Advanced Engineering Mathematics
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Lecture - 16
Laurent Series

Hello friends, welcome to my lecture on Laurent Series. In various applications it is necessary to expand a function f(z) around points where f(z) is similar meaning that f(z) is not analytic.

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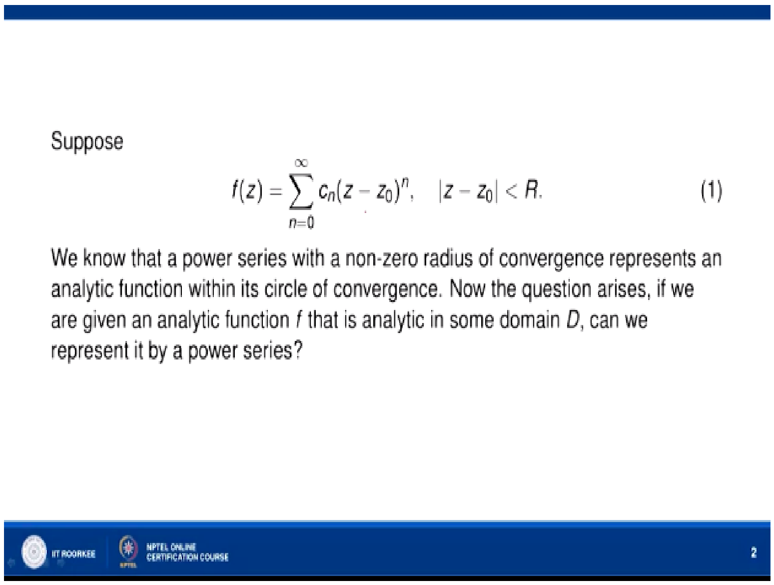

Now Taylor's theorem cannot be applied in such cases because in the Taylor series we lead the function to be analytic in a neighbourhood of that point. Now, Laurent series is named after the French engineer in mathematician, Pierre Alphonse Laurent and the theorem goes like this. If f(z) is analytic on two concentric circles C1 and C2 with center z0 and in the annulus between them, then f(z) can be represented by the Laurent series f(z)=sigma n=0 to infinity bn z-z0 to the power n + sigma n=1 to infinity Cn / z – z0 to the power n.

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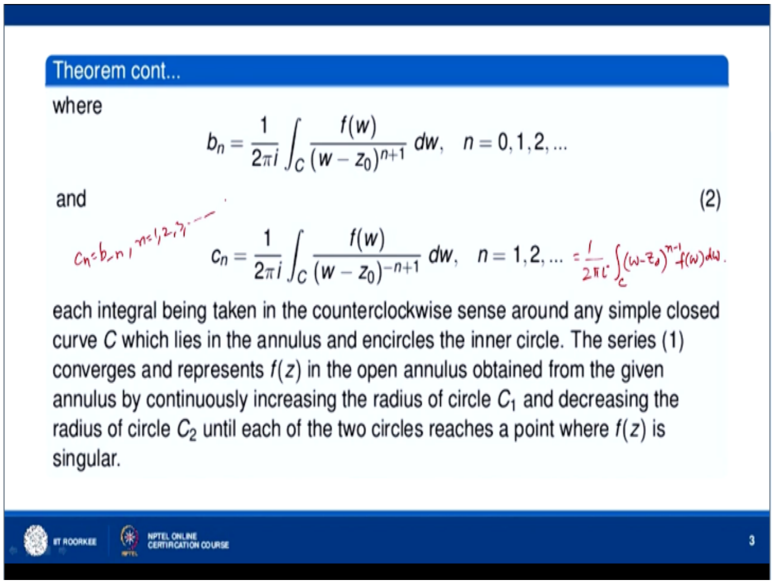

Now, then; where the values of bn and Cn are given by integrals, bn = 1/2pi i integral/c f(z) / w-z0 to the power n+1 dw, n varies from 0 and takes values 1,2,3 and so on so n=0, 1, 2, 3 and so on and Cn=1/2pi i integral/c f(w) dw / w-z0 to the power – n+1. We can also write this Cn as = 1/2pi i integral / c w-z0 to the power n-1 f(w)dw. So we can also write Cn like this. Now each integral in Dn and Cn is being taken in the counterclockwise sense around any simple closed curve C which is lies in the annulus and encircles in the inner circle.

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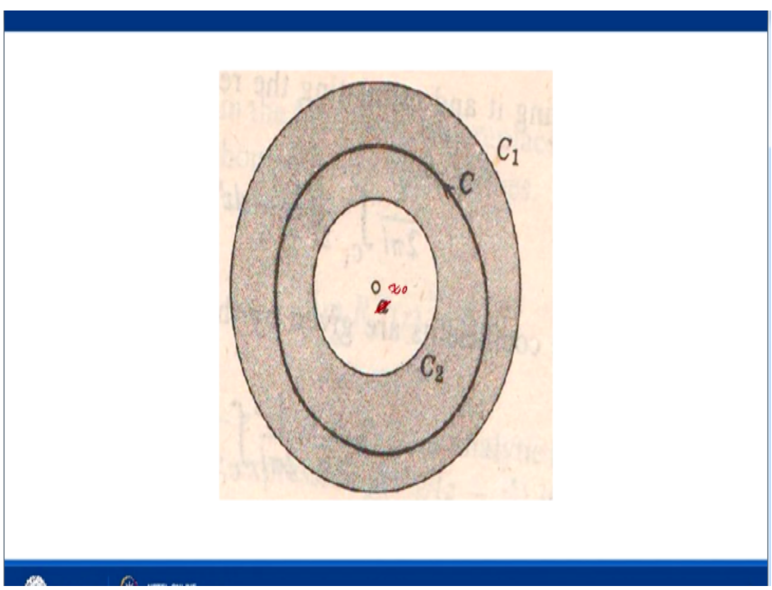

We can see the figure this one, okay. Here you can see there are two circles, C1 and C2 concentric circles with center at this is not A this is z0. So the two circles C1 and C2 which are concentric with center at z0 are given, and the function the shaded region means the function is analytic in this area in this shaded portion which is the annular region between C1 and C2 and C is any simple closed curve which lies in the annulus and encircles the inner circle.

So the series 1 then converges, the Laurent series 1 then converges and represents f(z) in the open annulus obtained from the given annuals. So the series then converges and represents f(z) in the open annulus obtained by increasing the radius of C1 and decreasing the radius of C2 till we reach similar point or we reach a point where the function is not analytic. So that is the region of convergences of the Laurent series. Now let us see how we prove this theorem. We can write the series 1 in an alternate form.

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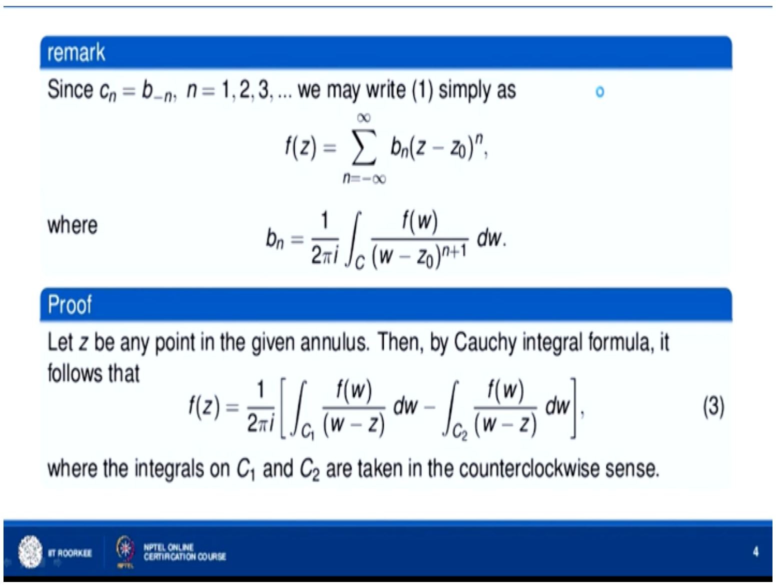

This series one can be written in an alternate form like this the f(z)=sigma n= - infinity to bn z-z0 to the power n. If you notice that Cn here, Cn is nothing but b-n okay, you can see Cn; you can see the expression of Cn and the expression of bn, okay. So when you replace n/-n in bn you get Cn, okay. So Cn=b-n when n takes values 1,2,3 and so on, okay. So when we use Cn=b-n here, okay so then what will happen.

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In various applications it is necessary to expand a function $f(z)$ around points where $f(z)$ is singular. Taylor's theorem can not be applied in such cases. A new type of series, known as, Laurent series named after the French engineer and mathematician, Pierre Alphonse Laurent is obtained in such a case.

Theorem 1

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with center z_0 and in the annulus between them, then $f(z)$ can be represented by the Laurent Series

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{c_n}{(z - z_0)^n} \quad (1)$$

Cn=b-n this will be equal to sigma n=0 to infinity bn z-z0 to the power n and then we have Cn=b-n so b-n z-z0 to the power –n n=1 to infinity. Now, when n runs from 1 to infinity –n runs from – infinity to -1 so we can combine this and this series and then we write sigma n=- infinity to infinity bn z-z0 to the power n, okay. So f(z) can be expressed thus, sigma n= - infinity to infinity bn z-z0 to the power n where bn's are given by 1/2pi i integral over C, f(w)/w-z0 to the power n+1 dw, okay. So that is an alternate form of the Laurent series.

Now, let us take z to the any point in the given annulus okay. So let us take z to the any point in the given annulus, okay. Let take z to be any point in the given annulus then by Cauchy integral formula. Okay. Now that we have assume that the function f(z) is analytic on C1, analytic on C2 and in the annular region will C1 and C2. So by using the Cauchy integral formula we can write f(z) as 1/2pi i integral over C1 f(w)/w-z dw – integral over C2 f(w)/w-z dw. What we do there?

That, we take a cross-cut okay, we take a cross-cut like this and then we move along the cross-cut say this is A, this is B, we move along AB then along C1 in the counter-clockwise then we move along BA and then we move along C2 in the clockwise direction total integral is 0; then the; we can apply the Cauchy integral theorem. When we apply the Cauchy integral theorem it turns out that the integral; by Cauchy integral formula then f(z) can be written as 1/2pi i integral over C1 f(w)/w-z dw.

Because z will lie inside the simple closed curve which we get by (()) (06:43) the cross-cut. So f(z) can be written 1/2pi i integral over C1 f(w) dw/w-z – integral over C2 f(w)dw/w-z. And integrals along C1 and C2 are taken in the counterclockwise sense, okay. If you recall, what we do their, f(w)/w-z, this function is analytic.

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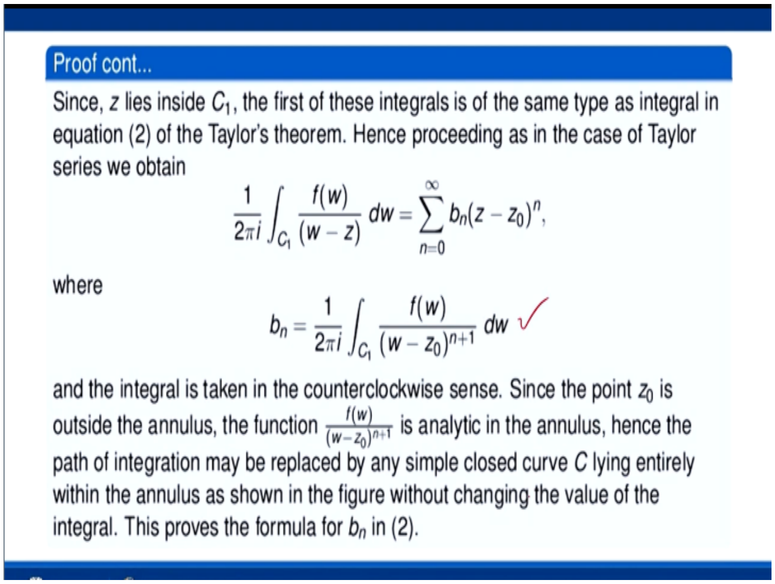

So what we do is since z lies inside C1 the first of this integral is of the same type. The first of this integral is of the same type as integral in equation 2 of the Taylor's theorem hence proceeding as in the case of Taylor series we obtain 2pi i integral over C1 f(w) dw/w-z = sigma n=0 to infinity bn z-z0 to the power n where bn is given by 1/2pi i integral over C1 f(w) dw/w-z0 to the power n+1 and the integral is taken in the counterclockwise sense. Now the point z0 is outside the annulus.

You can see, the point z0 is outside this annular region, so the function f(w)/w-z0 to the power n+1 is analytic in the annulus and hence the part of integration maybe replaced by any simple closed curve C lying in the annulus as shown in this figure, okay. C1; the integral along C1 can be replace by integral along any simple closed curve which lies in the region of the annular region as shown in the figure without changing the value of the integral. So this proves the proofs the formula for this one, okay, this formula for bn.

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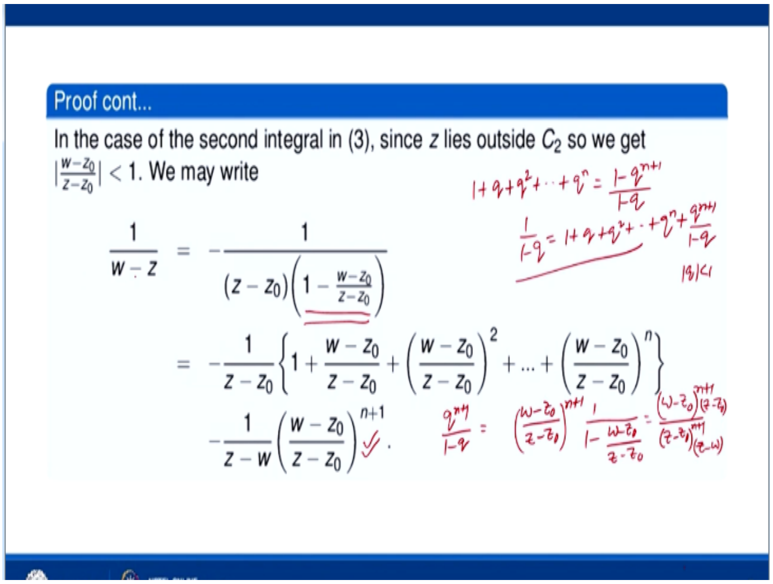

No in the case of the second integral. Let us take the second integral now, this one, okay. In the case of the second integral in 3, since z lies outside C2, okay we can see here this point z lies outside C2, so mod of z-z0 mod of z-z0 > mod of w-z0 where w belongs to C2, okay. w is the variable of integration along C2, so if you take the point w here, okay mod of w-z0 < mod of z-z0. So in the case of the second integral, since z lies outside C2 we get mod of w-z0 over z-z0 < 1 and therefore 1/w-z, okay.

Let us again recall that, 1+q+q square and so on q to the power n = 1-q to the power n+1/1-q. Or we can write it as 1/1-q=1+q+q square and so on q to the power n + q to the power n+1/1-q okay, so where q is; mod of q < 1, okay. So 1/w-z = -; okay we can write it as 1/w-z can be written as 1+q+q square q to the power n you can see here, 1/w-z I am writing as -1/z-z0 * 1/w-z0 / z-z0. Now 1/1-q, this is q okay can be written as 1+q+q+square q to the power n and then q to the power n+1 upon 1-q * z-z0, so that gives you this one, okay, q to the power n+1/1-q, okay.

That gives you how much, q=w-z0/z-z0 raise to the power n+1/1-w-z0 over z-z0, okay that is equal to w-z0 to the power n+1/z-z0 to the power n+1*z-z0/z-w, okay. So when we multiply by -1/z-z0 this z-z0 and the z-z0 cancel minus sign and make this w-z; become -1/z-w; w-z0 = over z-z0 to the power n+1. So by using this formula, okay we get here. Now therefore, -1/2pi i f(w)/w-z dw=; now from here we can see, we multiply, we integrate over C f(w)dw/w-z and multiply by 1/2pi i.

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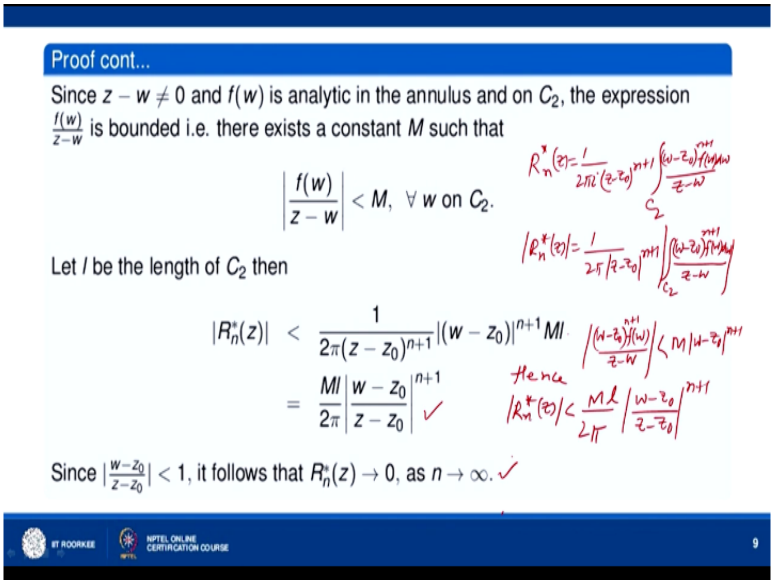

So we come here; 1/2pi i, 1/z-z0 integral over C2 f(w)dw, 1/z-z0 whole square integral over C2 w-z0 * f(w)dw + and so on. 1/z-z0 to the power n+1 this term and we have integral over C2 w-z0 to the power n f(w)dw+Rn*z, okay. Now, Rn*z = 1/2pi i, 1/z-z0 to the power n+1 from here we are getting, okay from this term, okay. So 1/2pi i * z-z0 to the power n+1 integral over C2 w-z0 to the power n+1 f(w)dw/w-z-z.

Now in each of these above integrals, integrals over C2 can be replace by integral over C because the function is analytic on C2 and in the annular region between C1 and C2. Now we have to show that Rn* goes to 0 as n goes to infinity. So what we do is, since z lies in the annular region, okay you can see z lies in the annular region and w varies on C2, z lies in the annular regions and w varies along C2.

So z is not equal to w, so z is not equal to w, f(w) is analytic in the annular region and also on C2 therefore, f(w)/z-w is continuous, okay along C2 and so it is bounded. And therefore they exists constant and such that mod of f(w)/z-w < m for all w on the curves on the circle C2. Now let us say l be the length of the circle C2 then mod of Rn* z by Cauchy inequality, let us apply Cauchy inequality here. Rn*z is this one, okay. 1/2pi i z-z0 to the power n+1 Rn*z=1/2pi i z-z0 to the power n+1 integral over C2, w-z0 to the power n+1 f(w)dw/z-w. Okay.

So mod of Rn*z=1/2pi mod of z – z0 to the power n+1 and then modulus of integral over C2 w-z0 to the power n+1 f(w)dw/z-w, okay. Now mod of w-z0 f(w)/z-w okay this is < m times mod of; this is n+1, okay mod of w-z0 to the power n+1, okay. So hence, mod of Rn*z < 1/2pi okay * m; L is the length of C2 then mod of w-z0/z- z0 to the power n+1, okay. So this is what we get, mod of Rn*z is < this quantity.

Now mod of w-z0 over z-z0 is < 1, okay it follows that mod of w-z0 over z-z0 to the power n+1 goes to 0 as n goes to infinity and therefore Rn*z goes to 0 as n goes to infinity. Thus, the representation 1, this one, okay. Thus, this representation with coefficients bn and Cn given by these integrals is established, okay.

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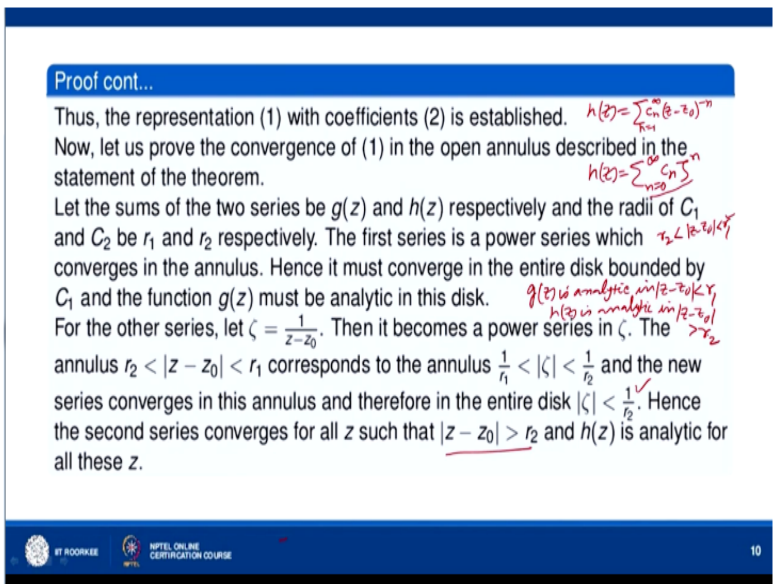

Now let us proof the convergence of the representation 1 in the open annulus described in the statement of the theorem that is we can go on continuously increasing the cycle of; radius of circle C1 and decreasing the radius of circle C2 until we reach a similar point, that is the region of convergence. So let the sums of the two series be g(z) and h(z) respectively. Let us denote this sum by g(z), okay. So let us take this as g(z) and this as h(z), okay, so then f(z)=g(z)+h(z). First series sum, we are writing as g(z) and next series sum we are writing as h(z).

So let the sums of the two series be g(z) and h(z) respectively and the radii of C1 and C2 be r1 and r2 then the first series is a power series, okay which converges in the annulus, and therefore, it must converge; because the; in this first series, okay z is any point in the annulus so and it is; sum taking as g(z), since z is any point in the annulus, so we can say that the first series converges in the annulus.

Now hence, it must converge in the entire disc bounded by C1 and the region and the function g(z) must be analytic in this disc because there is no other similar point inside C1 of the function; of this series g(z), so it must converge in the entire disc bounded by C1 and the function g(z) must be analytic in this disc. Now for the other series whose sum is h(z) let us take h(z)= sigma Cn z-z0 to the power –n n=1 to infinity, okay.

So for the other series let us take zeta = 1/z-z0 then it becomes a power series in zeta, this is; if you take zeta = 1/z-z0 then h(z)=sigma n=0 to infinity Cn zeta to the power n, okay. So it becomes a power series in zeta. The annular region is described by r2 < -z mod of z0 < r1, r1 is the radius of C1 circle, r2 is the radius of C2 circle, okay. So we can say 1/r1 taking reciprocal here, 1/r1 < mod of zeta < 1/r2. And the new series converges in this, because new series is now this one, sigma n=0 to infinity Cn zeta to the power n which is a power series.

And this power series then converges in this region, 1/r1 < mod of z < 1/r2. And therefore, in the entire disc mod of zeta < 1/r2, okay. So the second series converges for all z; now mod of zeta < 1/r2 means mod of z – z0 > r2. So the second series converges for all z such that mod of z-z0 > r2 and h(z) is analytic for all these z, okay. So g(z) is analytic then mod of z-z0 is < r1 okay, g(z) is analytic in the disc mod of z-z0 < r1 and h(z) is analytic in mod of z-z0 > r2, okay.

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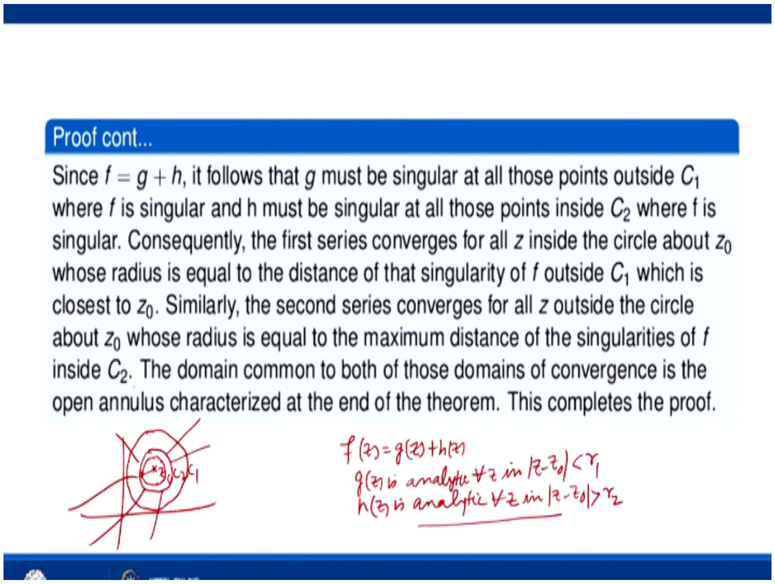

Now, since f=g+h it follow that g must be similar at all those points outside C1 where f is similar. Why? Because f(z)=g(z)+h(z), okay g(z) is analytic for all z in mod of z-z0 < r1, okay. And h(z) is analytic for all z in mod of z-z0 > r2. So if this is your z0 point, okay, this is circle C2 and this is circle C1, okay. So for all z such that mod of z-z0 < r1, h(z) is analytic inside the circle C1 everywhere.

And g(z) is analytic in the circular disc mod of z-z0 < r1 and h(z) s analytic in the region outside the disc mod of z-z0 circular disc mod of z-z0 <= r2. So h(z) is analytic everywhere here, okay. So it says that, g must be similar at all those points outside C1 where f is similar. If f is similar outside C1, okay then g will be similar because h(z) is analytic for all z outside mod of z-z0 = C2, okay.

Consequently, the first series converges for all z inside the circle about z0 who is radius is equal to the distance of that similarity of f outside C1. You can increase this radius of the circle C1 till we reach a similar point of f(z) that is the distance of the region. The circle, this radius of C1 can be enlarged so much that, the radius will be the distance of z0 from the nearest similarity of f(z), okay. So similarly, the second series converges for all z outside the circle about z0 whose radius is equal to the maximum distance.

Now you can see, the second series converges for all z, okay outside the circle about z0 whose radius is equal to the maximum because the whatever the function f(z) will be similar, okay. Since h(z) is analytic for all z inside mod of z-z0 > r2, so if f(z) is similar inside the disc mod of z-z0 < r2 then your h(z) will also be similar there. So the second series converges for all z inside the circle about z0 whose radius is equal to; suppose there are 3, 4 points inside the circle C2 at which f(z) is similar, then the; you have to take the distance of z0 from the further similarity which lies inside C2, okay.

Suppose this is the further similarity, okay. So then you have to take the distance of further similarity from z0 and you can reduce the radius of circle C2, you can go on reducing this radius of circle C2 till you reach this point, okay which is the; from this point inside the circle C2 from the point z0. So similarly, the second series converges for all z outside the circle about z0 whose radius is equal to the maximum distance of the similarity of f inside C2.

The domain common to both of those domains of convergence is the open annulus, characterized at the end of the theorem which says that, we can go on increasing the radius of the circles C1 and go on decreasing the radius of the circle C2 until we reach a similar point. So this completes the proof of the Laurent theorem.

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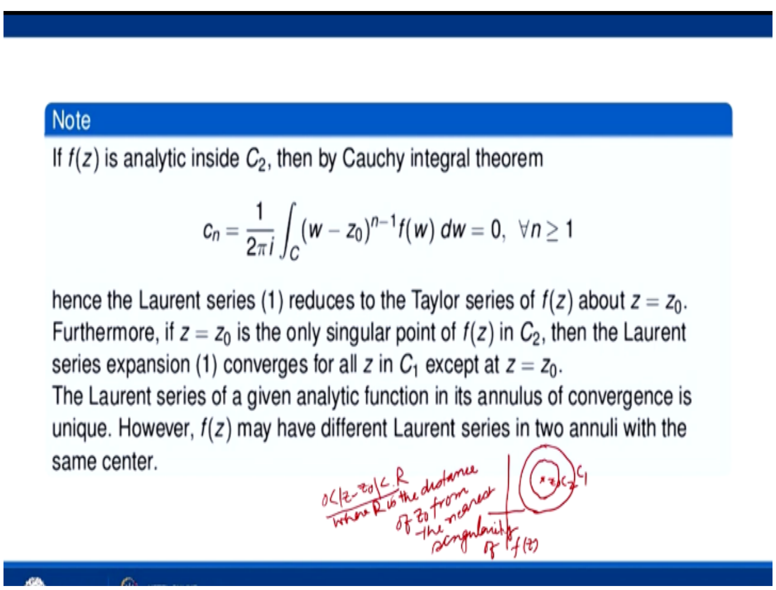

Now, in the Laurent theorem we can notice that if f(z) is analytic inside C2, okay let us notice this, if f(z) is analytic inside C2 then f(z) will be analytic inside on the simple closed curve C, okay and therefore, this Cn okay, Cn=1/2pi i integral over C w-z0 to the power n-1 f(w)dw. Now this is a polynomial in w of degree n-1, okay f(w) is (w) is analytic inside C and on the simple closed curve C therefore, the product of w-z0 to the power n-1 f(w) is analytic inside and on the simple closed curve C and therefore by the Cauchy integral theorem Cn will be equal to 0.

So then what will happen, this part of the Laurent series which contains the negative powers of z-z0 it will be 0, it will vanish and therefore we will have f(z)=sigma n=0 to infinity bn z-z0 to the power n which is the Taylor series of the function f(z) which center at z=z0. So if the function f(z) is analytic inside C2 then the Laurent series reduces to the Taylor series of f(z) about z=z0. So this follows the Taylor Laurent series reduces to the Taylor series.

Now further more if f(z)=z0; now if it so happens that, this is your z0 say, this is circle C2 and this circle let us say C1, okay. If z=z0 is the only similarity inside the circle C2 then we can go on decreasing the radius of C2 till we reach the point z0. And therefore, the region of convergence of the Laurent series will be 0 < mod of z – z0 < r where r is the distance of z0 from the nearest similarity of f(z), okay.

So where, r is the distance of f: z0 from the nearest similarity of; nearest similarity of f(z), okay. So if z=z0 is the only similar point of f(z) in this circle C2 then the Laurent series expansion converges for all z in this region, okay. That is the deleted neighbourhood of z=z0. Now the Laurent series of a given analytic function in its annulus of convergence is always unique. However, it may have different Laurent series in two annually with the same center.

Okay, so in different annually; with the same center it can have different Laurent series but in a given annular region it will have a unique expansion.

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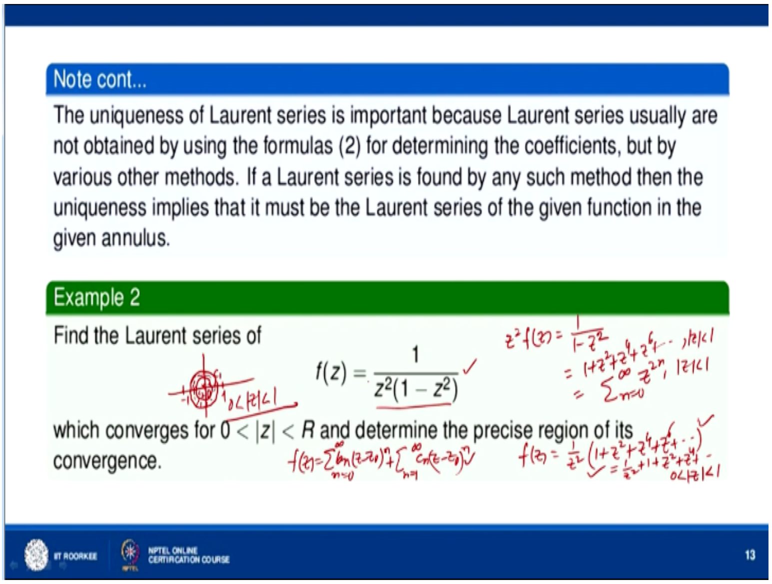

Now the uniqueness of the Laurent series is important because Laurent series usually are not obtained by evaluating the coefficients bn and Cn by the integrals. So usually, Laurent series f(z) of the function f(z) is not obtained by evaluating the value of bn and Cn, okay by this integrals but by alternate method, so there we use the uniqueness of the Laurent series.

So if a Laurent series is found by any method any such method where if you are expanding a function f(z) the center f(z)=z0 in the form of the Laurent series that is, it contains positive and negative integrals powers of z-z0 then it will represent the Laurent series of that function in that annular region, okay. So if a Laurent series found by any such method then the uniqueness implies that it must be the Laurent series of the given function in the given annulus.

For example, let us consider f(z)=1/z square * 1-z square. Let us see how we find the Laurent series of this function. So we can write it as z square * f(z)=1/1-z square. Now, z square * f(z) is an analytic function okay, it is 1/1-z square it is analytic everywhere except that; I mean analytic except that you call to +-1, 1/1-z square can be expanded by Taylor series and this we know, this is equal to 1+z square + z to the power 4, z to the power 6 and so on.

And this region of convergence is mod z < 1, okay. Now; or we can also write it as sigma n=0 to infinity z to the power 2n then mod z is < 1. Now, we can write; so therefore f(z)=; now we divided by 1/; 1 multiplied by 1/z square, so 1/z square 1+z square + z to the power 4, z to the power 6 and so on, okay, so this equal to 1/z square + 1+ z square + z to the power 4 and so on and the regions of convergence. Now z=0 as to be excluded because f(z) is not analytic at z=0, so the region of convergence will be 0 < mod z < 1.

We can see it like this also, say this function f(z) is not analytic at z=0 and z=+-1, so this is 0 here and 1 is here, -1 is here. Okay. Let us take two concentric circles which center at z=0, okay. So the function f(z) is analytic in the annular region, okay, and on the circle C1 and C2. We are taking the radius of C1 to be less than 1 and C2 to be having radius between 0 and 1. So C1 and C2 are two concentric circles which center at z=0 and in the annular region between them.

Now the Laurent series the circle, radius of C2 can be go; we can go on increasing till we reach the similar point 0 and we can go on increasing the radius of C1 till we reach 1 and -1, 1 and -1 both are at the same distance from 0 that is 1, so the radius of C1 can be made as large as 1 and the radius of C2 can be made as small as 0. So the region of convergence will be 0 < mod of z < 1.

And we arrive at the series expansion of f(z) 1/z square + 1 + z square + z to the power 4 which is of the type 1, okay sigma bn z-z0 to the power n + sigma Cn z-z0 to the power n, okay. So this expansion is of this type where z0 = 0. And therefore, this expression, this expansion of f(z) is Laurent series of f(z) about that equal to 0. So; and region of convergence is 0 < mod z < 1, that is the deleted neighbourhood of z=0.

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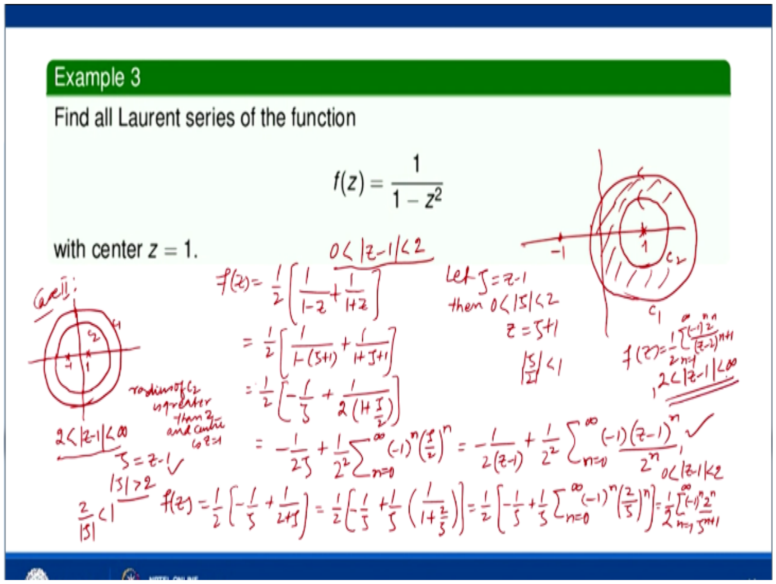

Now let us consider another example, f(z)=1/-z square, okay. So here we need to find all Laurent series of this function. So the function f(z) is not analytic at z=1 and -1, okay. We want to expand this function which center at z=1, so z=1. So what we will do, let us construct two concentric circle with center at z=1 such that the function is analytic between the; in the annular region between C1 and C2 and on the circle C1 and C2.

So let us consider one circle like this, this is C2, another circle like this C1, okay. Now we have taken the radius of the circle C1 to be smaller than 2, the distance of 1 from -1 is 2. So let us take this radius of C1 to be less than 2 and radius of C2 to the between line between 0 and 2, okay. So then the function f(z) is analytic in the annular and on the circles C1 and C2. And then what we can do we can go on decreasing the radius of C2 till we reach a similar point.

So we can go on and decrease the radius of C2, there is no similar point except z=1, okay. So the radius of C2 can be made as small as 0 and the radius of C1 can be made as large as 2, because we can go on increasing the radius of C1 till we reach the point -1 and the distance of -1 from 1 is 2, so the radius of convergence in this case will be 0 < mod of z-1 < 2. What we will do, we will write the partial fraction of f(z).

So 1/2 times, 1/1-z – 1/1+z, okay. So we can write here +, okay. So 1/1-z+1/1+z will be 2/1-z square * 1/2. Okay, now what we do, let us write let zeta be equal to z-1, okay. Then, 0< region of convergence will be 0<mod of zeta < 2. Okay. And here z will be equal to zeta+1. So let us put the value here, so 1/2 1/1-zeta+1. And here we will have 1/1+zeta+1. So this will be equal to 1/2 – 1/zeta, okay. And here we will have 1/2+zeta.

Now mod of zeta/2, okay is < 1. Okay, mod of zeta/2 is < 1, so I can write it as 2 times 1+zeta/2, okay. And then write it as -1/2 zeta + 1/2 square. Now this is 1/1+zeta2 where mod of zeta/2 is < 1 and therefore, we can expand it by Taylor series, so this is sigma n=0 to infinity -1 to the power n zeta/2 raise to the power n and then I can put the value of zeta here z-1, so -1/2 times z-1+1/2 square summation n=0 to infinity – 1 to the power n z-1 to the power n/2 to the power n, okay.

And region of convergence is 0 < mod of z-1 < 2. So this is the Laurent series in the case where we have the region of convergence 0 < mod of z-1 < 2. Now let us consider another situation, okay. In the other situation what will happen, we can take the circles like this. Suppose this is 1 and this is -1 okay. Then you take the inner circle the center at z-1 of radius more than 2, okay. Let draw it again. So let us draw the circle, inner circle, this is inner circle, okay. And this is outer circle. This center is z=1, okay. This is -1, okay.

So a drawing is circle with center C1 of radius more than 2, okay. This is C2 and this is C1. Radius of C2 is > 2, okay. And center is z-1. Okay. So then what will happen, we can go on increasing the radius of C1 since there is no similar point other than 1 and -1 the radius of C1 can be made infinity and the radius of C2 can go with, we can go on decreasing till we reach the point -1, okay. And the distance of -1 from 1 is 2.

So we will have the region of convergence as 2 < mod of z-1 < infinity. Now let us find the; this is case 2, okay. So in this case if you want to find the Laurent series expansion of f(z) then what we will do, we will again consider zeta = z-1, okay. Now here what will happen, mod of z will be > 2, okay. So what we will do here, we have f(z)=1/2, 1/1-z + 1/1+z then we put z=zeta+1, so after putting z=zeta+1 what we have, f(z)=1/2 – 1/zeta, okay and then 1/2+zeta, okay we have 2+zeta here, right.

So what we do, now this is 2/mod of zeta < 1. Okay. Mod of zeta > 2 gives you 2/mod of zeta < 1. So what we do here, 1/2-1/zeta and then we have + 1/zeta 1+2/zeta, okay. So then we shall expand this, this is equal to 1/2-1/zeta+1/zeta sigma n=0 to infinity – 1 to the power n and 2/zeta to the power n, okay. So when you put n=0 here what will happen, we will get -1 to the power 0 which is 1, 2/zeta to the power 0 which is 1, so 1/zeta will get.

So first term will cancel from -1/zeta here, so we shall write 1/2 times sigma n=1 to infinity – 1 to the power n, 2 to the power n / zeta to the power n+1, okay. So 1/2 I have written outside. So 1/2, okay sigma n=1 to infinity – 1 to the power n 2 to the power n upon zeta the power n * 1/zeta, so zeta to the power n+1. And then we put zeta= your z-1. So we will get f(z)=; so in the second case f(z)= will get z=1/2 sigma n=0; 1 to infinity -1 to the power n 2 to the power n upon z-2 to the power n+1. This is the Laurent series when 2 < mod of z-1 < infinity.

So we have discussed both the cases which are possible, in the case of f(z)=1/1-z square. We have two different annually, one annually is 0 < mod z-1<2 where we get this series expansion, this Laurent series and we have another case where the annular region is 2< mod of z-1<infinity. And there we get this Laurent series. So in different annually f(z) may have different Laurent series. But in a given annular region f(z) has a unique Laurent series, so that uniqueness of Laurent series we use to find the Laurent series expansion of f(z) in a gain annular region.

As we said, we do not find the coefficient bn Cn usually, we use alternate methods to determine the Laurent series expansion of f(z) in a given annular region of convergence by implying some other methods. With this I would like to end my lecture. Thank you very much for your attention.