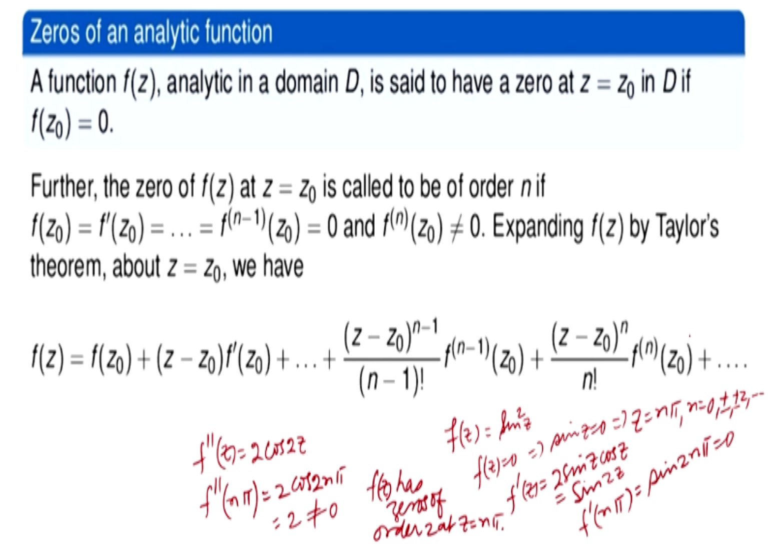
Advanced Engineering Mathematics
Prof. P.N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 17
Zeros and Singularities of an Analytic Function

Hello friends. Welcome to my lecture on Zeros and Singularities of an Analytic Function. Suppose we are given an analytic function f(z) in a domain D, then it is said to have a zero at z=z0 in D if the value of f at z=z0 is 0.

(Refer Slide Time: 00:46)



Now the zero of f(z) at z=z0 is said to be of order n if f(z0) f prime z0 and n-1 of the derivative of f(z) at z=z0 is 0 and n-th derivative of f(z) at z=z0 is non-zero. Say for example, we can consider f(z)=say sin square z, okay, then we know that f(z)=0 gives sin(z)=0, which means that z=n pi, okay, n=0, +/-1, +/-2, and so on. Now if you take the derivative so that at z=n pi, f(z) has zeros. Now let us find f prime z, f prime z is 2 sin(z)*cos(z), which is equal to sin (2z), okay.

So again at z=n pi f prime n pi=sin 2 n pi. So this is equal to 0. But if you find f double prime z, f double prime z is 2*cos 2z, so then f double prime n pi will be equal to 2 cos 2 n pi, okay and cos 2n pi=1, so we will get f double prime n pi=2, which is non-zero. So we can say that f(z)=sin square z has zeros of order 2 at z=n pi, okay. So f(z) has zeros of order 2 at z=n pi where n=0, +/-1, +/-2 and so on. Now if you write the Taylor series expansion of f(z) about the point z=z0.

Then we can write f(z)=f(z0) + z-z0 f prime z0+z+z-z0 to the power n-1/n-1 factorial fn-1(z0) + z-z0 to the power n/n factorial fn(z0) and so on. Now here if your function f(z) has a zero of order n, then f(z0) will be zero, f prime z0 will be 0, fn-1(z0) will be zero. So first n terms will vanish and we will have f(z)=z-z0 to the power n/n factorial fn(z0), then z-z0 to the power n+1/n+1 factorial*fn+1(z0) and so on.

(Refer Slide Time: 03:35)

If $f(z)$ has a zero of order n at $z = z_0$ then we have

$$\begin{aligned} f(z) &= (z - z_0)^n \left[\frac{f^{(n)}(z_0)}{n!} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z - z_0) + \dots \right] \\ &= (z - z_0)^n g(z) \end{aligned}$$

where $g(z)$ is a analytic in $|z - z_0| < R$ and $g(z_0) = \dfrac{f^{(n)}(z_0)}{n!} \neq 0.$

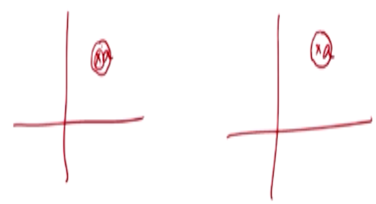
So there we can take z-z0 to the power n common and we will get f(z)=z-z0 to the power n*fn(z0)/n factorial fn+1(z0)/n factorial z-z0n so on. Now the bracketed expression we can denote by the function g(z). It is a function of z. we can write it as g(z). Now g(z) given by this power series in z-z0. So it is an analytic function and this analytic function has the region of convergence of the Taylor series of f(z) about z=z0, okay.

So g(z) is analytic in the region mod of z-z0 less than R and you can see that if you write z0 for z in the bracketed expression, then g(z0) becomes fn+fn(z0)/n factorial, okay. g(z0) becomes fn(z0)/n factorial and fn(z0) is not equal to 0, okay. fn(z0) is not equal to 0, so g(z0) will be non-zero and therefore, if an analytic function f(z) has a 0 of order n at z=z0, then it can already be represented as f(z)=z-z0 to the power n*g(z) where g(z) is analytic in the region mod of z-z0 < R, where R is the distance of the point z0 from the nearest inlet of f(z).

(Refer Slide Time: 05:07)

Isolated point

A point *a* of a point set *S* is called an isolated point of *S* if there is a neighbourhood of *a* which contains no other point of *S* except *a*. On the other hand if every neighbourhood of *a* contains a point of *S* other than *a* (hence infinitely many points of *S*), the point *a* is said to be a limit point of *S*. Note that, *a* may be, but need not be, a point of *S*.

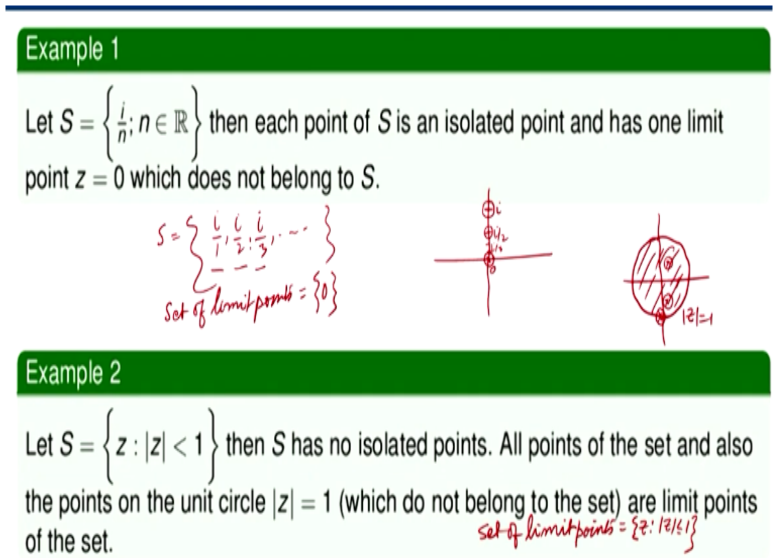


Now let us discuss isolated point. A point A, okay a point A of a point set S is called an isolated point of S if there is a neighborhood of A, which contains no other point of S except A. So if you take in the complex, then suppose this is your point A, it will be called an isolated point if you can find a small neighborhood of this point, okay, which contains no other point of A, except A, which contains no other point of S except A.

On the other hand, if every neighborhood of A contains a point of S other A, hence infinitely many points. Suppose in the other situation, suppose, this is your point A, you take any neighborhood of z=A, okay. It contains a point of S other than A, then the point A is called the limit point of S. Now if it contains a point of S other than A, you can take even a smaller neighborhood. That will also contain a point of S other than A.

You take still smaller neighborhood of A, okay it contains another point of S except A. So any neighborhood of z=A contains infinitely many points of S. So then the point A is called limit point A, which is the limit point of S may or may not belong to S.

(Refer Slide Time: 06:28)



For example, let us consider S=i/n where n belongs to n, n is a natural number. So S contains the points i/1, i/2, i/3, and so on, okay. Now if you plot these points in the complex plane, then suppose this is i, this is i/2, then i/3, then i/4 and so on, okay. So each point of S is an isolated point. You can always find a small neighborhood of the point i or small neighborhood of the point i/2, okay, such that it does not contain any other point of S.

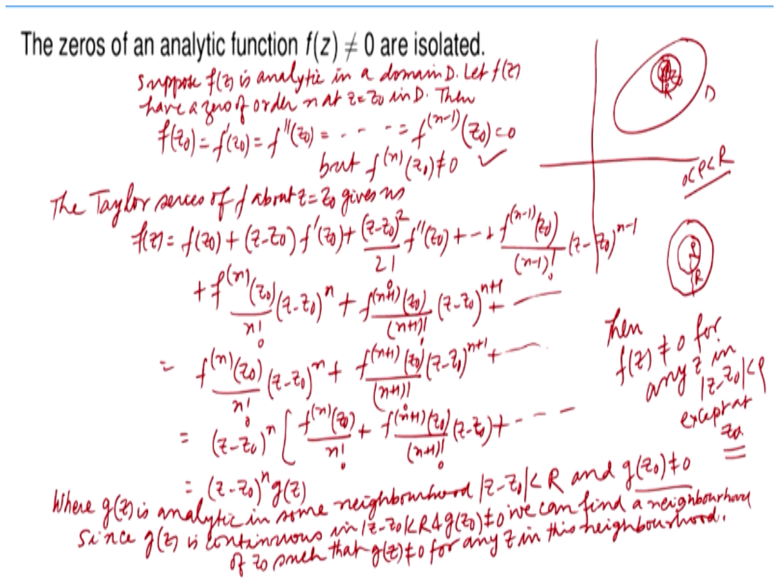
So each point here, okay, is an isolated point of S, but z=0 is the limit point of S because how sober a small neighborhood of z=0 you take, infinitely many members of S will belong to that neighborhood, okay. So z=0 is the limit point of S and you can see that z=0 does not belong to S. Now if you take S=set of all elements z, such that mode of z<1. Then, this is your circle mode z=1, so interior of the circle we are considering, okay.

Interior of the circle is the region given by mode z<1, then S has no isolated points. You take any point here, okay, you cannot find a small neighborhood of this point, which contains no other point of S, okay. So this set S has no isolated point and all points of this set and also the points on the boundary that is mode z=1, they are limit points of this set, because you take any point in the interior of mode z=1.

How sober a small neighborhood you take, it will contain infinitely many points of S and if you take any point on mode z=1, you can always how sober a small neighborhood of that point you take, it will contain again infinitely many points of S. So set of limit points of S will be the set of limit points of S here with the set of all z such that mode z is <=1. You can see the set of limit points is bigger than the set S itself, okay.

And here set of limit points is singleton set 0, set of limit points is equal to singleton set 0, okay. Each point here is an isolated point, but here no point is an isolated point, okay.

(Refer Slide Time: 09:24)



Now let us show that 0s of an analytic function f(z) are isolated. Suppose f(z) is analytic in a domain D, okay. Suppose this is your domain, okay z0 is any point in D, okay. For that let f(z) have a 0 of order n at z=z0 in D. Then, f(z0), f prime z0, f double prime z0, and so on fn-1(z0)=0, but fn(z0) is non-zero, okay.

The Taylor series of f about z=z0 gives us f(z)=f(z0)+z-z0 f prime z0 z-z0 whole square/2 factorial f double prime z0 and so on fn-1(z0)/n-1 factorial z-z0 to the power n-1+fn(z0) n factorial z-z0 to the power n+fn+1(z0)/n+1 factorial z-z0 to the power n+1 and so on, okay. Now using these conditions, okay, we get fn(z0)/n factorial z-z0 to the power n+fn+1(z0)/n+1 factorial z-z0 to the power n+1 and so on.

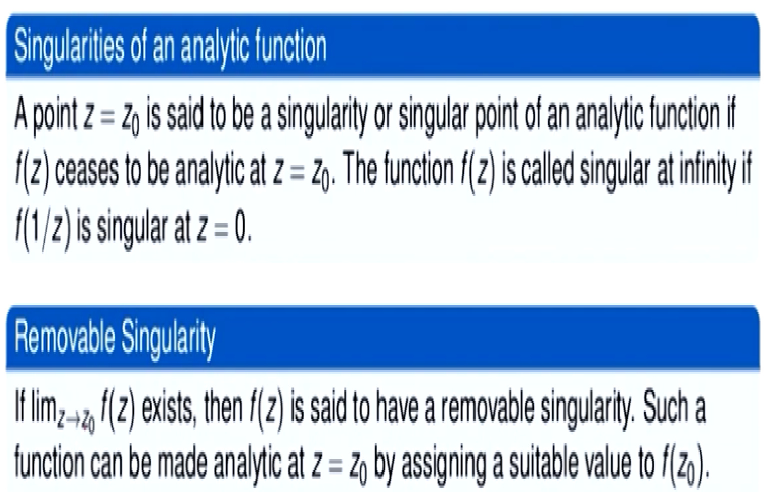
And we can then write z-z0 to the power n*fn(z0)/n factorial fn+1(z0)/n+1 factorial z-z0 and so on, okay. Now this I can write as z-z0 to the power n*g(z), then where g(z) is an analytic function because it is given by this power series g(z) is analytic and some neighborhood of mode of and some neighborhood of z0, okay. In some neighborhood mod of z-z0<R, okay and g(z0)S not equal to 0, okay, because g(z0)=fn(z0)/n factorial and fn(z0) is non-zero.

So this is not equal to zero. Now so we will get some neighborhood of z0, in which g(z) is analytic and g(z0) is not equal to 0. Now g(z) is an analytic function, so what will happen. It is continuous, okay. Since g(z) is continuous, okay and mod of z-z0<R and g(z) is not equal to 0, okay. We can find in neighborhood of z0 such that g(z) is not equal to 0 for any z in this neighborhood by continuity, okay. So we can find a neighborhood of this z0.

In which let us take its radius to be rho, okay. So 0<rho<R, okay. This radius is earlier, the radius was R, okay. Now we can get rho, okay like this. This is z0, this is radius rho and this is R, okay. So we can get neighborhood of z0, say of radius rho such that 0<rho<R and g(z) is not equal to zero for any z in this region, mod of z-z0<rho and then what will happen f(z)=z-z0 to the power n g(z), so f(z) will not be 0 for any z, except z=z0 in the region mod of z-z0<rho, okay.

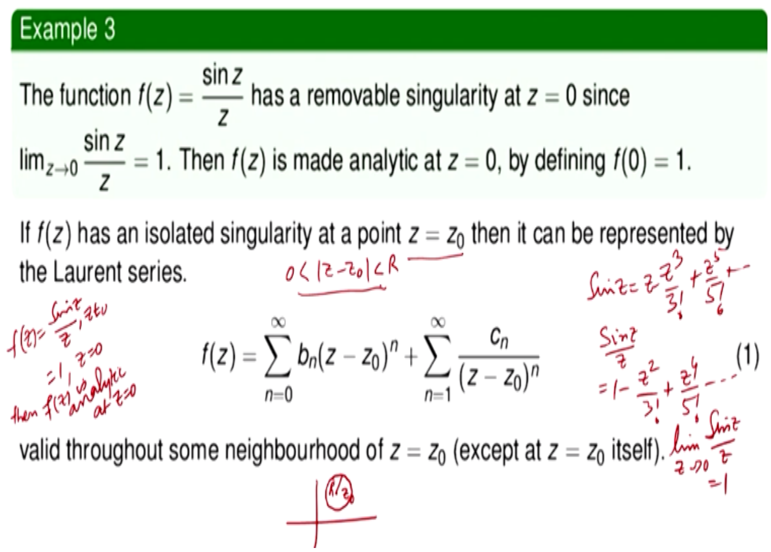
So then, f(z) is not equal to 0 for any z in mod of z-z0<rho except at z0 and so the zero of f(z) at z=z0 is an isolated 0, okay. So the zeros of an analytic function are isolated.

(Refer Slide Time: 16:25)



Now find z=z0 is set to be a singularity or singular point of an analytic function if f(z) ceases to be analytic at that point. The function f(z) is called singular at infinity if f(1/z) is singular at z=0. Now let us first consider removable singularity. If f(z) the limit of f(z) at z tends to z0 exist, then f(z) is said to have a removable singularity, such function can be made analytic at z=z0 by assigning a suitable value to f(z0).

(Refer Slide Time: 16:59)



Say for example, let us consider the function f(z)=sin z/z. It has a removable singularity at z=0, because when limit z tends to 0, sin z/z, okay sin z can be expanded as z-z cube/3 factorial+z 5/5 factorial and so on. So sin z/z=1-z square/3 factorial+z four/5 factorial and so on and therefore limit z tends to 0, sin z/z will be equal to 1. This limit is equal to 1. So we can make the function f(z)=sin z/z analytic by assigning the value 1 to f(z) at z=0.

That means if you write f(z)=sin z/z when z is not zero and 1 when z=0, then f(z) is analytic at z=0. So the singularity of f(z) is said to be removable singularity because it can be removed by assigning a suitable value to the function f(z) at z=0. Now suppose f(z) has an isolated singularity at a point z=z0. Isolated singularity means we can find neighborhood of z=z0 in which there is no other singularity of the function f(z).

So let us say there is a neighborhood, okay. There is a neighborhood means you can find circular neighborhood of radius R in which there is no other singularity of the function f(z). So we can take, say this is you z0, isolated singularity means we can find a neighborhood of z=z0, say of radius R in which there is no other singularity of the function. So we can write the region as 0<mod of z-z0<R. Then this is the annular region.

So function f(z) can be represented by the Laurent series f(z)=sigma n=0 to infinity bn(z-z0) to the power n, sigma n=1 to infinity cn/z-z0 to the power n, which is valid in this region 0<mod of z-z0<R, R is the distance of z0 from the nearest singularity of f(z). Now the second term, this term.

(Refer Slide Time: 19:47)

The second term on the right hand side of (1) is called the principal part of $f(z)$ at $z = z_0$. If from some n on, all the coefficients c_n are zero, say, $c_m \neq 0$ and $c_n = 0$ for all $n > m$, then (1) reduces to the form

$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n + \frac{c_1}{(z - z_0)} + \frac{c_2}{(z - z_0)^2} + \ldots + \frac{c_m}{(z - z_0)^m}. \qquad (2)$$

In this case, where the principal part of $f(z)$ consists of finitely many terms, the singularity of f at $z = z_0$ is called a pole, and m is called the order of the pole. Poles of the first order are also known as simple poles.

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^{n+m} + c_1 (z-z_0)^{m-1} + c_2 (z-z_0)^{m-2} + \cdots + c_{m-1}(z-z_0) + c_m$$

$$\lim_{z \to z_0} (z-z_0)^m f(z) = c_m \neq 0$$

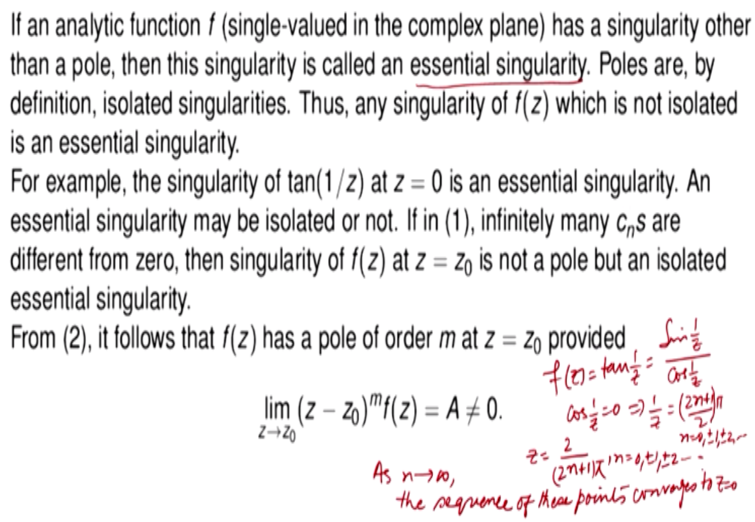
On the right side of equation 1 is called the principle part of z, okay. So this is called principle part, okay of f(z) at z=z0. Now if it so happens that from some n onwards, all the coefficients cns are 0, okay from some n onwards, all the coefficient cns are 0, okay that is suppose cm is not zero, but cn=0 for all n>m that is cm+1, cm+2, cm+3, they are all zeros, then the equation 1 will reduce to.

This equation will reduce to f(z)=sigma n=0 to infinity bn (z-z0) to the power n+cn/z-z0 c2/z-z0 square and so on + cm/z-z0 to the power m. In this case, we have the principle part of f(z) consists of only finitely many terms. There are only you can see m terms, okay even it may happen that some provisions c1, c2n so on c(m-1), okay. There may be zeros. So number of terms will be utmost m here.

Earlier we are saying here that cm is not zero. We are not saying anything about the provision c1, c2, c(m-1). There may also be zeros. So this principle part will contain at most m terms, so that is why we say that the principle part consists of finitely many terms. The singularity of f at z0 is called a pole. So if the principle part contains only finitely many terms, we say that the singularity of f(z) at z=z0 is a pole and the highest power of 1/z-z0.

That is m here. You can see Cm is non-zero, okay. So the highest power of 1/z-z0 is m that m is called the order of the pole. Now if it so happens that m=1, that means you only have here in the principle part one term, c1/z-z0. Then, the pole of first order we will have m will be equal to 1. So pole of first order is also called as simple pole.

(Refer Slide Time: 21:55)



Now if an analytic function f, say we are considering single valued functions in the complex plane has singularity other than a pole, okay. Suppose it so happens that an analytic function has a singularity other than a pole, then this singularity is called as an essential singularity, okay. Now poles are by definition isolated singularity, you can see while defining pole, we started with f(z) as an isolated singularity, okay at z=z0. So by definition poles are isolated singularity, okay.

Thus any singularity of object, which is not isolated is called an essential singularity. Now for example, the singularity of tan(1/z) at z=0. Let us look at tan(1/z). So f(z)=tan(1/z) can be written as sin(1/z)/cos(1/z), okay. Now singularity of tan(1/z) will be given by 1/ wherever cos(1/z) is 0. So cos (1/z)=0 implies 1/z=2n+1/2*pi where n=0,+/-1,+/-2, and so on, okay. So wherever z=2/2n+1 pi, okay, wherever z=this cos(1/z) will be 0, okay.

Now, you can see that as n goes to infinity, as n goes to infinity the sequence of these points converges to z=0, okay. At all these point, denominator is 0, cos(1/z) is 0, so they are all singularities of the function f(z), okay and you can also see that at these points cos(1/z) has a simple 0, that is 0 of order 1. If you take the derivative of cos(1/z), it will be –sin(1/z*1/z square), which will not be 0 at these points.

So cos(1/z) has a simple 0 at these points, z=2/2n+1 pi and therefore f(z) has a simple pole at all these points, z=2/2n+1 pi, okay and the sequence of these simple poles, okay converges to z=0 and therefore z=0 is a non-isolated singularity of f(z)=tan(1/z) because every neighborhood of z=0 contains a singularity of the function tan(1/z). So z=0 is a non-isolated singularity of tan(1/z) and therefore, we call it an essential singularity.

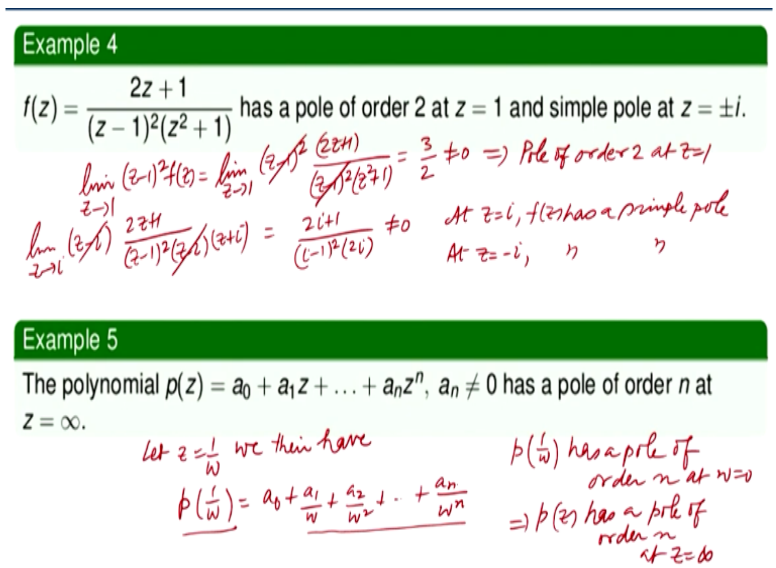
So the singularity of f tan(1/z) at z=0 is an essential singularity. Now an essential singularity may be isolated or not. Now when it can be isolated? It can be isolated provided the principle part contains infinitely many terms. So if the principle part of, if in one infinitely many cns are different from 0, then singularity of f(z) at z=z0 is not a pole, but an isolated essential singularity.

So if the principle part of the Laurent series contains infinitely many terms, then the function f(z) has an essential singularity at that point and it is an isolated essential singularity and non-isolated singularities occur as limits of sequences of poles like here. So now from 2 it follows that f(z) has a pole of order m at z=z0. Now you can see the following from here. This is the definition of a pole of order m. You can see here.

If you multiply this equation by z-z0 to the power m and take the limit at z tends to z0, then see what happens z-z0 to the power m*f(z) will be equal to summation n=0 to infinity bn(z-z0) to the power n+m+c1*(z-z0) to the power m-1c2 (z-z0) to the power m-2 and so on cm-1(z-z0)+cm, okay. Now take the limit at z tends to z0, okay, then what will happen. When you take n=0, we will have b0 (z-z0) to the power m.

So every term on the right hand side contains z-z0 as a factor, except this last term and therefore when z tends to z0, what we get is cm, okay. So the function f(z) has a pole of order m if z-z0 to the power m*f(z) at z tends to z0 is not equal to 0, cm must not be 0. So this is another definition of a pole of order m. From 2 it follows that f(z) as a pole of order m at z=z0 provided limit z tends to z0, z-z0 to the power m*f(z) is a finite quantity, which is non-zero, okay.

(Refer Slide Time: 27:52)



For example, f(z)=2z+1/z-1 whole square*z square+1. You can see here limit, now you can see f(z) is singular at the point z=1 and z=+/-i because z square+1=0 means, z=+/-i, so z has three singularities, z=i, z=-i and z=1. Let us see these nature of these singularities. So we here see that limit z tends to 1, z-1 whole square*f(z) is equal to limit z tends to 1, z-1 whole square*2z+1/z-1 whole square *z square+1. So this will cancel with this.

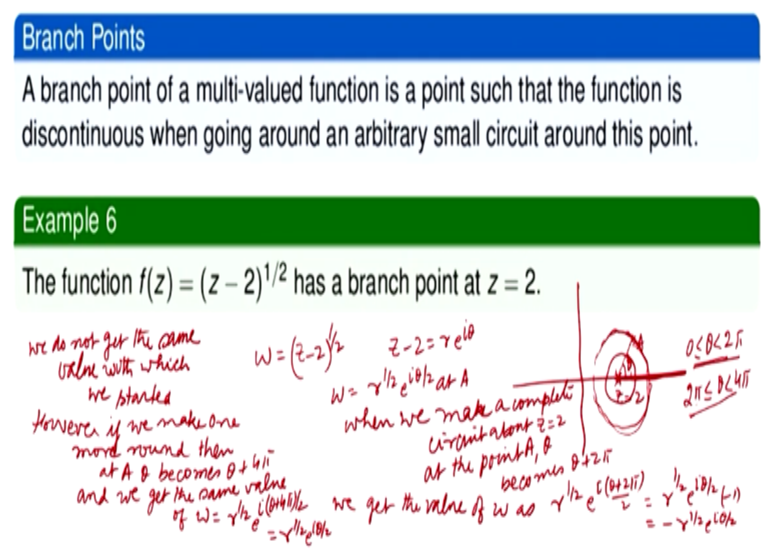
And when z tends to 1, we will get 3/2, okay, which is non-zero. So f(z) has a pole of order 2 at z=1. Now if you multiply here by z-1, instead of z-1 square, then what will happen, z-1 will cancel 1z-1 will cancel and in the denominator, we will get one more z-1, so at z tends to 1, it will become infinity. So we go on increasing the power of z-1 till we get a finite non-zero limit. So z-1 is not giving the non-zero limit. So we make z-1 square, which gives us a non-zero limit.

So we have pole of order 1 at z=1. Pole of order 2 at z=1. Now again, we have singularity at z=in-i, so limit z tends to i, we start with z-i power 1. So 2z+1/z-1 square* here we can factorize z-i z+i. So this will cancel with this and we will get limit as 2i+1/i-1 whole square*2i, okay, which is a non-zero quantity. So at z=i, we have a simple pole, f(z) has a simple pole. Similarly, we can show that at z=-i f(z) has a simple pole in a similar manner.

Now let us look at the polynomial p(z)=a0+a1z and so anz to the power n where an is not equal to 0. We know that this function is analytic for all finite z, because we can differentiate p(z) for any z, which is finite. Now it is a polynomial of degree n, an is not equal to 0, we want to show that it has a pole of order n at z=infinity. So let us take, let z be equal to 1/w, okay. So we will get, we then have p(1/w)=a0+a1/w a2/w square and so on an/w to the power n, okay.

And this is nothing but the Laurent series of p1/w, okay. We can see that this Laurent series has, this is the principle part of the Laurent series, okay. Since an is not 0, p1/w has a pole of order n at w=0, at w=0 and as a result of this pz has a pole of order n at z=infinity, okay.

(Refer Slide Time: 32:21)



Now let us look at the branch point. A branch point of a multivalued function is a point such that the function is discontinuous when going around arbitrary to small circuit around this point. Let us say for example, f(z)=(z-2) to the power 1/2. So this is your z=2. Let us take a point a, okay, say this angle is theta 1, okay, then let us say, let w=z-2 to the power 1/2, okay. This is theta, okay. So we will have, let us start making a complete round, starting with a around the point z=2.

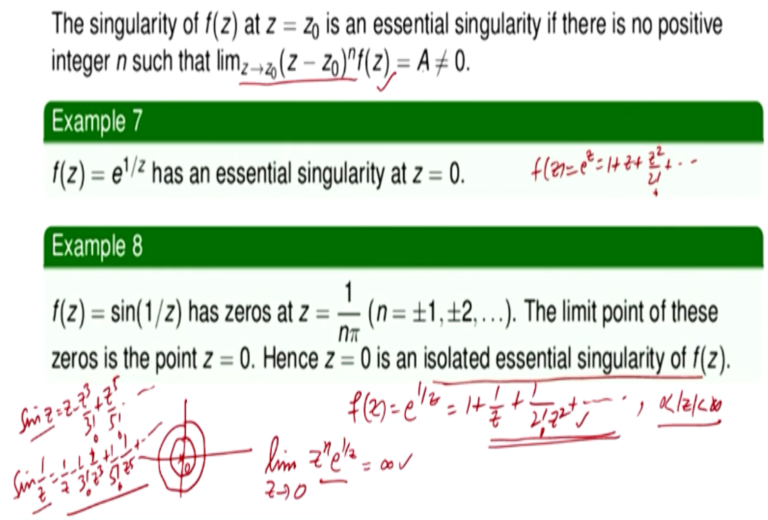
Okay, now at the point z=a, if you take z-2=r e to the power i theta, then what will happen w will be equal to r to the power 1/2 e to the power i, theta/2 at a, okay. So when we return back, okay at a, then what will happen, theta will become theta+2 pi, okay. So when we make a complete circuit about z=2, okay at the point a, theta becomes theta+2 pi. So what do we get, we get the value of w as r to the power 1/2, e to the power i theta+2 pi/2, which is equal to r to the power 1/2, e to the power i theta/2*-1.

That is –r to the power 1/2 e to the power i theta/2, so we do not get the same value. We do not get the same value with which we started. However, if we make one more round, however, if we make another one more round, then at a theta becomes theta+4 pi and we get the same value, same value of w that is r to the power 1/2 e to the power i theta+4 pi/2, which is equal to r to the power 1/2 2 to the power i theta/2.

Now this means that when 0 is <= theta <2 pi, okay, we get one value of w, okay. When 2 pi is <= theta<4 pi, we get another value of w, okay. So there are branches, okay. One branch is for the value of theta lying between 0 and 2 pi 0<=theta<2 pi and the other branch is for the values of theta lying between 2 pi to 4 pi and what we do is we drag this boundary, okay. This is boundary, so when we take a complete round here, we assume that we will not cross this boundary. We will go to the other branch.

After 1 complete round, okay, we will go to the other branch to get another single valued function. So 1 single valued function, we will get on 1 branch, 0 <= theta<2 pi and the other 1 branch, we will get for the other value of theta 2pi<=theta<4 pi. This line is called as the branch cut or branch line, okay. So we do not cross this. As soon as we reach here, we move to the other branch and this point z=2 is called the branch point of f(z).

(Refer Slide Time: 37:20)



So the singularity of f(z) at z=z0 is called an essential singularity. If there is no positive integer and such that limit z tends to z0 z-z0 to the power n f(z)=a. For example, if you take f(z)=e to the power z, 1/z, it has a essential singularity at z=0, you can see that, we can write f(z)=e to the power 1/z s1+1/z 1/2 factorial z square and so on, okay. This expansion of e to the power 1/z we have written using the expansion of f(z) as e to the power z=1+z+z square/2 factorial and so on. 0+z/1/z where we have found this expansion.

And this expansion since e to the power 1/z is analytic in the whole complex plane except at z=0, this series expansion of e to the power of 1/z will converge and will be convergent for 0<mod of z<infinity. You can take 2 concentric circles with center at z=0, the inner circle, you can go on reducing till you reach the point z=0, outer circle, we can go on expanding till we reach infinity.

So the region of convergence is 0<mod z<infinity and you can see this is the principle part of the Laurent series of e to the power of 1/z, which contains infinitely many terms in the powers of 1/z. So there are infinitely many terms, so the singularity of f(z)=e to the power 1/z at z=0 is an essential singularity. Now we can arrive at this conclusion by using this Laurent series expansion as well as by using this definition. In this definition, what we will do?

We have to see whether it has essential singularity at z=0. So let us take the limit z tends to 0, z to the power n*f(z) is e to the power 1/z. Now this limit is never a non-zero, okay. It is always infinity, okay. Whatever value of n you take, n=1, 2, 3, howsoever large value of n you take, z to the power n*e to the power 1/z is always infinity. So this function f(z) has an essential singularity at z=0 because we cannot find any positive integer and such that this limit is a, okay.

Now if you take f(z)=sin(1/z), then we can see that sin(1/z) has 0 at z=1/n pi, where n takes values +/-1, +/-2, and so on. The limit point of these 0s is the point z=0, where the function sin(1/z) has a singularity. So sin(1/z) has isolated essential singularity at f(z), okay. This is an isolated essential singularity of sin(1/z). You can also see this by the expansion of sin z. Sin z has this expansion. Sin z has this expansion z-z cube/3 factorial z5/5 factorial and so on, okay.

Then sin(1/z)=1/z-1/3 factorial*1/z cube 1/5 factorial 1/z to the power 5 and so on, okay. So this is the principle part of the Laurent series, which contains infinitely many terms. So this has essential singularity sin(1/z) has essential singularity at z=0 and we can also use this definition. If you multiply by sin(1/z)/z to the power n and take the limit as z tends to 0, okay no value of n, howsoever large you take will ever give you a non-zero finite quantity as its limit.

So sin(1/z) has an isolated essential singularity at z=0. With this, I would like to end my lecture. Thank you very much for your attention.