Advanced Engineering Mathematics
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Lecture - 18
Residue at a Singularity

Hello friends. Welcome to my lecture on Residue at a Singularity. The behaviour or singularities into poles and essential singularities.

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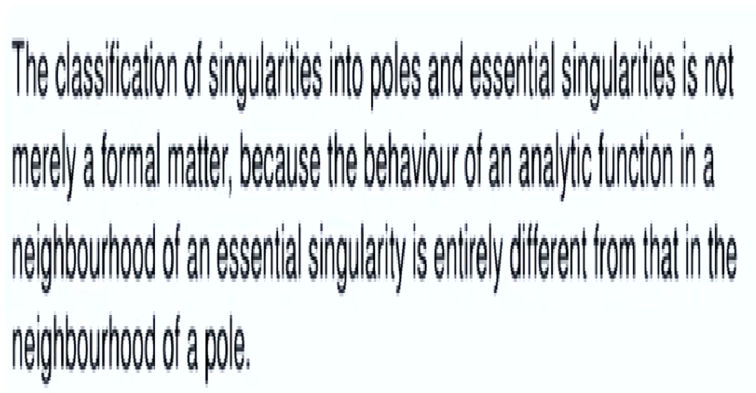

The classification of singularity into poles and essential singularities is not merely a formal matter. Because the behaviour of an analytic function in a neighbourhood of an essential singularity is entirely different from that in the neighbourhood of a pole. Let us see how is the nature of the function in the neighbourhood of a pole. Suppose f(z) is analytic and has a pole at z=z0, then mod of f(z) tends to infinity at z tends to z0 in any manner.

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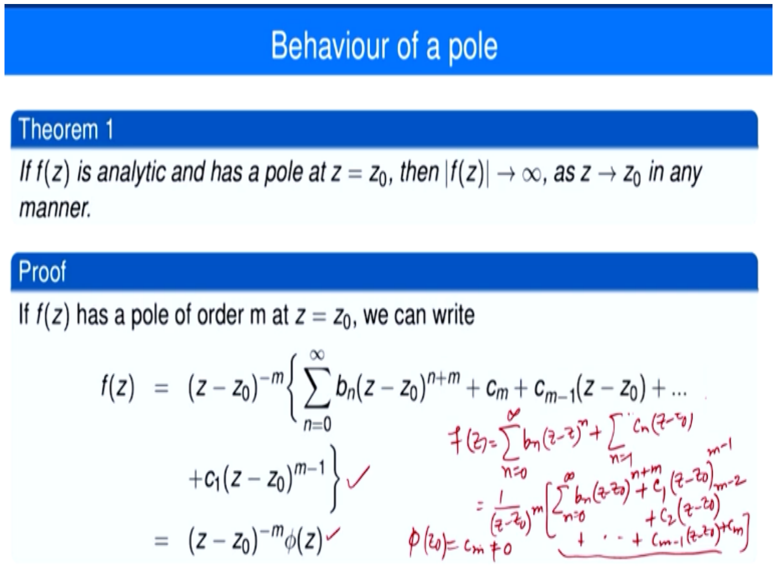

That means in a neighbourhood of z=z0 f(z) becomes indefinitely large. So if f(z) has a pole of order m at z=z0, then we can write f(z) like this, because if f(z) has a pole of order m at z=z0, then by Laurent series f(z)=sigma n=0 to infinity bn (z-z0) to the power n+sigma n=1 to m cn (z-z0) to the power –n. So here this is nothing but z1/z-z0 to the power m and then we shall have sigma n=0 to infinity bn(z-z0) to the power (n+m)+c1*(z-z0) to the power.

We have here c1(z-z0) to the power -1. So this will be equal to m-1, okay c2 (z-z0) to the power m-2 and so on cm-1 (z-z0) +cm, okay. So we can write like this, okay. If z has a pole of order m at z=z0. Now this is equal to (z-z0) to the power –m*Phi z and Phi z is this Taylor series. You can see here when n=0, we have b0 (z-z0) to the power m here. So we can start with this cm, cm-1, z-z0, then cm-2 (z-z0) square, then c1 (z-z0) to the power m-1.

Then the term z-z0 to the power m, which is b0 (z-z0) to the power m, then b1 (z-z0) to the power m+1 and so on. So this is the expression inside the square brackets is a Taylor series of some function, which we denote by Phi z.

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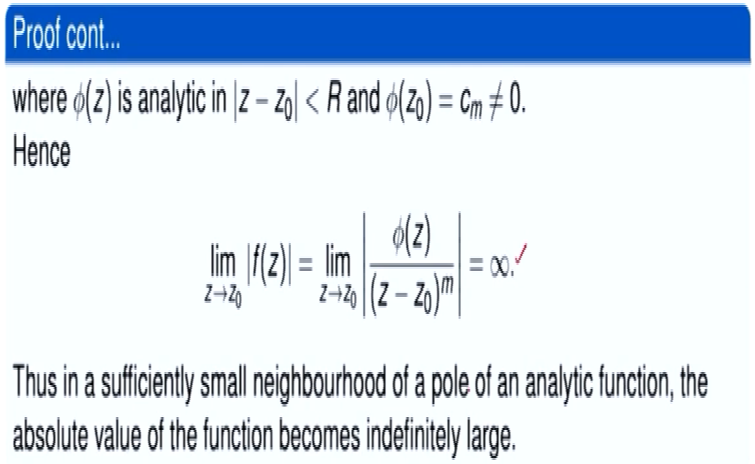

Now Phi z is also, you can see Phi z =z0=cm, which is non-zero, because f(z) has a pole of order m at z-z0. So 5z is analytic and mod of z-z0<R and Phi z0 =cm where cm is non-zero. Now we can take the limit of mod of f(z) as z tends to z0, okay. Mod of f(z)= mod Phi z /mod of(z-z0) to the power m, so f(z) goes to z0. Phi z is a continuous function, because it is analytic in this region. So it is continuous and also Phi z is not equal to 0.

So f(z) tends to z0, we can see that this expression mod of Phi z /z-z0 to the power m goes to infinity. So when f(z) has a pole of order m in a neighborhood f(z) in a neighbourhood of z=z0, f(z) becomes arbitrarily large, okay, so indefinitely large. So thus in a sufficiently small neighbourhood of a pole of an analytic function, the f th root value of the function becomes indefinitely large.

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Limit point of zeros and poles

Theorem 2

If $f(z)$ is an analytic function in a domain D and if $z_1, z_2, \ldots$ is a sequence of zeros of $f(z)$ in D having a limit point in the interior of D, then $f(z) \equiv 0$ in D.

Proof

Let z_0 be the limit point of the sequence of zeros $z_1, z_2, \ldots$ of $f(z)$. Let z_0 be an interior point of the domain D in which $f(z)$ is analytic. Then, since $f(z)$ is continuous at z_0, $f(z_0) = 0$. Since z_0 is a limit point of the sequence $z_1, z_2, \ldots$ every neighbourhood of z_0, no matter how small must have infinitely many points of the sequence. This implies that $z = z_0$ is not an isolated zero of f(z). Hence either f(z) is not analytic or $f(z) \equiv 0$. Since f(z) is given to be analytic, we have $f(z) \equiv 0$.

Now suppose, let us look at the limit point of zeroes and poles. If f(z) is an analytic function in a domain D and z1, z2, is a sequence of zeroes and f(z) and D having a limit point in the interior of D, then f(z) is identically 0 in D. So let us consider that z0 be the limit point of the sequence of zeroes, z1, z2 and so on of f(z). Since z0 is the interior point of the domain D, okay, it follows that f(z) is continuous at z0, okay, f(z) is analytic inside D.

And therefore it is continuous inside D and z0 is a point in the interior of D, so f(z) is continuous at z0 and therefore f(z0) must be 0. Now let us see how f(z0)=0.

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$z_1, z_2 \ldots$ is a sequence of zeros of $f(z)$
and z_0 is the limit point of $z_1, z_2 \ldots$
$f(z)$ is continuous at z_0
$\Rightarrow$ for a given $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$
$|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ — (1)
For $\delta > 0$ we can find an integer n_0 such that
$|z_n - z_0| < \delta, \ \forall\, n \geq n_0$ — (2)
From (1) & (2)
$|f(z_n) - f(z_0)| < \varepsilon \quad \forall\, n \geq n_0$
$\Rightarrow |0 - f(z_0)| < \varepsilon \quad \forall\, n \geq n_0$
$\Rightarrow |f(z_0)| < \varepsilon$
$\Rightarrow f(z_0) = 0$ since $\varepsilon > 0$ is arbitrary

See we have z1, z2, and so on. This is the sequence of zeroes of f(z), okay and z0 is the limit point of z1, z2 and so on. This is the sequence. Now f(z) is continuous at z0 implies that for a given epsilon>0 there exist a positive number delta, such that mod of f(z)-f(z0)<epsilon whenever mod of z-z0 is less than delta, okay. Now for delta>0, okay for delta>0, we can find and integer n0, such that mod of zn-z0<delta for all n>=n0, okay.

So what will happen, mod of zn-z0 is < delta for all n > R=n0. Let us combine the equation 1 and 2, okay. From 1 and 2, we find that mod of f(zn)-f(z0)<epsilon for all n>=n0, because all these points zn, for which n>=n0, they are at a distance delta less than delta from z0 and so by the definition of continuity mod of f(zn)-f(z)<epsilon. This implies that mod of 0-f(z)<epsilon for all n>=n0, okay. So this we mean that mod of f(z0)<epsilon, okay.

So this means that f(z0)=0 since epsilon>0 is arbitrary and thus we can say that because of the continuity of f(z), f(z)=z0 and z0 being the limit point of the sequence of zeroes at z1, z2, and so on, it follows that f(z0) =; so this f(z0)=0, now since z0 is a limit point of this sequence z1, z2 and so on, okay. So every neighbourhood of z0, howsoever a small be it, will have infinitely many points of the sequence z1, z2 and so on.

This implies that z=g(z0), okay, is not an isolated zero of f(z), okay. It is not an isolated zero of f(z) and therefore f(z) must be identically = 0. So if f(z) is an analytic function and z1, z2, zn is a sequence of zeroes, having a limit point in the interior of D, then f(z) must be identically 0.

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From the above theorem it follows that the limit point of the sequence of zeros of a non zero function, analytic in a domain D, can not be an interior point of D. If it is an interior point of D then either $f(z) \equiv 0$ or it can not be analytic at the limit point. If the sequence of zeros of a non-zero analytic function is a singularity of f(z), it is an isolated singularity. Since, it is not a pole, it is an isolated essential singularity.

Example 3

The function $f(z) = \sin\frac{1}{z}$ has zeros at $z = \frac{1}{n\pi}$, $n = \pm 1, \pm 2, \ldots$. The limit point of these zeros is the point $z = 0$. Hence $z = 0$ is an isolated essential singularity of $f(z)$.

$\sin\frac{1}{z} = 0 \Rightarrow \frac{1}{z} = n\pi$, $n = \pm 1, \pm 2 \ldots$

$\Rightarrow z = \frac{1}{n\pi}$, $n = \pm 1, \pm 2 \ldots$

The limit point of this sequence is $z=0$

Now from the above theorem, it follows that the limit point of the sequence of zeroes of a non-zero function, which is analytic in a domain D cannot be an interior point of D. If it is interior point of D, then f(z) must be identically 0. So if the sequence of zeroes of a non-zero analytic function is a singularity of f(z), it is an isolated singularity. No this isolated singularity cannot be a pole, okay.

Because in the neighbourhood of, in any neighbourhood of this singularity, f(z) is not becoming indefinitely large. Every neighbourhood of z=this singularity at z=say z0, okay, we have infinitely many zeroes of the function f(z), okay. Since f(z) does not become indefinitely large in any neighbourhood of this singularity at z=say z0, it follows that it is a not pole. Therefore, it must be an isolated essential singularity.

So to discard this case, that this is not a pole, we are using this theorem, okay, because poles are isolated singularities. So here we are finding an isolated singularity, but that isolated singularity cannot be a pole. If it is a pole, then in an arbitrarily small neighbourhood of z=z0, f(z) must tend to infinity as z tends to z0, but here what do we notice, howsoever a small neighbourhood of z0 be there, there are infinitely many zeros of the function f(z).

And therefore f(z) tends to infinity is not possible. So the singularity at z=z0 cannot be a pole, it must be an isolated essential singularity. So this means that if the analytic function f(z) is non-zero, then the sequence of its zeros will always tend to an isolated essential singularity of the function. So for example, let us consider this f(z)=sin 1/z. it has zeros at z=1/n pi, where n takes values +/-1, +/-2, and so on. You can see sin 1/z=0 means 1/z=n pi, okay.

So where n takes values +/-1, +/-2 and so on. So z=1/n pi where n=+/-1, +/-2 and so on. Now let this sequence z=1/n pi, okay for this sequence, the limit point is z=0, okay. Now since sin 1/z okay, sin 1/z is a non-zero analytic function, therefore z=0 must be an isolated essential singularity of sin 1/z.

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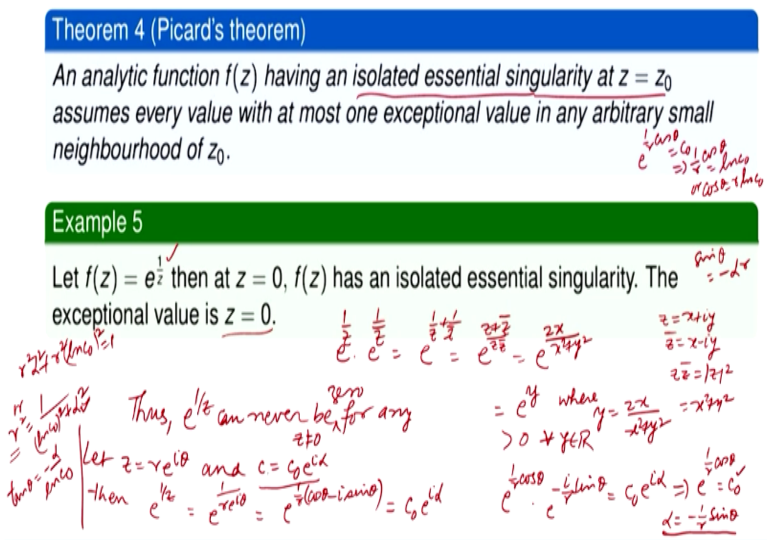

Now let us go to Picard's theorem, which tells us the nature of an analytic function in a neighbourhood of n isolated essential singularity. So an analytic function f(z) having an isolated essential singularity at z=z0 assumes every value with at most one exceptional value in any arbitrary small neighbourhood of z0. So it means that if f(z) has an isolated essential singularity at a point z=z0, then it will assume every value except at the most one value, okay, in an arbitrarily small neighbourhood of z0.

Let us illustrate this Picard's theorem for the function f(z)=e to the power 1/z. We are going to show that this f(z)=e to the power 1/z, which has an isolated essential singularity at z=0 assumes every value in an arbitrarily small neighbourhood of z=0, except just one value and that value is z=0. We can see that e to the power 1/z is never 0 for any complex number z. To show that, let us consider f(z), let us multiply e to the power 1/z/e to the power 1/z conjugate, okay.

Then this is into the power 1/z+1/z conjugate, okay, which is equal to e to the power z+z conjugate/z conjugate, okay and if z=x+iy, z conjugate will be x-iy. So this will be e to the power 2x/mod of z square. Z conjugate is mod of z square, so this is x square+y square, okay, so x square+y square and z is a non-zero complex number. So this e to the power 2x/x square+y square. This is the real function, okay 2x/x square+y square is a real number, okay.

So this is e to the power y where y is 2x/x square+y square and e to the power y, we know, e to the power y is always is >0 for every y, okay for every y belonging to R, okay. So e to the power z, thus e to the power 1/z can never be 0 for any z, for any complex number z, for any z0=0, okay. Now let us show that it assumes every value, okay, in an arbitrary small neighbourhood of z=0, okay, except of course can never be 0, okay.

Now let us say let z be equal to Re to the power i theta and c=c0 e to the power i alpha be any complex number, okay c=c0 e to the power i alpha where c is not zero be any complex number, then e to the power 1/z=e to the power 1/Re to the power i theta will be equal to e to the power 1/R cos theta-i sin theta, okay e to the power 1/R*e to the power –i theta is e to the power 1/R cos theta-i sin theta. Let us put it equal to c0e to the power i alpha, okay.

This c0e power i alpha, so then equating the absolute values and arguments on both sides, you see we have e to the power 1/R cos theta, left side is e to the power 1/R cos theta*e to the power-i/R sin theta=c0 e to the power i alpha. So taking absolute values both sides, we find that e to the power 1/R cos theta=c0 and equating the arguments what we get alpha=-1/R sin theta, okay. So now this equation gives you e to the power 1/R cos theta=c0.

E to the power 1/R cos theta=c0 implies that 1/R cost theta=lnc0, okay and alpha=-1/R sin theta gives you what or we can say cos theta=Rlnc0, okay and sin theta=-alpha R, okay sin square theta+cos square theta=1, so we can say that R square alpha square+R square lnc0 square=1or we can say R square=1/lnc0 square + alpha square and tan theta=-alpha R/Rlnc0, so tan theta=-alpha/lnc0, okay. Now let us see R can this R, R is the modulus of z, okay.

This R can be made arbitrarily small by adding multiples of 2pi to this alpha. Alpha is the argument of this complex number c and adding multiples of 2pi to alpha, this c will remain unaltered, okay. So we notice that we obtain.

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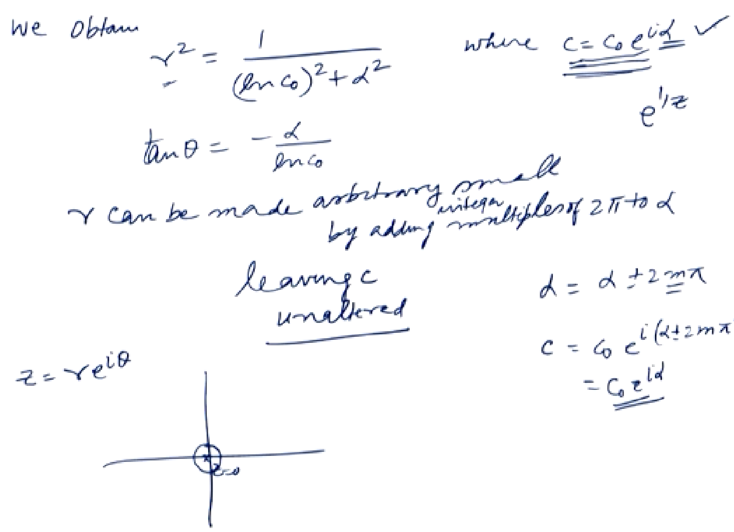

R square=1/lnc0 square+alpha square and tan theta=-alpha/lnc0, okay where we have taken c=c0e to the power i alpha, okay. Now what we notice here is that R can be made arbitrarily small by adding integer multiples of 2pi to alpha, okay. So we can replace alpha by alpha+2npi, okay where n is an integer, okay, but that will not change the value of c, c will remain c0e to the power i alpha+(-2npi) if you take, it will remain c0e to the power i alpha.

So without changing the value of this c, so c=c0e to the power i alpha again. So without, so R can be made arbitrarily small by adding integer multiples of 2pi to alpha, but c remains, leaving c unaltered, okay. Now what we have seen. You take any complex number c=c0e to the power i alpha where c is of course non-zero, okay then in an arbitrarily small neighbourhood of z=0, okay. See this Rz we have taken as Re to the power i theta, okay.

So in an arbitrarily small neighbourhood of z=0, because when you add integer multiples of 2pi to alpha, then this R will go on becoming small, so R can be made arbitrarily small by adding integer multiples of 2pi to alpha and this gives you a neighbourhood of z=0. So in an arbitrarily small neighbourhood of z, okay=0, e to the power 1/z assumes is an arbitrary complex number. This is an arbitrary complex number other than 0, okay. Now let us go to residue at a singularity.

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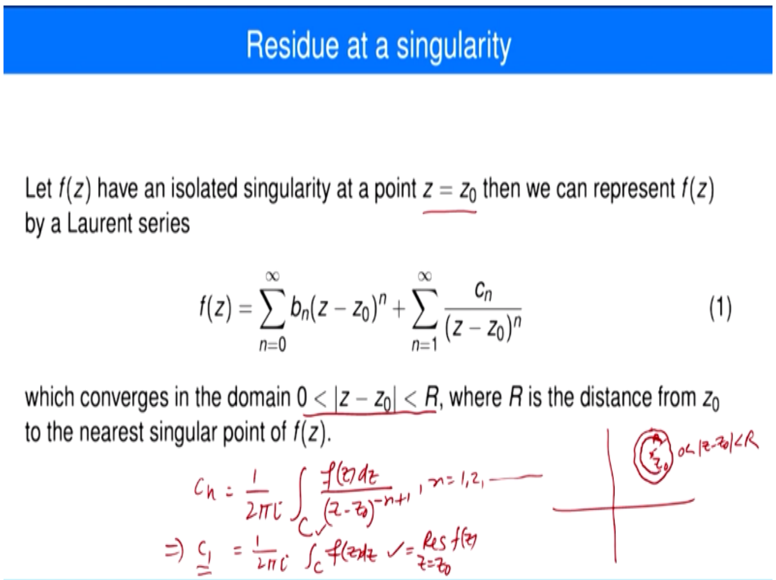

Let f(z) have an isolated singularity at a point z=z0. Isolated singularity means you can find a neighbourhood of z=z0 in which f(z) does not have any other singularity, okay. Then, f(z) can be represented by Laurent series f(z)=sigma n=0 to infinity bn(z-z0) to the power n+sigma n=1 to infinity cn/z-z0 to the power n. This series converges in a deleted neighbourhood of z0. So if you have z0 point here, okay.

Then you can find the neighbourhood of z0, okay, which is a deleted neighbourhood, okay. R is the distance of z0 from the nearest singularity of f(z). So in this neighbourhood f(z) can be represented by the Laurent series given by this.

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From the Laurent series expansion, we find that the coefficient c_1 of $\frac{1}{z-z_0}$ in this development is

$$c_1 = \frac{1}{2\pi i}\int_C f(z)dz$$

or $$\int_C f(z)dz = 2\pi i c_1, \qquad (2)$$

where the integration being taken in the counter-clockwise sense around a simple closed path C which lies in the domain $0 < |z - z_0| < R$ and contains the point $z = z_0$ in its interior. The coefficient c_1 in the development (1) of $f(z)$ is called the residue of $f(z)$ at $z = z_0$. We shall denote it as $c_1 = Res_{z=z_0} f(z)$.

From the Laurent series expansion, we find that the coefficient c1 of 1/(z-z0)=1/2pi i integral/c f(z) d(z) or we can say integral over cf(z)d(z)=2 pi ic1. Let us call that in the Laurent series cns are given by 1/2pi i integral/c f(z)d(z)/(z-z0) to the power –n+1, okay n taking values 1, 2, 3 and so on, okay. So here if you take n=1, here taking n=1 what do you get 1/2pi ic1, okay gives you 1/2pi i integral/c f(z)d(z), okay. So when you take n=1 here, you get the value of c1.

This c1 is called the residue of f(z) at z=z0. Residue of, can be denoted like this, residue of f(z) at z=z0, okay. This is coefficient of 1/z0 you can see. This is c1/z-z0 like that. The coefficient of 1/z-z0 is c1 here, okay and the c is the simple closed curve, c is any simple closed curve, which lies in the annular region this 1, annular region and in circles z0 that is z0 lies in its interior. So it can be taken any simple closed curve like this, okay, which lies in the annular region 0<mod of (z-z0)<R and encircles the inner circle, point circle.

So we can say this is integral/c f(z)d(z)=2pi ic1 where the integration being taken in the counter clock by sense around a simple close past c, which lies in the region 0<mod of (z-z0)<R and contain the point z=z0 in its interior. The coefficient c1 in the development 1 of f(z), in this development the coefficient of c1, the coefficient c1 is called the residue of f(z) at z=z0 be denoted by residue of f(z) at z=z0 like this.

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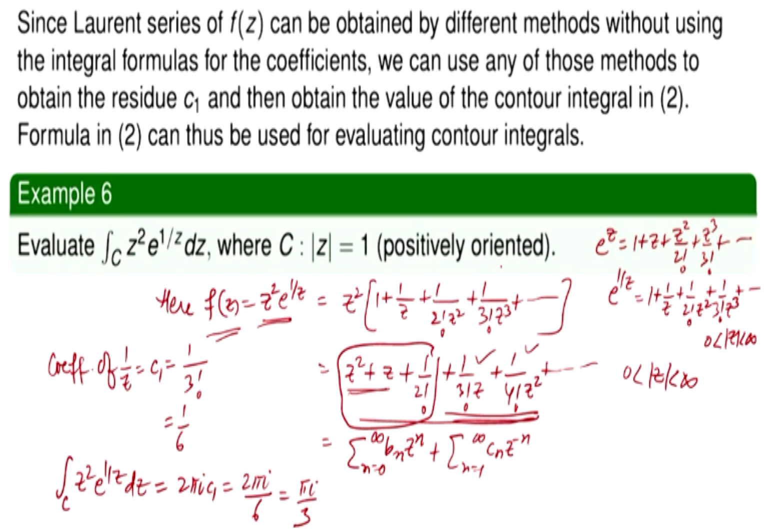

Now Laurent series of f(z) can be obtained by different methods without using the integral formulas for the coefficients. You know, we have the integral formulas for the coefficients bn and cn. For cn we have this formula and for bn we have the formula 1/2pi i integral/c f(z)d(z)/z-z0 to the power n+1 where n=0, +/-1, +/-2 and so on, okay. So Laurent series of a function f(z) are obtained by different methods without using rarely the formulas for the coefficients are used.

Okay, we use alternate methods to find the Laurent series of f(z) in a given annular region. So we obtain the Laurent series of the function f(z) by any alternate method and then we determine the coefficient of 1/z-z0 in that, which gives us the value of c1. Then the integral of f(z)d(z) around the curve c is given by 2pi i*c1, okay. We can use that Laurent series to determine the contour integral. So for example, let us consider this function.

Integral/c f(z) square e to the power 1/z here f(z)=z square e to the power 1/z, which we can write as z square*let us write the expansion of e to the power 1/z. We know that e to the power z is 1+z+z square/2 factorial z cube/3 factorial and so on. So replacing z by 1/z we get e to the power 1/z as 1+1/z 1/2 factorial z square 1/3 factorial z cube and so on, which converges in the region 0<mod of z<infinity. So here 1+1/z 1/2 factorial z square 1/3 factorial z cube and so on.

We get this. So this is z square+z+1/2 factorial 1/3 factorial*z 1/4 factorial*z square and so on we get, okay. Now this is the Laurent series of the function f(z) in the region z<mod of

z<infinity, okay. z=0 is the only singularity of f(z) and this is an essential singularity you can see, because the principle part, this is the part, which is sigma this part. This part is corresponding to sigma n=0 to infinity bn z to the power n.

Here we are taking z0=0 and this part, okay corresponds to sigma n=1 to infinity cnz to the power –n, okay. So the principle part contains infinitely many terms and therefore the singularity at z=0 is n essential singularity. So f(z)=z square/1/z has an isolated essential singularity at z=0 and the coefficient of, now let us find the coefficient of 1/z-z0, z0 is 0 here. So coefficient of 1/z=c1=1/3 factorial, which is 1/6, okay.

So integral/cz square e to the power 1/zd(z) will be equal to 2pi ic1, okay. So 2pi i/6 and we get pi i/3, okay, but the integral around c is being taken in the counter clock by chance. Now let us take one more example.

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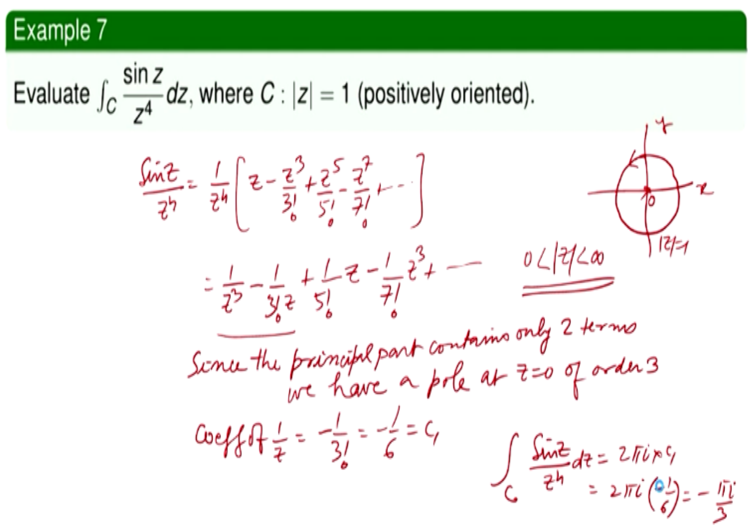

Let us take one more example, say integral/c sin z/z to the power d(z), okay. You can see that sin z/z to the power 4 can be expressed as 1/z to the power 4 and we know the expansion of sin z (z-z cube/3 factorial z5/5 factorial and so on and this is therefore z/z cube, so 1/z cube-z/3 factorial and then we have 1/3 factorial*z, okay. Then 1/5 factorial*z, okay and then we have -1/7 factorial into z to the power 3 and so on. Now here you can see.

The principle part contains only finitely many terms. There are only 2 terms in the principle part 1/z cube-1/3 factorial z, okay. Since the principle part contains only finitely many terms we have a pole at z=0. We have a pole at z=0 of order 3. We have -1/3 factorial*1/z+1/z cube. So it is of order 3, the coefficient of 1/z=-1/3 factorial that is -1/6. So this is the value of c1 and therefore the integral/c or sin z/z to the power 4, okay d(z)=2pi i*c1, which is 2pi i*-1/6.

And this is –pi i/3 and you can notice that the region of convergence is 0<mod z<infinity. Mod z=1 is the circle, this 1. This is mod z=1, okay. So this is 0, okay, mod z=1 is the simple closed curve, which lies in this region, okay 0<mod z<infinity and contains the point z=0 in its integral, same is the case here in the previous example, the region of convergence is 0<mod z<infinity and mod z=1 lies in this annular region and encircles the point z=0, this point.

This is mod z=1, okay. This is how we can use the Laurent series to determine a contour integration. In our next lecture, we shall study the residue theorem, which will be an extension to of this case. Here, we have only 1 singularity inside the simple closed path c. So this method will be extended to the case where the curve c contains finite number of singularities inside it, okay and then it is a very simple and elegant method to determine the contour integration in case the curve c has infinite number of isolated singularities inside it.

So with this I would like to end my lecture. Thank you very much for your attention.