Advanced Engineering Mathematics

Prof. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 12
Sequences and Series

Hello friends welcome to my lecture on sequences and series of complex functions. Let us consider an infinite series of functions of a complex variable. We shall see that most of the definitions and theorems obtained for infinite series of real terms can be applied with little or no change to the series whose terms are complex. Let us consider a sequence f and z, n = 1, 2, 3 and so on be a sequence of functions defined on a set D of complex numbers.

Then consider the series sigma n = 1 to infinity fn z = f1 z + f2 z and so on, fn z and so on.

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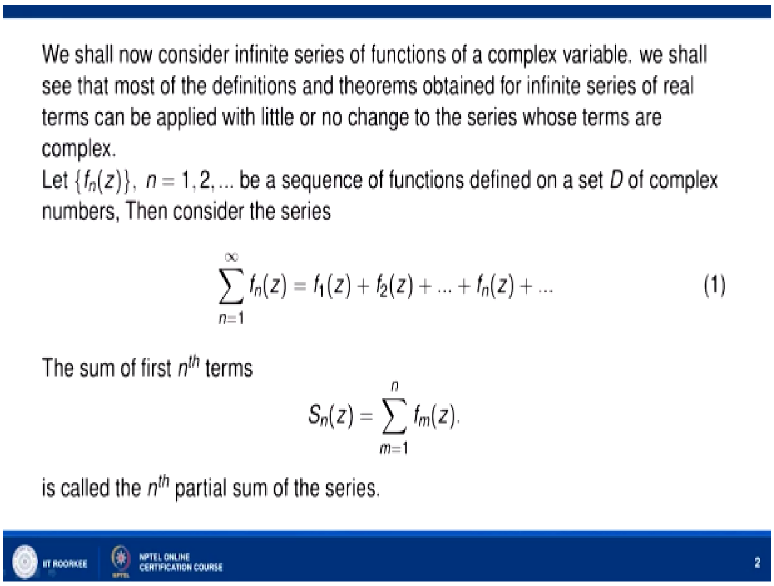

Then the sum of the first n terms of the series is given by the function Sn z. Sn z = sigma m = 1 to n, fmz and it is called the nth partial sum of the series.

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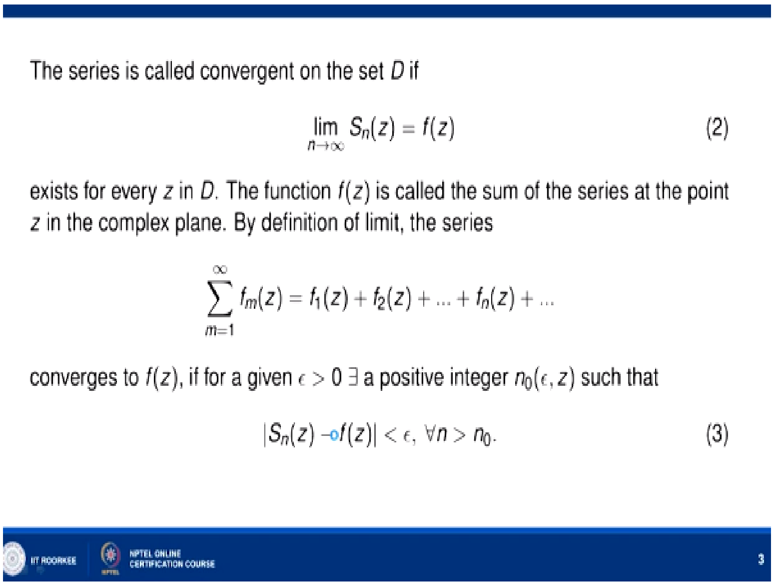

The series is said to be convergent on the set D if limit n tends to infinity Sn z = fz exists for every z in D. The function fz is then called the sum of the series at the point z in the complex plane by definition of limit the series sigma m = 1 to infinity f m z = f1 z + f2 z and so on fn z converges to fz for a given epsilon > 0, we can find positive integer n0 depending on epsilon and z, such that mod of Sn z – fz is < epsilon for all n greater than n0.

You can see that the definition is here is same as we have in the case of series of functions of a real variable.

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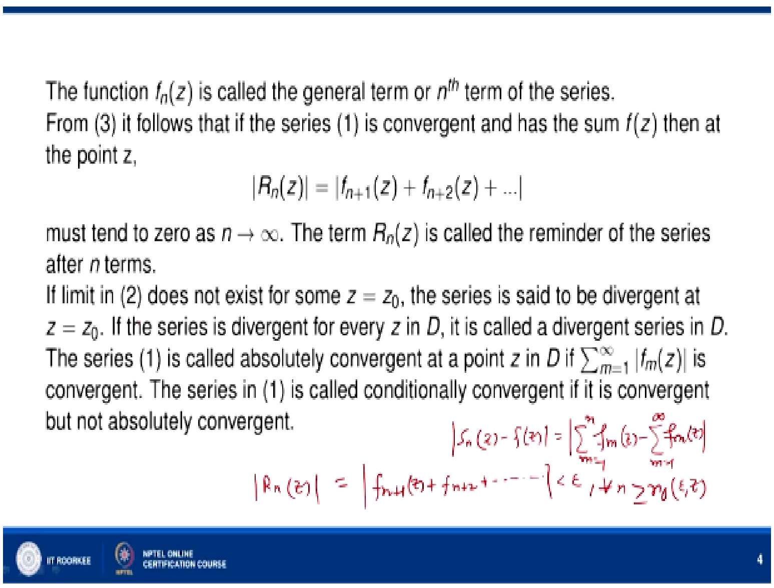

Now the function fn z here is called the general term or the nth term of the series sigma fmz where m = 1 to infinity. Now from 3 it follows that let us look at 3 from this inequality it follows that mod of Sn z – fz means, Sn z is the sum of the first terms of the series okay, so

Sn z – fz means you will have Sn z – fz = fn + 1 z + fn + 2 z and so on, first n terms from the sum function fz will be removed and we will have the terms fn + 1 z and so on, fn + 2 z and so on, which we denote by Rn z.

Rn z is called the reminder of the series after n terms. So mod of Rn z = mod of fn+1 z + fn + 2 z and so on. Modulus of this quantity, that is this must go to 0 as n goes to infinity. Mod of we can see, mod of Sn z – fz. This is = mod of sigma m = 1 to infinity fmz – sigma m = 1, 2, this is n, here M = 1 to infinity fm z. So this is = mod of fn + 1 z, fn + 2 z and so on okay. So mod of Rn z – fz is said to be the sum remainder after n terms of the series and we denoted by Rnz.

So Rn z mod of Rn z = this, this should be < epsilon for all n >= n0 depending on epsilon nz and this means that mod of Rn z should go to 0 as n goes to infinity Rn z is called the remainder of the series after n terms. Now if limit in 2, this is your 2 okay, if limit in 2 exist, if the limit in 2 does not exist for some z = z0 then we say that the series is divergent at z = z0 and if this occurs for every z in D that is the series is divergent for every z in D we call it a divergent series in D.

Now the series 1 is called absolutely convergent, we call this series, this series 1 to the absolutely convergent, if the corresponding series of absolute value is of the sequence functions fm z is convergent. So sigma m = 1 to infinity, mod of fm z if it is convergent we say that the given series is absolutely convergent at the point z. Now the series given by equation 1, this series is called conditionally convergent if it is convergent but not absolutely convergent.

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Since the series $\sum_{n=1}^{\infty} |f_n(z)|$ is a series of positive real terms, the criteria for convergence for a series of real positive terms can be applied to this series. Thus, if $\sum_{n=1}^{\infty} f_n(z)$ is absolutely convergent then

$$R_n(z) = |f_{n+1}(z)| + |f_{n+2}(z)| + \ldots \quad \to \quad 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad f_{n+1}(z) + f_{n+2}(z) + \ldots \quad \to \quad 0 \text{ as } n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} f_n(z) \text{ is convergent.}$$

Thus absolute convergence implies convergence but converse is not true.

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Since the series sigma n = 1 to infinity mod of fn z is the series of positive real terms okay, the criteria for convergence for a series of real positive terms can be applied to this series. So if sigma n = 1 to infinity fn z is absolutely convergent it means that sigma n = 1 to infinity mod of fn z, these are convergent series okay and therefore the remainder after n terms of this series r and z will be mod of fn + 1z, mod of fn + 2z and so on.

So if the series is convergent then the remainder after n terms of this series r and z must go to 0 as n goes to infinity. So when Rn z goes to 0 then goes to infinity and Rn z consist of the absolute values of fn + 1 z, fn + 2 z and so on then it follows that fn + 1 z, fn + 2 z and so on that also goes to 0 because mod of fn + 1 z, mod of fn + 1z + fn + 2z and so on, okay, so when fn + 1 z, mod of fn + 1 z, mod of fn + 2 z and so on goes to 0 then this will also go to 0.

Okay, so we will have sigma n = 1 to infinity fn z is convergent and therefore absolute convergence implies convergence, but the converse is not true. We shall see an example on this fact.

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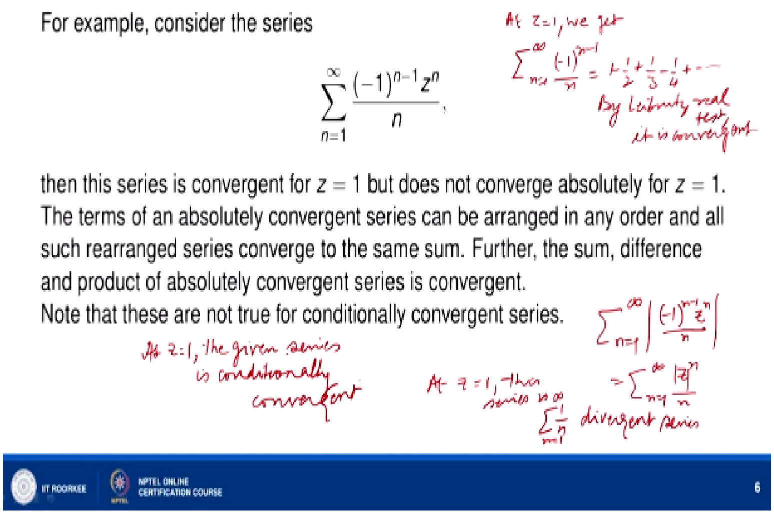

For example, let us consider the series sigma n = 1 to infinity -1 to the power n-1 z to the power n/n. We can see that this series is convergent for z = 1, because that z = 1 we get sigma n = 1 to infinity -1 to the power n – 1/n, which is 1 – 1/2 + 1/3 and so on, which is an alternating series and by Liouville's real test we know that this series is convergent, okay so at z = 1 this series is convergent, but if you take the absolute value of each term that is you consider sigma n = 1 to infinity, mod of -1 to the power n-1 z to the power n/n okay.

Then you get sigma n = 1 to infinity z to the power n, mod of z to the power n/n and at z = 1 this series is harmonic series sigma 1/n n = 1 to infinity, which is known to be a divergent series and therefore at z = 1 this series is conditionally convergent okay. So at z = 1 the given series is convergent but not absolutely convergent so it is the given series is conditionally convergent.

So this series is convergent but does not converge absolutely for z = 1. Now the terms of an absolutely convergent series can be arranged in any order and all such rearranged series converge to the same sum, further the sum difference and product of absolutely convergent series is convergent. Now these results are not true in the case of a conditionally convergent series.

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Cauchy's convergence criterion holds for series of complex numbers. Thus, a necessary and sufficient condition for the series (1) to be convergent is that for a given $\epsilon > 0$ $\exists$ a positive integer n_0 such that

$$|S_{n+p}(z) - S_n(z)| < \epsilon, \ \forall n > n_0 \text{ and } p = 1, 2, \ldots$$

$\sum_{n=1}^{\infty} f_n(z)$

Since

$$|S_{n+1}(z) - S_n(z)| = |f_n(z)|,$$

it follows that a necessary condition for the convergence of the series $\sum_{n=1}^{\infty} f_n(z)$ for any fixed z is

$$\lim_{n\to\infty} f_n(z) = 0$$

but the condition is not sufficient.

Now Cauchy's convergence criterion holds for series of complex numbers. So a necessary and sufficient condition for the series sigma n = 1 to infinity fn z to be convergent is that for a given epsilon > 0, we can find positive integral n0 such that mod of Sn + pz – Sn z is < epsilon for all n > n0 and for a very p = 1, 2 and so on. Now is you put p=1 here then mod of Sn +1 z – Sn z is = mod of fn z.

Because Sn + 1 z is the sum of the first n + 1 terms of the series and Sn z is the sum of the first n terms of the series. So mod of Sn + 1 z – Sn z will be = mod of fn z and this is < epsilon for all n greater than n0, so this means that fn z must tend to 0 as n goes to infinity. So a necessary condition for the convergence of the series sigma n = 1 to infinity fn z is that the nth term of the series must go to 0 as n goes to infinity but this condition is not sufficient.

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Again, let us consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{n-1}}{n}.$$

At $z=-1$, we get $\sum_{n=1}^{\infty} \frac{(-1)^{2(n-1)}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

Note that for $z = -1$

$$f_n(z) = \frac{1}{n} \to 0, \text{ as } n \to \infty$$

$f_n(z) = \frac{(-1)^{n-1} z^{n-1}}{n}$

$\Rightarrow f_n(-1) = \frac{1}{n} \to 0,$ as $n \to \infty$

but the series

$$\sum_{n=1}^{\infty} f_n(-1) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$$

the series $\sum_{n=1}^{\infty} f_n(-1) = \sum_{n=1}^{\infty} \frac{1}{n}$ = divergent series

is a divergent series.

Let us take an example on this. You can consider again the series sigma n = 1 to infinity – 1 to the power n -1 z to the power n-1 over n, at z = -1 we can see sigma, the series becomes at z = -1 we get sigma n = 1 to infinity – 1 to the power n -1 * -1 to the power n -1. So -1 to the power 2 times n -1/n okay, which is = sigma n = 1 to infinity 1/n and so the nth term of the series.

fn z, fn z was the nth term of the series, fn z was = - 1 to the power n -1 z to the power n-1/n and this gives us fn – 1 = 1/n which goes to 0 as n goes to infinity. So nth term of the series goes to 0 as n goes to infinity but what we notice is that if you put n = -1 in the series the series is sigma n = 1 to infinity fn-1 is = sigma 1/n and which is a divergent series.

So the series satisfies the necessary condition for convergence that is nth term of the series goes to 0 as n goes to infinity, but the series is not convergent. So it is just the necessary condition for the convergence of the series, it is not sufficient.

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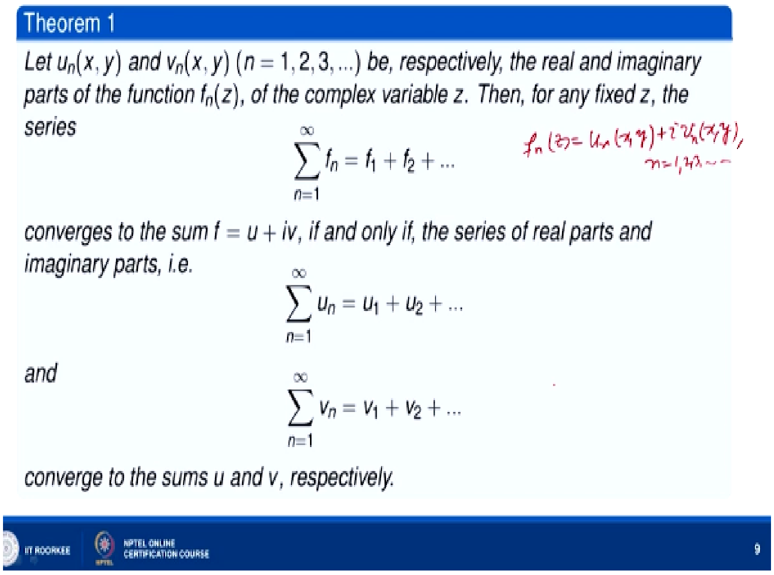

Now let us consider, let Un xy and Vn xy be respectively the real and imaginary parts of the function fn z, fn z = let us assume that fn z = Un xy + i Vn xy for all n = 1, 2, 3 and so on. So Un xy and Vn xy are real and imaginary parts of the function fn z of the complex variable z. Then for any fixed z the series sigma fn z, sigma fn z = f1 z + f2 z and so on, converges to the function f whose real part is say Uxy, imaginary part is Vxy, if and only if the series of real part sigma Un xy, that is Un xy + U2 xy and so on.

And sigma Vn xy which is v1 xy + v2 xy and so on converge to the sums Uxy and Vxy respectively.

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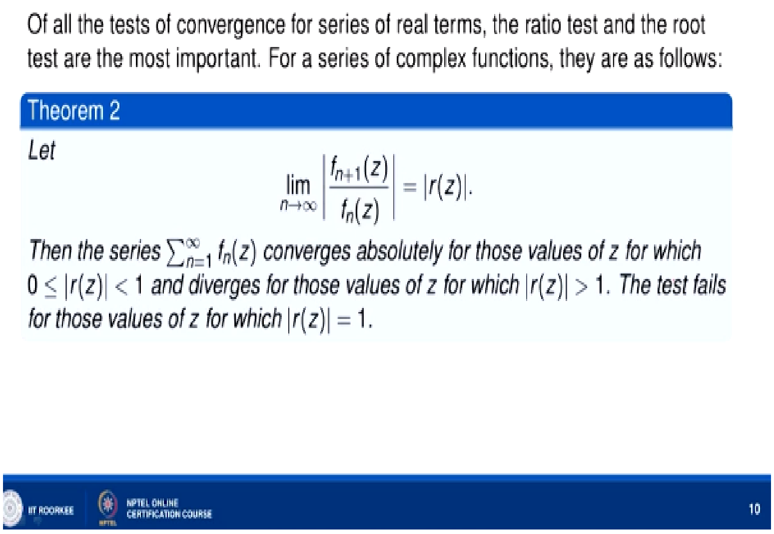

Now of all the test for the convergence of a series of real terms. The ratio test and the root test are we know they are the most important ones, for a series of complex functions they are as follows. So let n go to infinity, limit n tends to infinity, mod of fn + 1 z over fn z = mod of Rz then the series sigma n = 1 to infinity fn z converges absolutely for those values of z, for which 0 is <= mod of Rz < 1 and diverges for those values of z for which mod of Rz is >1. The test fails for those values of z for which mod of Rz is = 1.

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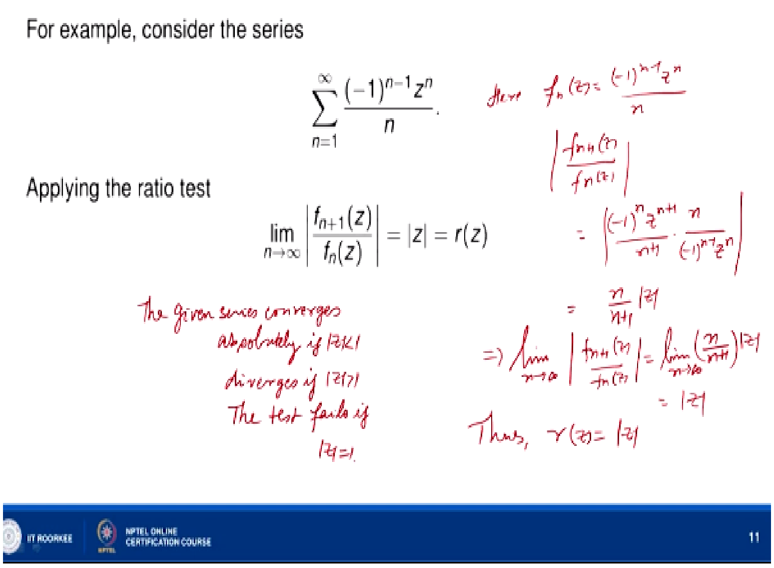

So let us consider the series sigma n = 1 to infinity -1 to the power n-1 z to the power n/n, then here fn z = -1 to the power n-1 z to the power n/n and therefore if you find mod of fn + 1 z over fn z, it comes out to be -1 to the power n z to the power n+1/n+1 * n/-1 to the power n-1 * z to the power n, and this is = n/n+1 mod of z to the power n+1/mod of z to the power n, so we get mod of z.

And therefore limit n tends to infinity mod of fn + 1 z/fn z = limit n tends to infinity n/n+1 * mod of z, n/n+1 as n goes to infinity tends to 1, so we get mod of z and thus Rz here is = mod of z and so we can say that the given series converges absolutely if mod of z is <1 diverges if mod of z >1. The test fails if mod of z = 1 okay. So we have to see what happens in the case mod z = 1 separately.

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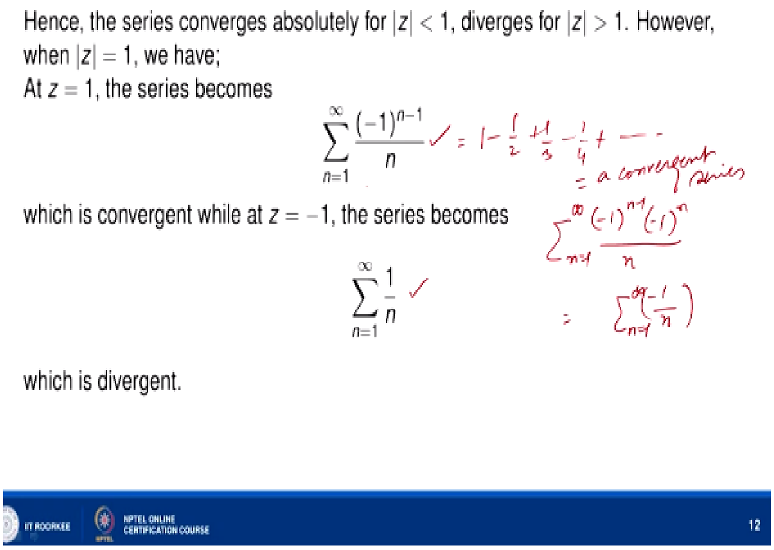

So the series converges absolutely for mod z < 1 diverges for mod z >1, however, when mod of z = 1, the test fails and why it fails we can see that. At z = 1 let us look at z = 1 what happens, z = 1 the series becomes sigma n = 1 to infinity -1 to the power n-1/n you can see, this is what we get at z = 1 and this series is what, 1-1/2 + 1/3 -1/4, so that is a convergent series by Liouville's real test.

Okay so this is convergent at z = 1 and at z = -1 if you find the series at z = -1 what will happen, sigma n =1 to infinity – 1 to the power n -1 * -1 to the power n, we will get this series. Sigma n = 1 to infinity – 1 to the power n – 1 * -1 to the power n/n and this will be = -sigma n = 1 to infinity 1 upon n okay or we can say we will get the series okay. Sigma n = 1 to infinity -1/n and this series we know this series is divergent because sigma 1/n is a divergent series, okay.

So we can say that whenever mod of z = 1 here we cannot conclude about the convergence or divergence of the series.

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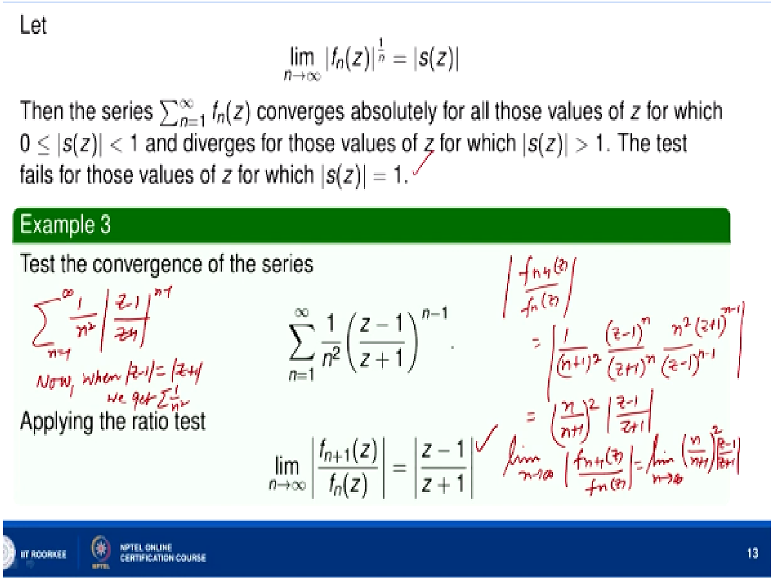

Now let us consider root test for the series of complex functions. So if limit n tends to infinity mod of fn z to the power 1/n = mod of Sz then the series sigma n = 1 to infinity fn z converges absolutely for all those values of z for which 0 is <= mod of Sz < 1 and diverges for those values of z for which mod of Sz is > 1. The test fails for those values of z for which mod of Sz = 1.

Let us test the convergence of this series. We are going to consider an example of a series sigma n = series of complex functions sigma n = 1 to infinity 1/n square z-1/z+1 to the power n-1 and we can apply here ratio test. So when you apply ratio test what we get mod of fn + 1 z/ fn z let us find this is = mod of 1/n+1 square z-1 to the power n/z+1 to the power n and this is fn + 1 z and divided by fn z.

So n square z-1 to the power n-1 and z+1 to the power n-1, okay, so this will be = n square/n + 1 whole square and we will get here mod of z – 1/z + 1 okay. So limit of this ratio okay, so limit n tends to infinity mod of fn + 1 z/fn z will be = limit n tends to infinity, n/n+1 whole square * mod of z-1 over z + 1 okay. So this is = n/n+1 goes to 1, so this is = mod of z-1/z+1 and therefore we can say that when mod of z-1/z+1 is < 1, the given series converges absolutely and when mod of z – 1/z + 1 is >1 the given series diverges okay.

When mod of z – 1/z + 1 is = 1 we shall have to see what the nature of the series separately.

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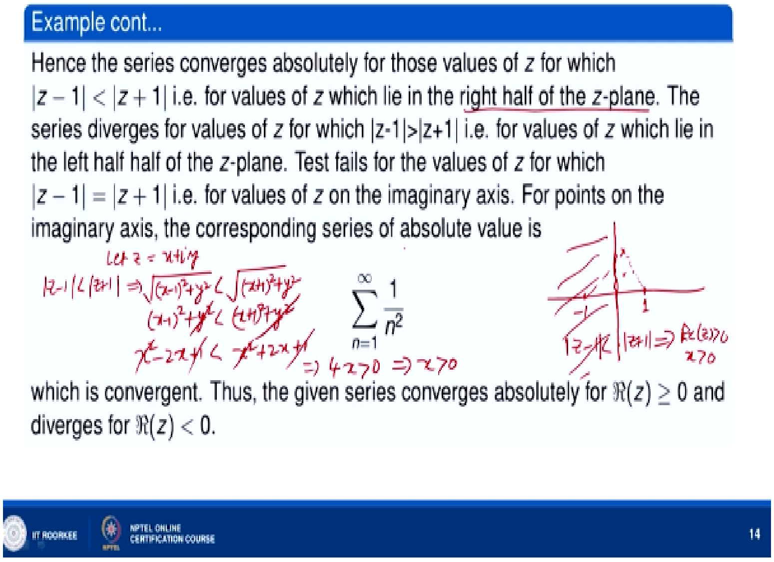

So hence the series converges absolutely for those values of z for which mod of z-1 is < z+1. Now let us see what are the points z where these points are located. Mod of z – 1 is the distance of z from the point 1. So z = 1 is here and z = -1 is here. Now this inequality is valid for all those z whose distance from 1 is < the distance from -1. Okay so this means that the points which lie to the right half of the z plane.

Okay the distance of this z from -1 will always be greater than it is distance from 1 okay. So this means mod of z -1 < mod of z +1 means real part of z >0 okay. So this inequality tells us that the values of z which lie to the right half of the z plane. Right half of the z plane means for which x is > 0. We can do it analytically also, here we have done it geometrically. We can do it analytically also.

So let us if we want to do analytically let us put, let z be = x + iy okay then mod of z – 1 < mod of z + 1 implies that under root x – 1 whole square + y square is < square root x + 1 whole square + y square. Squaring both sides, we get x – 1 whole square + y square < x + 1 whole square + y square. This y square and this y square cancel. You can now open these, so we have x square – 2 x +1 < x square + 2x + 1.

We can cancel 1, 1 x square, x square and then what we get 4x >0, so which implies that, so x > 0 means that the points which lie to the right half of the z plane satisfy the inequality mod of z-1 < mod of z + 1 and the series diverges for values of z for which mod of z – 1 is > mod of z + 1 that means left half of the z plane. So that means x > 0, for values of z which lie to the left half of the z plane that means the values of z for which x is < 0.

Now test fails for the values of z for which mod of z – 1 = mod of z + 1, mod of z – 1 = mod of z + 1 means x = 0 okay. If you put here instead of less equality, then we will have 4x = 0. So x = 0, so mod of z – 1 = mod of z + 1 means the points in the z plane whose distance from 1 and -1 are same. So that is the points which lie on the imaginary axis okay. Any point on the imaginary axis has it is distance from 1 and -1 same okay.

So for values of z which lie on the imaginary axis for points on the imaginary axis the corresponding series of absolute value is now let us see what happens. So if you take modulus here okay. Sigma n = 1, 2 let us consider the series of absolute values, sigma n = 1 to infinity, 1/n square mod of z-1 upon z + 1 raise to the power n-1 okay. So now when mod of z -1 upon z + 1 = 1 okay.

We get sigma 1/n square series okay, which we know is a convergence series. So when mod of z – 1/z + 1 is = 1 okay we see that this series is convergent and therefore the series is absolutely convergent for values of z which satisfy mod of z – 1 <= mod z + 1, and for mod of z-1 > mod of z + 1 this series is divergent okay. So the series converges absolutely for real part of z >= 0 that is x >= 0.

For x > 0 we find from here okay, for x = 0 we have seen separately that this series is absolutely convergent and diverges for x < 0.

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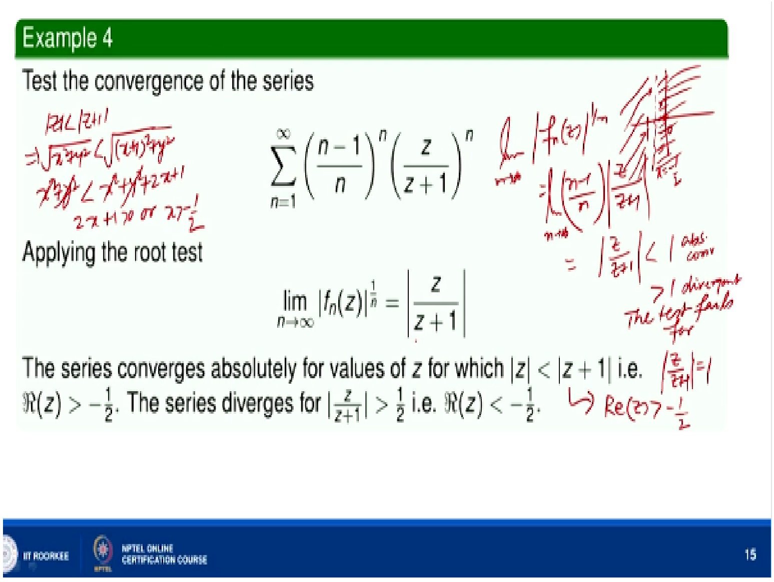

Now let us consider the series sigma n = 1 to infinity, n – 1/n to the power n okay, z/z + 1 to the power n. So here when we apply root test then limit n tends to infinity mod of fn z to the power 1/n, when you take 1/n as root of this fn z we get mod of fn z raise to the power 1/n = n-1/n * z/z + 1 as you know this root test we apply when the nth term of the series contains n in the exponent okay.

So here we have n in the exponent of n-1/n and therefore it will be convenient to apply root test. So this is what we have. Now then limit n tends to infinity if you find n -1/n goes to 1 as n goes to infinity so we get mod of z/z+1 okay and therefore the series converges absolutely for values of z, for which mod of z/z + 1 is < 1, so absolute convergence for mod of z < mod of z + 1 and divergence for mod of z > mod of z +1.

The test fails for mod of z/z+ 1 = 1 okay. The test fails for this, so mod of z < mod of z + 1 means what? It is the set of those points z whose distance from 0 is < this distance from -1 okay and if you take this is 0 here and -1 here. So distance from -1 is < the distance from 0 okay. So this is -1/2, okay, this is -1/2, so this means that if you take a point to the left of this is x = -1/2.

If you take a point to the left of x = -1 is distance from -1 will be < is distance from this okay. So this means that we take the right half of this x = -1/2, if you take the right 1/2 then distance of a point from -1 will be more than the distance from 0. So this actually tells us it is real part of z > -1/2. We can see it also directly z = x + iy will give you square root x square + mod of z < mod of z + 1.

If you want to see analytically we can write under root x square + y square for mod of z and here we can write under root x + 1 whole square + y square. So squaring we have x square + y square < x square + y square + 2x + 1. So these 2 cancel and we get 2x + 1 > 0 or x > -1/2. So right half of the, right side of the line x = -1/2 okay this part. Okay so all points z must lie to the right side of x = -1/2 and similarly when mod of z is < mod of z + 1 that is real part of z is < -1/2 okay.

So this side okay for all any z to the left of x = - 1/2 the series is divergent and the series test fails when mod of z/ mod of z + 1 is = 1. So test fails for points z which lie on the line x = -1/2.

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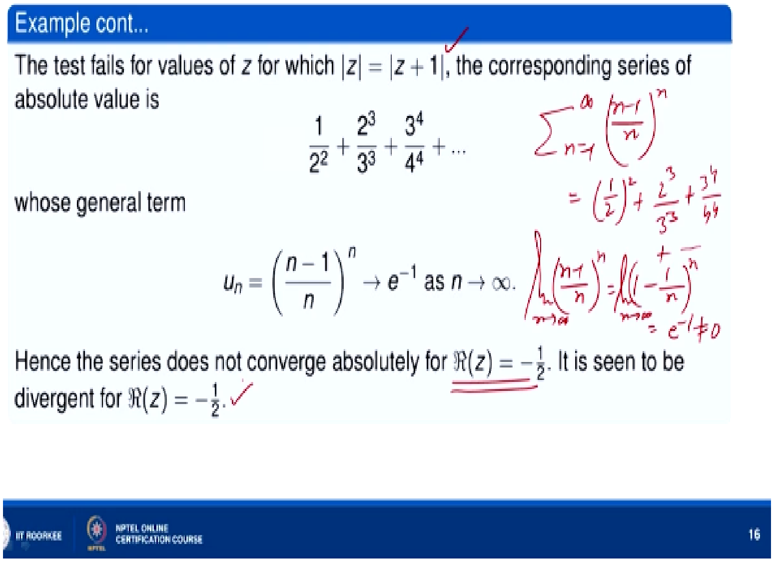

Now the test fails for this okay. So the corresponding series absolute value is what let us see when mod of z/z+1 = 1 we get the series sigma n = 1 to infinity, n-1/n to the power n okay. So when you put n = 1 it is 0, when you put n = 2 it is n = 2 will give you 1/2 raise to the power 2 okay and n = 1 will give 0, so this is what we have 1/this gives you 1/2 raise to the power 2 and then n = 3 will give you 2 to the power 3/3 to the power 3.

And then 3 to the power 4/4 to the power 4 and so on. So and we can see that n-1/n, the nth term of the series n-1/n to the power n we can write as 1-1 upon n raise to the power n okay and so limit n tends to infinity this gives = e to the power -1 okay which is not 0, okay this is not 0 so nth term does not go to 0 and therefore the series is not absolutely convergent okay. So series is not absolutely convergent for real part of z = -1/2.

It can be shown that it is divergent for real part of $z = -1/2$. With this I would like to end my lecture. Thank you very much for your attention.