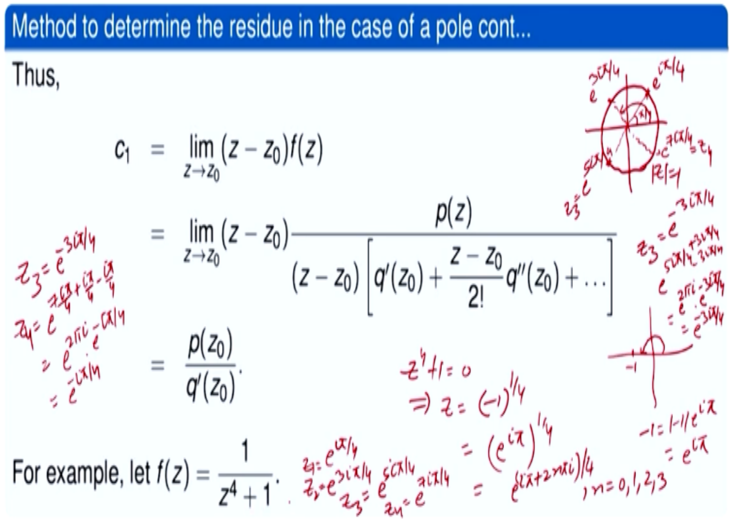

Then we see that c1, c1 we have seen in the case of a simple pole, c1 is given by limit z tends to z0, z-z0*f(z). So limit z tends to z0, z-z0*f(z) is pz/qz. So this is the expression for qz, okay. we can write the expression for qz as z-z0*q prime z0+z-z0/2 factorial q double prime z0 and so on, so that we have written here. Now z-z0 we can cancel and then as z tends to z0, pz goes to pz0, because pz is analytic, okay and therefore it is continuous, so pz0.

And similarly qz is analytic, so it is continuous, q prime z is also analytic, so q prime z is also continuous. So q prime z0 we get. So pz0/q prime z0. Now so this formula is also sometimes useful to determine the residue in the case of a simple pole at z=z0. For example, suppose we have this function f(z)=z to the power 4+1. Now this function has got poles at the points where the denominator of this f(z) vanishes, okay.

So z to the power 4+1=0, z to the power 4+1=0 gives us z=-1 to the power 1/4. Now we know that -1 lies here on the real axis and its magnitude, -1 if you write in the polar form, then -1=mod of -1*e to the power theta where theta is the argument of -1. So this argument is pi, so we have e to the power i pi. So-1 e to the power i pi. So this is equal to e to the power i pi raised to the power 1/4.

Now we can write its general value e to the power i pi+2npi i/4 where n takes value from De Moivre's theorem. So applying the De Moivre's theorem, the 4 roots of -1 to the power 1/4 are given by e to the power ipi+2npi i/4 where n=0,1,2,3 that means that when you put n=0, you get e to the power ipi/4, let us call it as z1. Now z2 is e to the power 3ipi/4, which you get by putting n=1 here. When you put n=2, you get z3, z3=e to the power 5ipi/4 and z4=e to the power 7ipi/4.

Which you get by putting n=3, okay. Now let us plot these 4 roots z1, z2, z3, z4 in the organ diagram, okay.

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Residue of $f(z)$ in the case of a pole of order $m(>1)$

If $f(z)$ has a pole of order $m > 1$, at a point $z = z_0$, the corresponding Laurent series expansion is of the form

$$f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n + \frac{c_1}{z-z_0} + \frac{c_2}{(z-z_0)^2} + \ldots + \frac{c_m}{(z-z_0)^m} \qquad (1)$$

where $c_m \neq 0$ and the region of convergence is $0 < |z-z_0| < R$.
From (1) we obtain

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^{n+m} + c_1(z-z_0)^{m-1} + c_2(z-z_0)^{m-2} + \ldots + \frac{c_{m-1}}{(z-z_0)} + c_m.$$

So if you draw them in the organ diagram, since the modulus of z, mod of z=1, okay -1 to the power 1/4 mod of z=1, so all the 4 roots lie on the unit circle. Mod z=1, okay, the first root is z1=e to the power ipi/4, so it lies here, e to the power ipi/4 its argument is pi/4 and f-th root value 1, okay. Then the second is e to the power 3ipi/4, it lies here. This is 3pi/4 angle, okay, this is 3pi/4 angle and then e to the power 5ipi/4. So it is e to the power 5ipi/4, okay.

So this is this 1, e to the power 5ipi/4 and this is e to the power 7ipi/4, okay. This is z4 and this is z3. Now z3 can also be written as e to the power -3ipi/4, okay because e to the power 5ipi/4, e to the power 5ipi/4+3ipi/4-3ipi/4 gives you e to the power 5ipi/4+3ipi/4 is 8ipi/4. So e to the power 2pi i and then into e to the power -3ipi/4, e to the power 2pi i=1, so we get e to the power -3ipi/4. So z3 is also equal to e to the power -3ipi/4, okay and similarly z4.

Z4 is e to the power 7ipi/4. So here also let us 8ipi/4 and subtract ipi/4. So this is e to the power 8ipi/4 means e to the power 2pi i * e to the power –ipi/4 and we get e to the power 2pi i is 1, so we get e to the power –ipi/4. Now let us find the, there are 4 factors of z4+1 f(z) we can write as.

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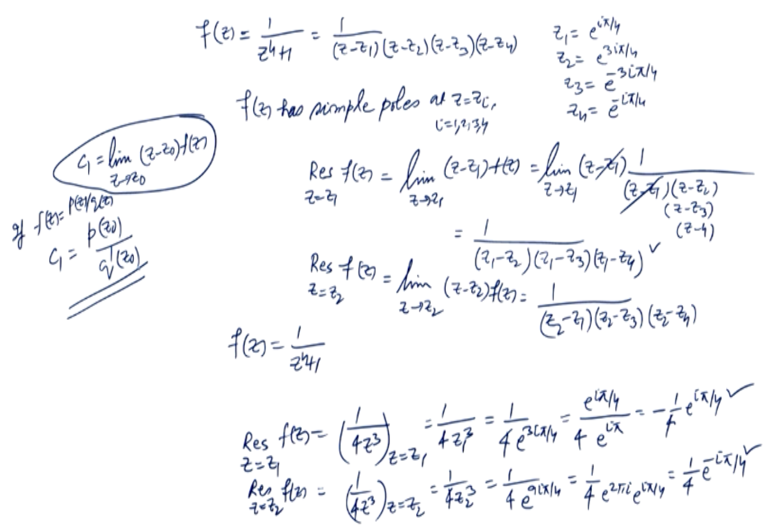

1/(z-z1) (z-z2) (z-z3) (z-z4), okay let us recall z1 is e to the power ipi/4, z2 is e to the power 3ipi/4, z3 is e to the power -3ipi/4 and z4 is e to the power –ipi/4, okay. Now suppose at each of these points, okay z1, z2, z3, the denominator has a simple 0, okay and therefore f(z) has simple poles at z=zi i=1,2,3,4. Now if you want to determine the residue at z1, z2, z3, z4 and you apply the first formula. The first formula was c1=limit z tends to z0 z-z0*f(z), okay.

To determine the residue in the case of a simple pole at z=z0, if you apply this formula, okay to determine the residue at the point z=z1 from here, what we get residue at z=z1 of f(z) will be equal to limit z tends to z1, okay and then z-z1*f(z), so this will give you what limit z tends to z1 z-z1*1 upon (z-z1) (z-z2) (z-z3) (z-z4), okay. We can cancel z-z1 in the numerator and denominator and then we have (z1-z2) (z1-z3) (z1-z4), okay. This is the residue at z=z1, okay.

Similarly, residue at z=z2 if you find, then we have limit z tends to z2, (z-z2*f(z) and what we get now? We have 1/z-z2 will now cancel in the numerator and denominator. So we will have (z2-z1) (z2-z3) (z2-z4), okay. Let us first see how difficult it is to determine the residue at z=z1 and z=z2 here. You have to put the values of z1, z2, z3, z4 here and then you have to multiply them and then you have to find 1/(z1-z2) (z1-z3) (z1-z4).

So it is not easy to determine the values of f(z) at z=z1 and z=z2, if you apply the other formula. The other formula is c1=pz0/q prime z0, okay. If we apply this formula, then you will see it is easy to determine the residue at each 1 z=z1, z=z2, z=z3 and z=z4. What we have to do is, simply we have to differentiate the denominator f(z) is 1/z to the power 4+1. Here we have taken f(z) as pz/qz. If f(z) is pz/qz, then we know that c1=pz0/q prime z0.

Denominator only we have to differentiate. So here this will be 1/4zq and so the residue at z=, so residue at z=z1 will be of f(z) will be 1/derivative of the denominator that is 4zq and you put z=z1 and you get 1/4z1 cube. Similarly, if you find the residue at z=z2, you do not have to do anything. You know already the derivative of the denominator of f(z). So again 1/4z cube and you put z=z2. So this is 1/4z2 cube and similarly for z=z3 and z=z4.

We will get the residue as 1/z3 cube and 1/4z4 cube and so here you can see very easy 1/4z1 is e to the power ipi/4, so we get e to the power 3ipi/4, okay and if you multiply by e to the power ipi/4 in the numerator and denominator what we get, e to the power 4ipi/4 means e to the power ipi, e to the power ipi=1, e to the power ipi is -1, so -1/4e to the power ipi/4 and here what we get 1/4 e to the power 3ipi/4 to the power 3. So that means 9ipi/4.

So we get 1/4, this is e to the power 2pi i*e to the power ipi/4, e to the power 2pi i=1, so we get 1/4e to the power –ipi/4. So you can see it is very easy to determine the residue if we differentiate the denominator and put the value of z, but here if you find the residue using this formula, then it is not easy to determine because we have to put the values of z1, z2, z3, z4, then you have to multiply them and then you have to find the reciprocal of that.

So it is easy to determine this. We use this formula, this 1, okay, this formula we use when z-z0 is given as a factor of f(z), like say, we will see some examples later. There it is better to use this formula, not this 1. We will see the problem like that, so but here in this particular problem, where f(z) is 1/z to the power 4+1, we need to use this formula to obtain the residues easily, okay. So this formula is very useful in some cases.

Now let us consider the residue of f(z) in the case of a pole of order m>1. If f(z) has a pole of order m, say >1 or >=2 let us say, m>=2 at a point z=z0, then the corresponding Laurent series expansion is of the form f(z)=sigma n=0 to infinity bn(z-z0) to the power n c1/(z-z0) c2/(z-z0) square and so on cm/(z-z0) to the power m where cm is non-zero and the region of convergence is 0<mod of z-z0<R because the poles are isolated singularities.

Now from this, when you multiply this equation by (z-z0) to the power m, what we get is (z-z0) to the power m f(z)=sigma n=0 to infinity bn (z-z0) to the power n+m c1(z-z0) to the power m-1 c2(z-z0) to the power m-2 and so on cm-1*(z-z0), okay cm-1*(z-z0)+cm, okay. Now this right hand side then is nothing but a power series with center at z=z0. Why because this is constant term, first term cm, then c-1*z-z0, then c2 z-z0, cm-2 (z-z0) square and so on.

C2(z-z0) to the power m-2 c1(z-z0) to the power m-1 and here we have b0(z-z0) to the power m b1(z-z0) to the power m+1, so right hand side defines an analytic function, okay that is z-z0 to the power m*f(z) is an analytic function.

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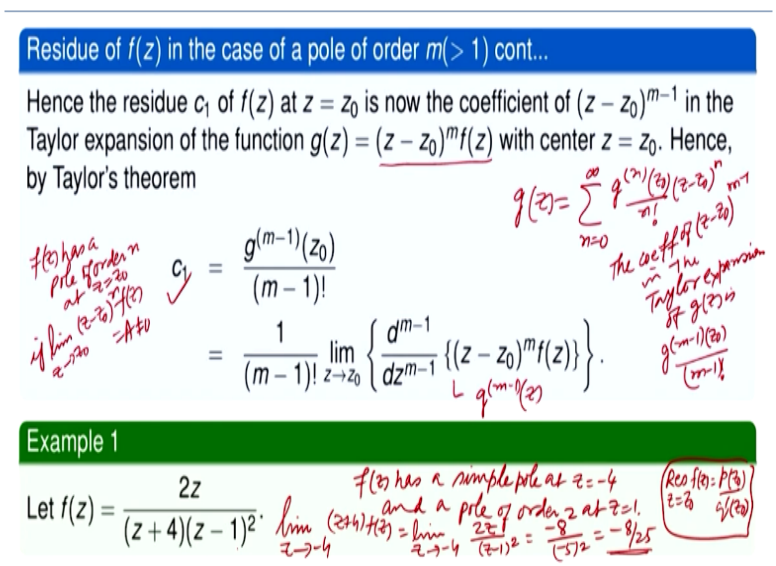

So the residue c1 of f(z), okay residue c1 of f(z) at z=z0 is now the coefficient of z-z0 to the power m-1, okay. This, this c1 is the residue of f(z). It is now the coefficient of z-z0 to the power m-1 and as I said the right hand side is the Taylor series, okay. This right hand side is the Taylor series of the function z-z0 to the power m*f(z) because it is a power series with center at z=z0. So let us call g(z) to be z-z0 to the power m*f(z), okay.

So right hand side, this 1 is the Taylor series of the function g(z). We are calling this function as g(z). So now we know that if the right hand side is the Taylor series of g(z)=then sigma n=0 to infinity g/n factorial*z-z0 to the power n. So if we want the coefficient of z-z0 to the power m-1 in the Taylor expansion of g(z), then the coefficient of z-z0 to the power m-1 in the Taylor series, Taylor expansion of g(z) is gm-1, okay, you put n=m-1.

So gm-1(z0)/m-1 factorial. So this is c1, okay. This c1 is the coefficient of z-z0 to the power m-1. So c1 is g(m-1)z0/m-1 factorial, so we get 1/m-1 factorial, now g(z) is z-z0 to the power m*f(z). This is m-1-th derivative of g(z). We are writing limit z tends to z0 because f(z) has z-z0, f(z) has a pole at z=z0 of order m. So f(z) is of this form, okay, where z-z0 occurs in the denominator. So we are multiplying z-z0 to the power m.

So directly we cannot put z=z0, we should take the limit at z tends to z0. So c1 is 1/m-1 factorial, limit z tends to z0. This is m-1-th derivative of g(z). So this is g(m-1)z. This is g(m-1)z and we are finding the limit of g(m-1)z at z tends to z0. Now let us consider this function f(z)=2z/z+4*(z-1) whole square. Then you can see that f(z) has a simple pole at z=-4 and it has a pole of order 2 and pole of order 2 at z=1. We can see this easily.

If you recall the definition of a simple pole, of a pole of order m, we had said that f(z) has a pole of order n at z=z0 if limit z tends to z0 z-z0 to the power n*f(z)=A, which is non-zero and finite, okay. Here you can see if you multiply f(z) by z+4 and take the limit as z tends to -4, okay, limit z tends to -4 z+4*f(z), then what will happen, f(z) z+4 will cancel and this will be equal to limit z tends to -4 and we will have 2z/(z-1) square. So this will be equal to 2*-4, so -8.

Divided by (-5) square, so we get -8/25. Now here in the case of the simple pole at z=-4, it is not advisable to use the formula for the simple pole residue at simple pole as reside at z=z0 of f(z), which is given by pz0/q prime z0. So it is not advisable to use this formula here. Because here we, if you differentiate the denominator and then put z0, z0 means -4, then it will be little bit cumbersome.

If you multiply f(z) by z+4 and take the limit as z tends to -4, then it becomes easy to determine the limit. So we have to decide from the definition of the function f(z), which formula for simple pole we have to use to get the residue, okay. So now let us find the residue at the double pole at z=1, okay for this function. So let us use.

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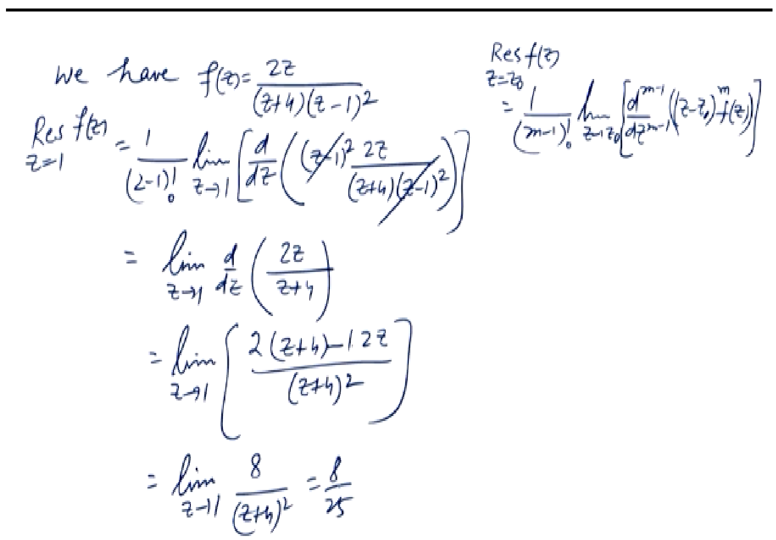

We have f(z)=2z/z+4*(z-1) square. So we are finding the residue of f(z) at z=1, it has a pole of order 2, okay. So we have in the case of a pole of order m, the formula is residue at z=z0 f(z) when f(z) has a pole of order m. It is m-1/m-1 factorial limit z tends to z0 dm-1/dzm-1 and then z-z0 to the power m*f(z), okay. So we like that formula. So here we have a pole of order 2. So m=2, so 1/2-1 factorial, okay limit z tends to z0 is 1, so z tends to 1 d/dz.

Because m=2 d/dz, then we have (z-1) square for z-z0 to the power m, we write (z-1) square*f(z), f(z) is 2z/z+4*(z-1) square and we get 1/1 factorial is 1, so limit z tends to 1 d/dz (z/z+4), okay. So we get limit z tends to 1. Let us differentiate. So we have derivative 2z is

2*z+4 derivative of z+4 is 1, so we get 1*2z, so we get this divided by (z+4) square. So limit z tends to 1 and 2z, 2z will cancel, we have 8/(z+4) square, so 8/25.

So this is how we get the residue in the case of a pole of order m where m>1, okay.

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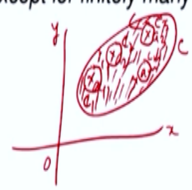
We know how to evaluate the contour integrals whose integrands have only a single isolated singularity inside the contour of integration. Now, we shall see how this method can be extend to the case where the integrand has several isolated singularities inside the contour.

Theorem 2

Let $f(z)$ be analytic inside and on a simple closed path C except for finitely many singular points $a_1, a_2, \ldots, a_m$ inside C. Then

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{m} Res_{z=a_j} f(z),$$



the integral being taken in the counterclockwise sense around the path C.

Now let us go to the residue theorem. I mean we are now going to discuss the residue theorem. We have seen how we can evaluate the contour integral whose integrant has only 1 isolated singularity inside the contour of integration. There we simply find the coefficient, the residue of f(z) at that point and then multiply by 2pi i. Now if your contour of integration has finite number of isolated singularities, then how we will extend that method.

So this theorem actually talks about that. So let f(z) be analytic inside and on a simple closed path c, except for finitely many similar points a1, a2, am inside c, then if you find the integral of f(z) around c, it is 2pi i*sigma j=1 to m residue of f(z) at z=aj. So let us say suppose this is your contour, okay c, you have similar points at a1, a2, a3 and so on am, okay. These are isolated singularities of f(z), which occur inside c, okay.

Then how we will find the integral of f(z) around c. So this theorem tells us that you find the residue of f(z) at each of these isolated singularities a1, a2, a3 and so on am, sum them up and then multiply by 2pi i. Here integral around c is to be taken in the counterclockwise sense.

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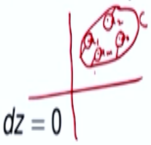
Proof

We enclose the n singularities within small circles $C_1, C_2, \ldots, C_m$ with their centers at $a_1, a_2, \ldots, a_m$, respectively, with radii small enough such that the circle do not overlap and lie inside C. Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, C_2, \ldots, C_m$ and on the entire boundary of D. From Cauchy's integral theorem, we have



$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_m} f(z)\,dz = 0$$

the integral along C being taken in the counterclockwise sense and the other integrals in the clockwise sense.

Now what we do is, we enclose the singularities, m singularities c1, c2, cm, have been at a1, a2, am within a small circle c1, c2, cm. What we do, let us take a small circle, okay with center at a1, a2, a3 and so on am. This is circle c1, this is c2, this is c3, this is cm, okay. So let us construct circles of sufficiently small radius with center at a1, a2, a3, and so on am. We call these circles as c1, c2, c3 and so on up to cm.

Such that the circles do not overlap and are completely inside the simple closed path c. So we enclose the m singularity with the small circle, c1, c2, cm with their centers at a1, a2, am respectively with radii small enough such that the circles do not overlap and lie inside c. Then f(z) is analytic in the multiply connected domain bounded by c and c1, c2, cm, okay and on the entire boundary of d. So then we can see that.

Then f(z) is analytic everywhere inside this simple closed paths inside this c, okay, this region I mean, this shaded region it is analytic everywhere here, okay as well as on c1, c2, cm and the curve c. Now what we do, we join these points, these paths by cross cuts. Let us draw another figure. Let me draw here. This is your c, okay. This is a1, this is a2, this is a3 and this is am, let us say, okay. We draw small circles with center at these points.

Now join them by cross cuts with the boundary and then, we can see that integral over c f(z) d(z) +integral over cj where j runs from 1 to m, summation over cj j runs from 1 to m=0, so by cross integral theorem, let us show it here.

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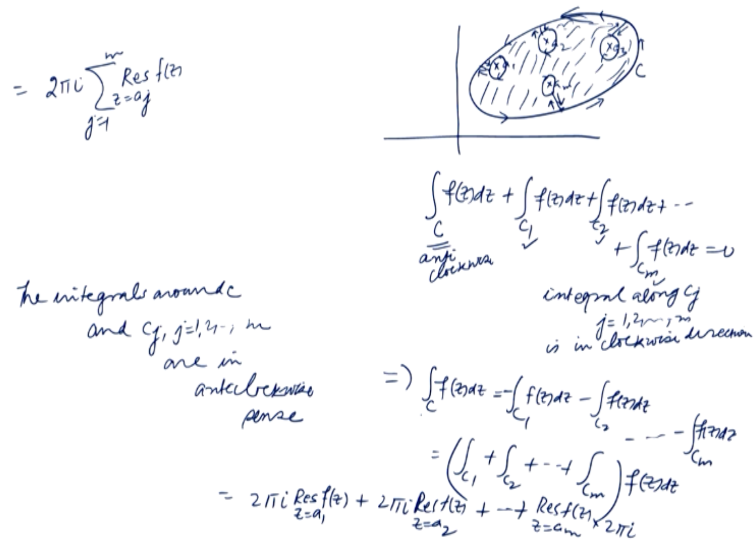

So let us draw a bigger figure. This is your contour c, this is a1, a2, a3 and here let me write am, okay. So this is small circle let us draw, okay and then join them with the boundary of c, boundary c, okay. So then what we do, let us start from here, okay. So we start from this point, let me go this way, this way and when I reach here, I come this way and then I move clockwise and then I go here and then when I reach here, this point, okay.

I go this way, then this and then this way and then I move on, I come here, I move this way then, clockwise, then this way and then this way and when I reach here I move this way, then this and then this, okay and I come back to the original point, okay with which I started. Then you can see, now the shaded region is bounded by this path, okay, which consists of c1, c2, cm, c and these things the crosscuts.

But you notice that along the crosscuts we are moving in opposite directions, okay. We are moving this way, then we are coming this way, so integral along the opposite directions will cancel and then along c we are moving in the anticlockwise direction, okay. So integral over c f(z) d(z)+integral over c1, c2, cm we are moving in the clockwise direction. So integral over c1 f(z) d(z)+integral over c2, f(z) d(z) and so on integral over cm f(z) d(z), okay will be equal to 0.

Because this path encloses a simply connected domain, okay and so integral along c is in anticlockwise direction by c1, c2, cm, okay. Integral along cj j=1, 2, and so on up to m is in clockwise direction. So this equation then gives us integral over c f(z) d(z)=integral over c1-integral over c1, okay f(z) d(z)-integral over c2 f(z) d(z) and so on –integral over cm f(z) d(z), okay. So this will give you integral over c1+integral over c2 and so on integral over cm f(z) d(z).

Where now all the integral along c and integral along cj, okay j= 1, 2, 3 and so on up to m are in the anticlockwise direction because here these integrals were in the clockwise direction, when you put a negative sign, then this integral along anticlockwise is equal to negative of integral over clockwise. So this is now, so the integrals around c and cj j=1, 2, so on up to m are in anticlockwise sense, okay.

Now you can see that c1, c2, c3, and so on up to cm. Each 1 encloses just 1 singularity, singularity at a1, singularity at a2, singularity over a3, so we know that integral over c1 f(z) d(z) is 2 pi i*residue at z=a1f(z) and then integral over c2 will be 2pi i residue of f(z) at z=a2 and similarly integral over cm f(z) d(z) will be residue of f(z) at z=am, okay, so multiplied by 2pi i, okay so this is nothing but 2pi i*sigma residue of f(z) at z=aj where j runs from 1 to m.

So this is how we evaluate the integral of f(z) around c. We simply find the residues of f(z) at its isolated singularities at the points a1, a2, am and then sum them up and multiply by 2pi i. So we had this equation where the integral along c is anticlockwise direction while integral along c1, c2, cm is the clockwise sense when we take these terms integral over cj j=1, 2, to m on the other side, we get the integrals along c1, c2, cm in the anticlockwise direction and we get this result.

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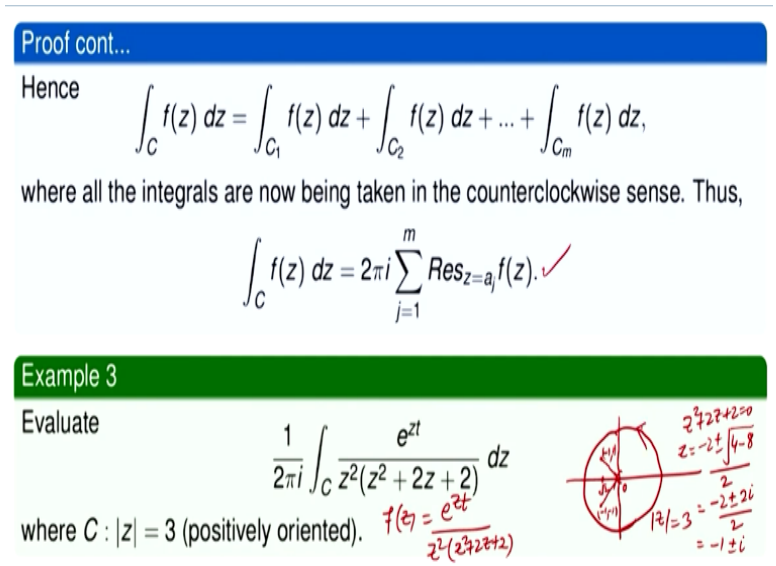

Now let us consider this function. We have f(z)=e to the power zt/z square, z square +2z+2, we have to find the value of this integral. C is the circle with center at origin and radius 3, mod z=3, okay. Now let us notice that here f(z)=e to the power zt/z square* this is z square+2z+2 and z square+2z+2 if you put equal to 0, then you get z=-2+/- under root b square means 4-4ac means 8/2, so we will get here -2, +/-4i, +/-2i, okay divided by 2.

So -1+/-i, so the roots of the denominator, okay denominator here is z square*z square+2z+2, the roots Rz=0 and z=-1+/-i, so let us plot them z=0 is here, -1+i is here, -1+i, this is -1 1 point, -1 1 point, okay. So this is 1 singularity and there is -1-1, so this is -1-1, okay so all the 3s. You can see the distance of -1 1, R-1-1, okay, whether you take this point or you take this point, distance from the origin is root 2, okay and the radius of the circle is 3.

So all the 3 singularities of f(z)=e to the power zt/z square*2z+2 lie inside c, okay.

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Let $f(z) = \dfrac{e^{zt}}{z^2(z^2+2z+2)}$

At $z=0$, $f(z)$ has a pole of order 2, and at $z=-1\pm i$, it has simple poles

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} z^2 \frac{e^{zt}}{z^2(z^2+2z+2)} = \frac{1}{2}$$

Let $z_1 = -1+i$, $z_2 = -1-i$

$z_1 - z_2 = 2i$

$z_1^2 = (-1+i)^2 = 1-1-2i$

$$\operatorname*{Res}_{z=-1+i} f(z) = \lim_{z\to z_1} \frac{e^{zt}(z-z_1)}{z^2(z-z_1)(z-z_2)} = \frac{e^{z_1 t}}{z_1^2(z_1-z_2)} = \frac{e^{(-1+i)t}}{(-2i)(2i)} = \frac{1}{4}e^{(-1+i)t}$$

$$\operatorname*{Res}_{z=-1-i} f(z) = \lim_{z\to z_2} \frac{e^{zt}(z-z_2)}{z^2(z-z_1)(z-z_2)} = \frac{e^{z_2 t}}{z_2^2(z_2-z_1)} = \frac{e^{(-1-i)t}}{(-1-i)^2(-2i)} = \frac{e^{(-1-i)t}}{(2i)(-2i)} = \frac{1}{4}e^{(-1-i)t}$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz} z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$= \lim_{z\to 0} \frac{e^{zt}\cdot t(z^2+2z+2) - (2z+2)e^{zt}}{(z^2+2z+2)^2} = \frac{2t-2}{4} = \frac{t-1}{2}$$

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \text{sum of residues at } z=0 \,\&\, z=-1\pm i = \frac{t-1}{2} + \frac{1}{4}\left[e^{(-1+i)t} + e^{(-1-i)t}\right] = \frac{t-1}{2} + \frac{e^{-t}}{4}\left(e^{it}+e^{-it}\right)$$

$$= \frac{t-1}{2} + \frac{1}{4}e^{-t}(2\cos t) = \frac{t-1}{2} + \frac{e^{-t}\cos t}{2} = \frac{(t-1)+e^{-t}\cos t}{2}$$

So we have let f(z) be equal to e to the power zt upon z square*z square+2z+2, okay. So at z=0, f(z) has a pole of order 2 and at z=+/-z-1+/-i and at z=-1+/-i at z=-1+/-i it has simple poles. See z=, if you multiply f(z)/z square and take the limit as z tends to 0, you get what limit z tends to 0, z square*f(z) gives you. So e to power zt becomes e to power 0 that is 1/2. So it has a pole of order 2 and at z=-1+/-i, the denominator has simple 0s, so f(z) has simple poles.

Now let us find the residue at each of these points because all the 3 singularities, singularity at z=0 and z=-1+/-i lie inside c, okay. So first we find the residue at the simple poles, residue at z=-1+i, okay of f(z), we can find. So this is equal to e to the power z. Now in this case we can even, so we can write z square*z, let me write 1 root as z1 the other root as z2. Let us say z1=-1+i and z2=-1-i. So z-z1*z-z2 I write for z square+z 2z+2 and here I multiply this by z-z1, okay.

And take the limit at z tends to z1, okay z1 I take as -1+i so this will be e to the power z1t, okay, this cancels with this divided by z1 square*z1-z2, okay. Now z1-z2=2i, so this is e to the power -1+i*t/z1 square, z1 square is (-1+i) square. So this is (1+i) square, so 1-1-2i, so we get -2i, so -2i*z1-z2 is 2i i square is -1, so this is 1/4 e raised to the power -1+i*t and residue at z=-1-i similarly will be limit z tends to z2 e to the power zt*z-z2/z square *(z-z1) (z-z2).

This cancels and we get e to the power z2t*z2 square z2-z1. So z2 is -1-i so e to the power -1-i*t/(z-1-i)square and we get z2-z1 as -2i. So this is e to the power -1-i*t/this is (1+i)square so 2i*-2i and what we get is 1/4 e to the power -1-i*t. So we found the residue at z=-1+i and -1-i. Now let us find the residue at z=0. So residue at z=0 of f(z) will be equal to because it is a pole of order 2, so 1/2-1 factorial limit z tends to 0 d/d(z) of z square*f(z).

So e to the power zt/z square*z square+2z+2. So this will cancel and now we have to differentiate this expression with respect to z, so limit z tends to 0 and what we get e to the power zt when we differentiate with respect to z, e to the power zt*t we get and then we get z square+2z+2 and then we get derivative of z square+2z+2 is 2z+2*e to the power zt/z square+2z+2 whole square. So when you put, let z go to 0 means, e to the power zt goes to 1.

Then here we get t*this is 0, this is 0 2, so 2t and here we will get -2 because this is 0, this is 2 and this is e to the power 0, so 1 and then denominator is 2 square, so 4 we have. So t-1/2, okay. Now we add all these residues, okay because the value which we have to find is the value we have to find of 1/2pi i integral/c f(z) d(z), okay. So this is nothing but some of residues at z=0 and z=-1+/-i, so this is equal to 2-1/2 and then we have here 1/4 e to the power -1-i*t.

First we write this, e to the power -1+i*t+e to the power -1-i*t e to the power –t we can write outside so this is t-1/2, then e to the power –t and inside we get e to the power it+e to the power –it, okay. This is t-1/2 and then we have 1/4 e to the power –t*2 cos t, okay. I am treating t as a, it could be even a complex number, cos z is e to power iz+e to the power –iz, okay. So this is nothing but t-1/2+e to the power –t cos t/2. So t-1+e to the power –t cos t/2.

This is the answer, okay. So this is how we find this contour integration. With this, I would end my lecture. Thank you very much for your attention.