Advanced Engineering Mathematics

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Lecture - 14
Power Series

Hello friends, welcome to my lecture on Power Series. An important special case of a series of complex value functions is the general power series.

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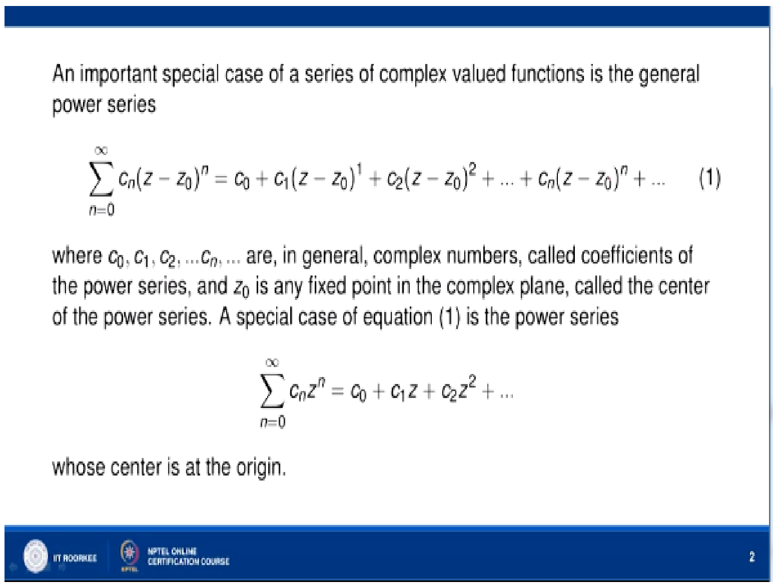

Sigma n=0 to infinity Cn z – z0 to the power n which can expressed as C0 + c1 z – z0 + C2 z – z0 to the power square and so on cn z – z0 to the power n and so on, where C0, C1, C2, Cn are, in general complex numbers and are called the coefficients of the power series, z0 is any fixed point in the complex plane called the center of the power series. A special case of this equation 1 is the power series where z0 = 0.

That is sigma n=0 to infinity Cn z to the power n = C0 + C1 z + C2 z square and so on. So the center of this power series is at the origin. Any complex polynomial say f(z) = C0 + c1 z + c2 z square and so on.

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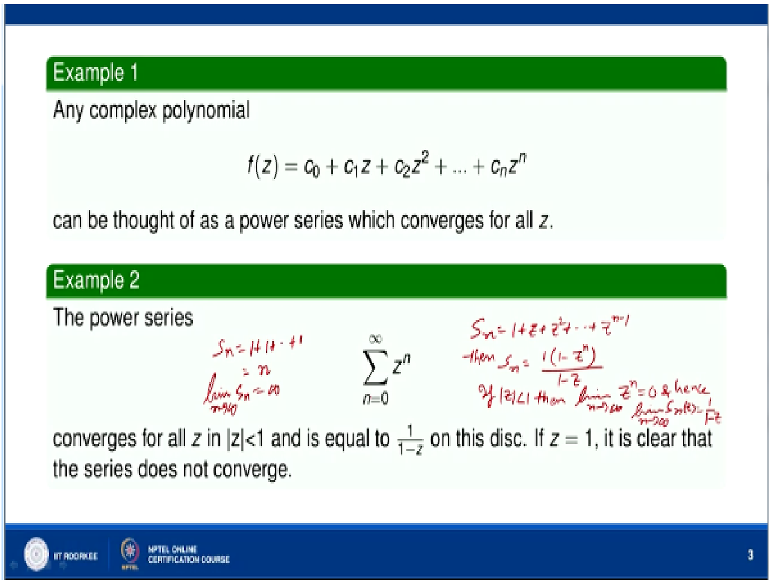

Cn z to the power n can be thought of as a power series which converges for all z. So here you can see that Cn+1, Cn+2 are the coefficient Cn+1 Cn+2 and so on all 0's so it represents a power series and it converges for all z because for any z in the complex series we can find the sum of the series. Now the power series, sigma n=0 to infinity z to the power n converges for all z in mod z < 1 and is equal to 1/1-z on the circular disc mod z < 1.

If z=1 then we will have the nth term of the power series as says 1 and so the nth term of the power series does not go to 0 and therefore the series does not converge. So we can see that this series converges for all z in mod z < 1, let us take the nth partial some of the series Sn=1+z+z square and so on z to the power n-1 then it is a geometry series and therefore Sn=1*1-z to the power n/1-z.

Now, if mod f z is <1 then limit n tends to infinity z to the power n goes to 0, okay it is equal to 0. And hence, limit n tends to infinity Sn(z)=1/1-z. So the sum of the series is 1/1-z whenever mod z is <1, okay. So we can say that the series converges for all z such that mod of z is <1 and this is equal to 1/1-z on this circular disc mod z <1 the center at z=0 and radius 1.

If z=1, then we can see that here also we can Sn=1+1 and so on 1 that is 1, so limit n tends to infinity Sn=infinity and therefore the series diverges for z=1. Hence the series converges for all z such that mod of z is <1 and diverges at z=1.

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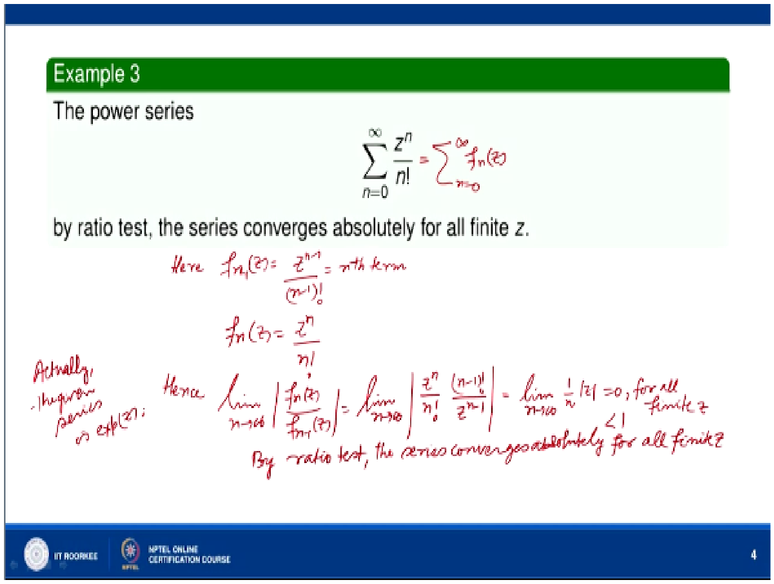

Now let us consider the power series. So sigma n=0 to infinity z to the power n over n factorial then we can see that this series converges absolutely for all finite z, we can do that by ratio test. So here fn(z) the nth term of the series fn(z)=z to the power n-1/n-1factorial, nth terms of the series fn(z) is; we are writing the series as sigma n=0 to infinity fn(z). So fn; this nth term of the series, we start with n=0 so nth term, this is nth term; actually this is nth term, okay.

So nth terms is actually here, it is fn-1, fn-1 is the nth term of this series, so fn-1 z is z to the power n-1/n-1 factorial and then fn(z)=z to the power n/n factorial. Hence, limit n tends to infinity mod of fn(z)/fn-1z= limit n tends to infinity z to the power n-n factorial * n-1 factorial divided by z to the power n-1. So this is equal to limit n tends to infinity 1/n times mod of z, okay. And this is equal to 0 for all finite z, okay.

So this limit is <1, okay. And therefore, the series converges absolutely by ratio test, the series converges absolutely, for all finite z. Actually it is nothing but the exponential z, okay this sigma n=0 to infinity z to the power n/n factorial is exponential z, the given series is exponential z.

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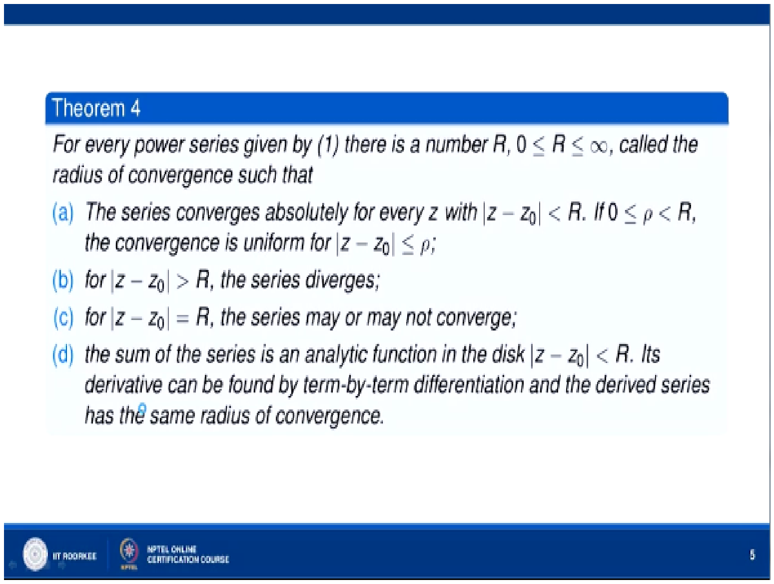

Now like in case of real calculus, we know that every power series has radius of convergence which lies between 0 and infinity, 0 <= R < infinity. So here also in the; we are considering a generalization of the power series for complex functions. So for every power series given by 1, let us see given by 1, for every power series given by this equation 1, okay there is a number R, 0 <= R <= infinity called the radius of convergences such that the series converges absolutely for every z mod of z – z0 < R.

If 0 is < R or = rho < R then the convergence is uniform for mod of z – z0 <= rho, if mod z – z0 is > R the series diverges and for mod of z – z0 = R the series may or may not converge. The sum of the series is an analytic function in the disc mod of z – z0 < R is derivative then we found by term by term differentiation and the derived series as the same radius of convergence. So this result we have.

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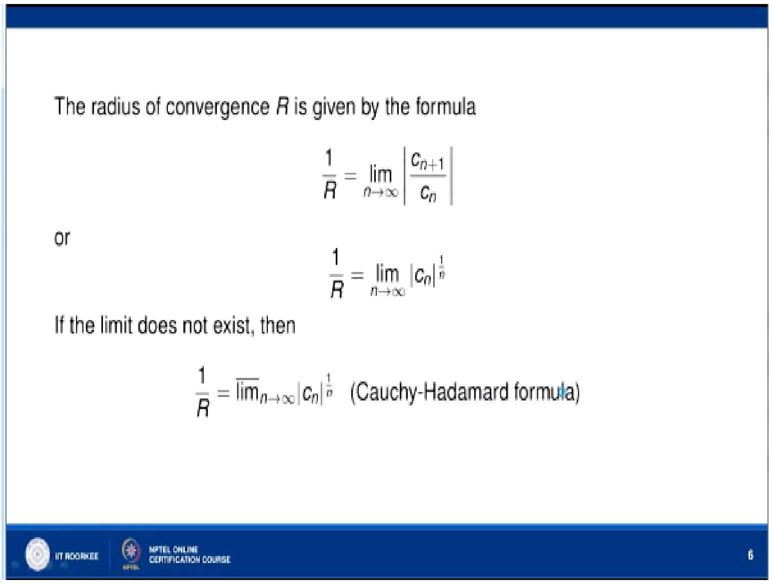

In case, the limit here, if the limit of mod of Cn+1/Cn exist okay then it is equal to 1/R. you can also find this radius of convergence or by using the ratio test so 1/R=limit n tends to infinity mod of Cn rest to the power by n, if this limit does not exist then the Cauchy-Hadamard formula says that 1/R=limit superior n tends to infinity mod of Cn raise to the power 1/n.

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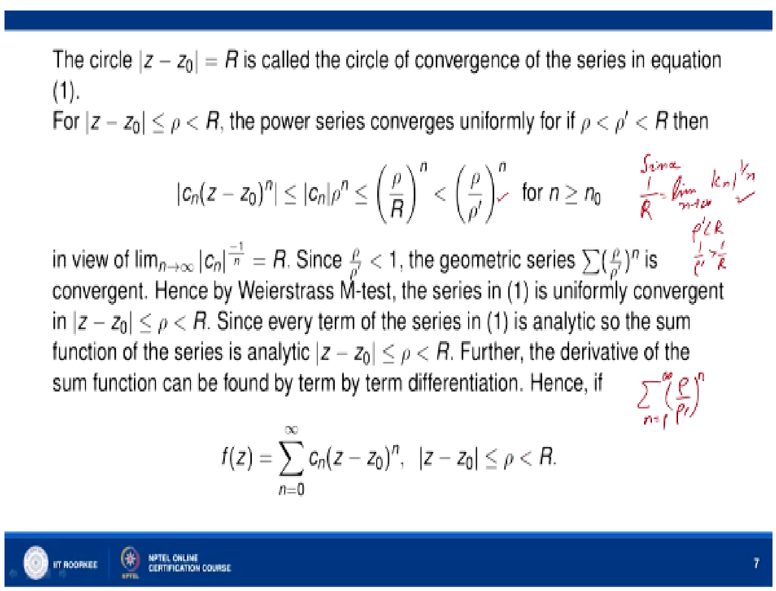

Now the session A, that is the power series converges absolutely for every z big mod of z – z0 < R can be easily seen by applying the ratio test to the series 1, okay. We shall show that the series converges uniformly whenever mod of z – z0 is <= rho and rho is strictly < R. So let us first show this. The circle mod of z – z0 = R is called the circle of convergence of the series in equation 1. So when mod of z – z0 is < R = rho and rho is < R we shall show that the power series converges uniformly.

Now, why it converges uniformly, the power series converges uniformly for if rho is < rho_ < R. So let us take a number rho_ which lies between rho and R. Let rho be < rho_ and rho_ is < R then the nth term of the power series mod of Cn z-z0 to the power n is <= mod of Cn, mod of z-z0 is <= rho, so rho to the power n and this is <= rho/R to the power n because, but since 1/R=limit n tends to infinity mod of Cn to the power 1/n, okay.

So since this is 1/R= limit n tends to infinity mod of Cn to the power of n/n there exists some n0 such that for n >= n0 mod of Cn is <= 1/R to the power n. So mod of Cn is <= rho/R to the power n and rho_ is < R, rho_ is < R so 1/rho_ is > 1/R or we can say 1/R is < 1/rho_, so this is < rho_ rho/rho_ to the power n for sufficiently large, okay for n >= n0, from here we are getting this inequality. So in view of limit n tends should mod of Cn z to the power 1/n=R we get this inequality.

Now since rho/rho_ is < 1 the geometric series sigma n=1 to infinity, rho/rho_ to the power n, okay. This is convergent and hence by Weierstrass M-test the series in 1 is uniformly convergent, okay. So this series converges uniformly in mod of z – z0 <= rho < r. Since every term of the series 1 is analytic we can see that every term of the power series sigma Cn z – z0 to the power n is analytic so the sum function of the series is analytic in mod of z – z0 <= rho and rho < R.

Further, the derivative of the sum function can be obtained by term by term differentiation. Hence, if fz=sigma n=0 to infinity Cn z – z0 to the power n and mod of z – z0 <= rho < r.

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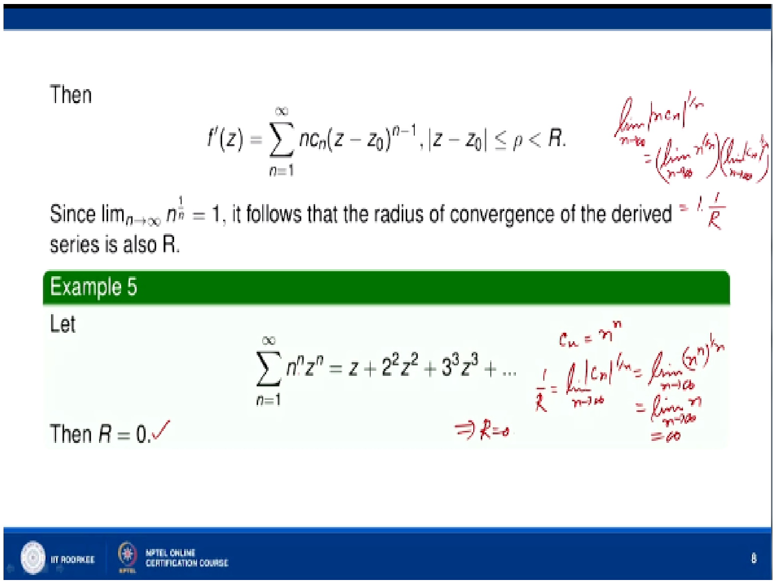

Then f_z will be equal to sigma n=1 to infinity and Cn z – z0 to the power n-1. And here you can see the coefficient of z-z0 to the power n-1 is nCn. So limit n tends to infinity and Cn raise to the power mod nCn raise to the power 1/n=limit n tends to infinity n to the power 1/n * limit n tends to infinity mod of Cn to the power 1/n.

Now n to the power n as n goes to infinity is = 1 so 1*1/R, okay. So you can see that the coefficient of z – z0 to the power n-1 is nCn and when we take the limit of mod of nCn to the power we get 1/R, so this means that the radius of convergence of the derived series is also R, okay. And this follows from the said that, limit n tends to infinity n to the power 1/n = 1, so radius of convergence of the derived series also R.

Now let us consider an example say, sigma n=1 to infinity n to the power n z to the power n. So this series can be expanded as z+ z square, 3 cube, z cube and so on. And here we can see that Cn=n to the power n. So mod of Cn raise to the power 1/n limit n tends to infinity = n to the power n whole to the power 1/n, so limit n tends to infinity n which is equal to infinity and this is equal to 1/R, okay. So this implies 1/R=infinity and so R=0, so radius of convergence of this series is R=0 that means that this series converges only at the point z=0 and nowhere else.

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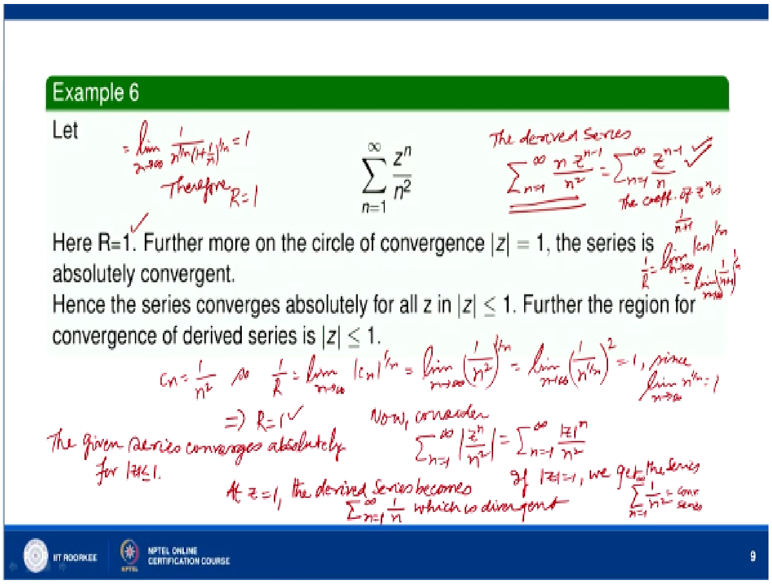

Now, let us consider another example. Sigma n=1 to infinity z to the power n/n square. Here we can see, here Cn = 1/n square, okay. So limit, so 1/R=limit n tends to infinity mod of Cn to the power 1/n=limit n tends to infinity 1/n square raise to the power 1/n, we can write it as limit n tends to infinity 1/n to the power 1/n whole square, okay. Now, we know that n to the power 1/n goes to 1 as n goes to infinity so this is equal to 1. Since limit n tends to infinity n to the power 1/n=1, okay. So what do we get? R=1.

Hence the radius of convergence of this power series is equal to 1. And now let us see, when you take, consider the series of absolute terms. Now consider, sigma n=1 to infinity mod of z to the power n/n square. So this is sigma n=1 to infinity mod of z to the power n/n square. Now if you take mod z=1, so if mod z=1 what we get the series sigma 1/n square, okay which is known to be convergent series, okay so convergent. It is a convergent series.

And there the given series convergent is absolutely for mod z <= 1. For mod z < 1 it converges because its radius of convergence is R=1, okay. So the given series converges absolutely for mod z <= 1. Now let us consider the derived series. The derived series is sigma n=1 to infinity and z to the power n-1/n square which is sigma n=1 to infinity z to the power n-1/n. Now here again, f you take the coefficient Cn, okay. The coefficient Cn is the coefficient of z to the power n, the coefficient of z to the power n is 1/n+1, okay.

So limit n tends to infinity, mod of Cn raise to the power 1/n = limit n tends to infinity 1/n+1 raise to the power 1/n, okay. And if you find this limit it is limit n tends to infinity 1/n to the power 1/n 1+1/n to the power 1/n, okay. So when n goes to infinity this goes to 1, okay. When this goes to infinity it goes to 1 because n to the power 1/n goes to 1 and 1+1/n to the power n also goes to 1.

So this is 1 and therefore R=1, because this is 1/R, 1/R= this. So R=1, okay and therefore the series, derived series, okay also converges for mod of z < 1. It converges absolutely for mod of z < 1. But if you take z=1 what do we get? At z=1, the derived series becomes sigma n=1 to infinity 1/n, okay which is a divergent series. So the derived series, okay converges absolutely for mod of z < 1.

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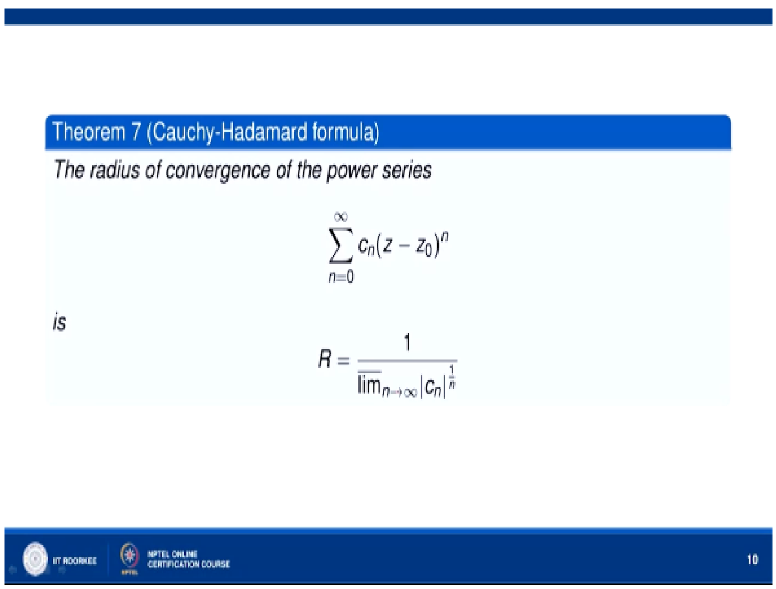

Now, let us consider the case where the limit of mod of Cn to the power 1/n as n goes to infinity does not exist, okay so there we use Cauchy-Hadamard formula. The radius of convergence of the power series sigma n=0 to infinity Cn z – z0 to the power n is R=1/limit superior n goes to infinity mod of Cn to the power 1/n. So let us see how we prove this.

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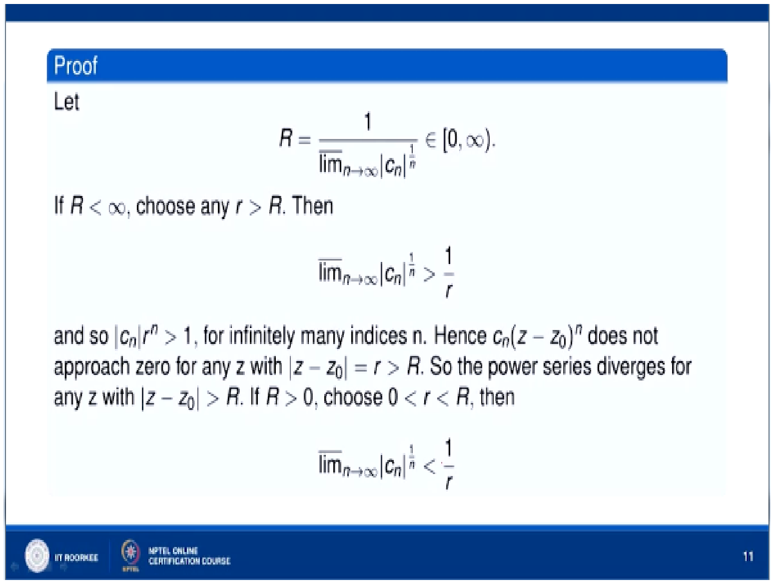

So let us assume that R=1/limit superior n tends to infinity mod of Cn to the power 1/n, we shall so that R is the radius of convergence of the power series sigma Cn and z – z0 to the power n. Now, since mod of Cn to the power 1/n is >= 0, limit superior will always be >= 0, so if it is infinity 1/infinity will become 0, so this R will belong to this limit, limit superior n tends to infinity of a sequence of non-negative real numbers will always be non-negative.

So R will always be non-negative and moreover if this limit is infinite 1/infinity will become 0 so this R will belong to this semi-closed interval 0 infinity, it is R is always >= 0. Now, first we discuss the case when R is < infinity. So if R < infinity let us choose any R > R, okay. Let us take if R is < infinity let us take any R > R then limit superior mod of Cn limit superior n tends to infinity mod of Cn to the power 1/n will be > 1/R okay.

Limit superior n tends to infinity mod of Cn to the power 1/n = 1/ R but 1/R is > 1/r, okay. So mod of Cn to the power 1/n*r > 1 for infinitely when it says n or we can mod of Cn * r to the power n>1 for infinitely many indices n. Hence Cn, z – z0 to the power n does not approach 0, this is nth term of the series, okay nth terms of the power series does not approach 0, for any big mod of z – z0 = R, R>R. Because if the series convergent then the nth term must to go 0.

Here what is happening is mod of Cn R to the power n>1 for infinitely many indices and therefore Cn mod z – z0 to the power n okay, does not approach 0 for any z with mod of z – z0 =

R where R is > R. So the power series diverges for any z with mod of z-z0 > capital R. If R is > 0, choose 0 < R < capital R, okay.

Then, if R is strictly > 0 then choose 0 < r < R then limit superior n tends to infinity mod of Cn to the power 1/n is < 1/R, okay. So this follows from, this is limit superior n tends to the infinity, mod of Cn power 1/n=1/R but 1/R is < 1/r, okay.

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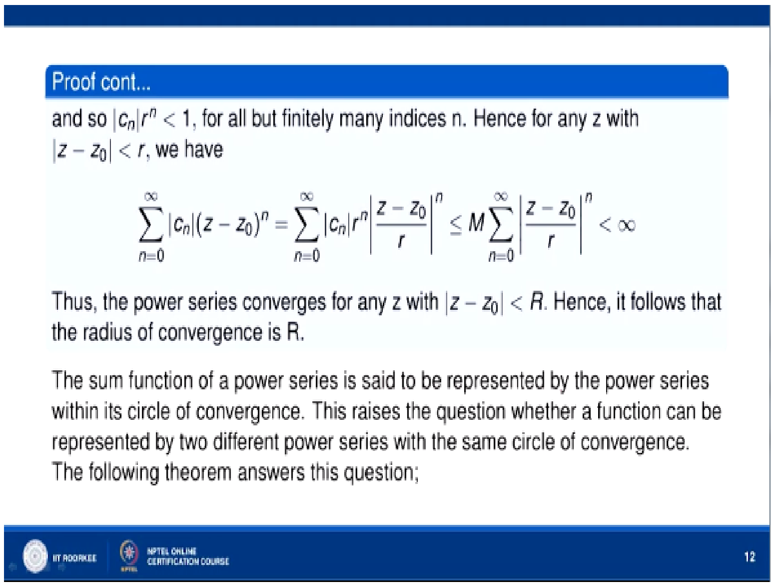

So if; and so mod of Cn * r to the power n is < 1. From here it follow that, mod of Cn * r to the power n is < 1 for all but finitely many indices n. And hence for any z, the mod of z – z0 < r we have sigma n=0 to infinity mod of Cn z mod of z – z0 to the power n = sigma n=0 to infinity mod of Cn * r to the power n mod of z – z0 over r to the power n which is <= m times. Because mod of Cn* r to the power n is < 1 for all but finitely 1 indices n.

So we can say that, mod of Cn* r power n can be <= m for all indices n, so this is <= m, sigma n=0 to infinity mod of z – z0 r to the power n which is < infinity because mod of z – z0 is over mod of z – z0 is < r, okay mod of z – z0 is < r, okay. So this is a convergent series and therefore this is < infinity that the power series converges for any z big mod of z – z0 < r. And hence it follows that the radius of convergence of the power series is R.

The sum function of a power series set to be represented by the power series within it circle of convergence. This raises the question whether function can be represented by two different power series with the same circle of convergence. Now the following theorem answers this equation.

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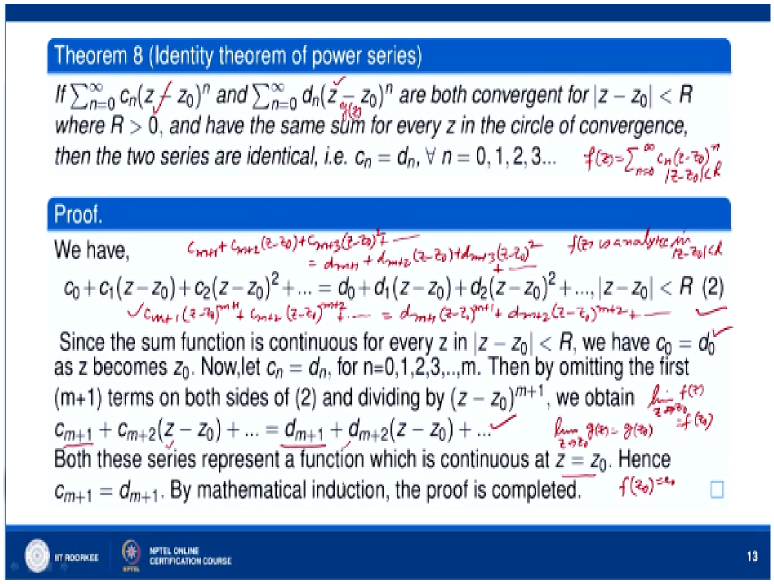

So if sigma n=0 to infinity Cn z – z0 to the power n and sigma n=0 to infinity Dn z - z0 to the power n are both convergent series for mod of z – z0 < r where r is > 0 and have the same sum for every z in the circle of convergence then the two series are identical. If there are two power series which are both convergent in the same region of convergence and have the same sum for every z in that, in the circle of convergence then the two series is must be identical, that is Cn=Dn for all n, this is called the Identity theorem for power series.

Now, so here what is happening is that this series, this series both the same sum therefore C0 + c1 z – z0 + c2 z – z0 whole square = d0 + d1 z - z0; d2 z – z0 whole square and so on and they converge both in the region mod of z – z0 < r. Now, since the sum function of both the series is, since they have the same sum, okay and sum function is analytic in the region mod of z – z0 < r and therefore the sum function is continuous for every z in mod of z – z0 < r.

And therefore, if the sum of the series is say f(z). Say, f(z)=sigma n=0 to infinity Cn z-z0 to the power n then f(z) is analytic in mod of z – z0 < r, okay. And f(z) is analytic means f(z) is continuous, okay. So continuity f(z) implies that limit z tends to z0 f(z)=f(z0). So when z tends to z0, okay when z tends to z0 f(z) must be equal to f(z0). And f(z0) is equal to here is C0, okay.

If this is g(z), if this is g(z) then limit, z tends to z; so again by the same argument g(z) is analytic in the region mod of z – z0 < r so limit z tends to z0 g(z)= g(z0), okay. So when z tends to z0 okay the left hand side tends to C0, the right hand side tends to d0 and so C0 must be equal to d0 okay by the continuity of the sum function. Now let us assume that Cn=Dn we are going to prove that C0 = d0 c1=d1; c2=d2 and so on. So let us assume that Cn=Dn for n=1,2,3 and so on up to m

Then we shall show that Cn=Dn for n=m+1 also. So then by omitting so when C0 = d0; c1=d1; c2=d2 and cm=dm then we can omit the m terms from both the sides, so by omitting the first m+1 terms, m+1 terms on both sides of this equation 2, what we will have, we will have cm+1 z-z0 to the power m+1+cm+2 z-z0 to the power m+2 and so on equal to dm+1 z-z0 to the power m+1+dm+2 z-z0 to the power m+2 and so on, okay. So by omitting the first m+1 terms on both the sides the series reduces to this equation.

And then we can divide by z – z0 to the power m+1. So when we divided by z – z0 to the power m+1 what we will get, we will get cm+1+cm+2 z – z0, okay, cm+1+cm+2 z – z0 cm+3 z – z0 whole square and so on equal to dm+1+dm+2 z – z0 dm+3 z – z0 whole square and so on, okay. So we obtained this one, this series. Now both these series represent a function which is continuous at z=z0.

Because when we remove the terms still the series represents analytic function, this series represent a analytic function, this also represent analytic function, so they are continuous functions at z=z0 n. And therefore, cm+1 must be equal to dm+1 as z tends to z0, this will tend to cm+1 and z tends to z0 this will tend to dm+1 and they are continuous so at z=z0 they must be equal to cm+1 and dm+1. So this is cm+1 and this is dm+1.

So cm+1 must be equal to dm+1 and therefore, by mathematical indexation the theorem is proved.

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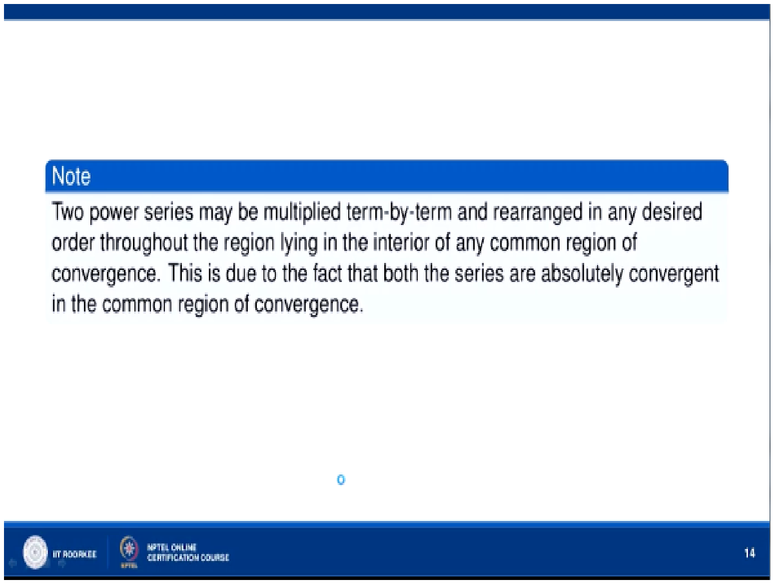

Now two power series maybe multiplied by term-by-term or rearranged in any desired order throughout the region lying in the interior. Thereof any common region of convergence because the both the series are absolutely convergent in the common region of convergence. With this, I would like to end my lecture. Thank you very much for your attention.