**Advanced Engineering Mathematics**
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**Lecture – 11**
**Winding Number and Maximum Modulus Principle**

Hello friends, welcome to my lecture on winding number and maximum modulus principle. Suppose that gamma is the closed contour in the complex plane okay and C be a given point is C - gamma okay.

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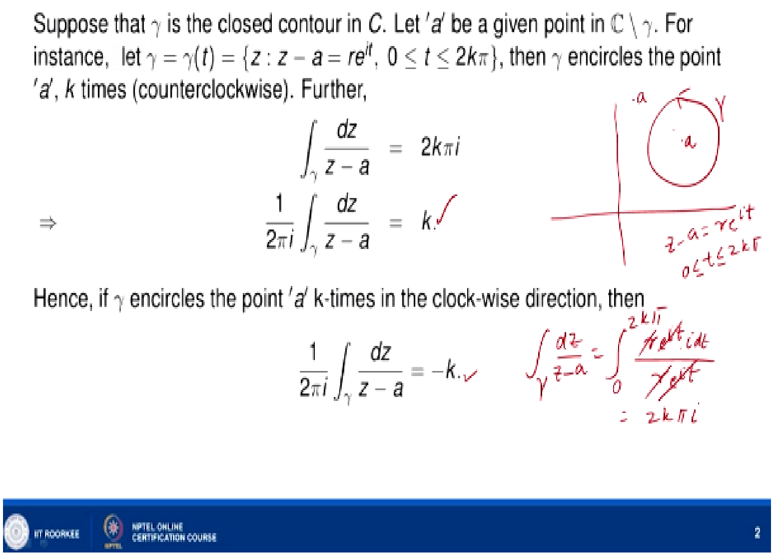


So let us take a close contour C okay. Let gamma be a close contour in C okay and a be a given point in C – gamma okay. So this is a is any given point here okay which does not belong to gamma okay. It can be either inside gamma or it is outside gamma, for instance let us say gamma = gamma t, that is this sort of all complex number z such that z – a = r e to the power it where 0 is <= t, <= 2 k pi.

So by this we are actually considering here gamma to be the circle with center at a and radius r. So here we are considering the particular case where this a is inside the circle okay, gamma and the circle, equation of the circle gamma is z – a = r e to the power it. Now t varies from 0 to 2 k pi, this means that we are moving along gamma k times okay.

We are taking round of the point a along gamma k times that means gamma encircles the point a k times in the counterclockwise direction and the further we notice that integral over gamma dz/z – a = 2 k pi i why because integral over gamma dz upon z – a will be = integral over 0 to 2 k pi and dz = r e to the power i t * idt/z-a which is r e to the power it. So we will get 2k pi i okay.

Or we can say 1/2 pi i integral over gamma dz upon z – a = k okay. So hence if gamma encircles the point a k times in the counterclockwise direction then we are getting the value as k, but if gamma encircles the point a k times in the clockwise direction okay, in the opposite direction then we will have the value as –k okay. So here we are considering a particular case where gamma is the circle with center at the point a and the radius of gamma is r.

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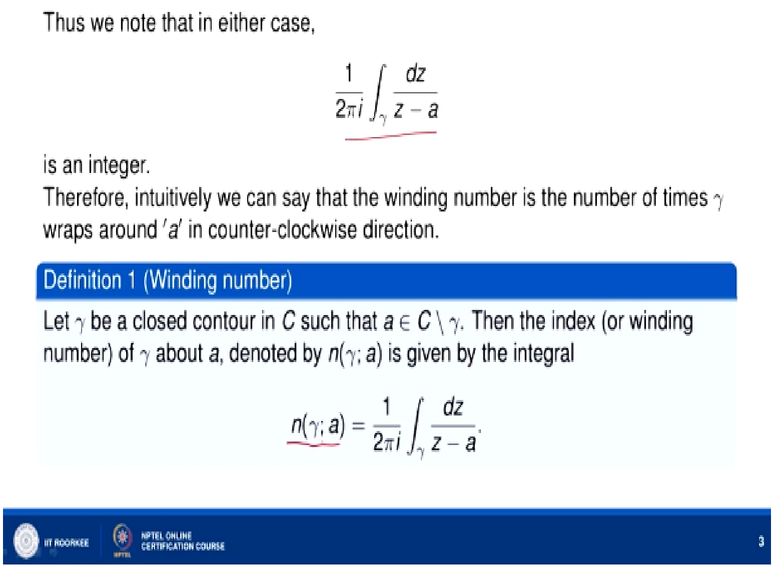


In the general case okay where gamma is any closed contour okay we can intuitively say that the winding number is the number of times so this actually give 1/2 pi i okay. Integral over gamma dz over z – a whether we are moving anticlockwise or we are moving clockwise it is a integer. In the anticlockwise direction we are getting k in the clockwise direction we are getting –k, so it is always an integer.

Now intuitively we can say that the binding number is the number of times gamma wraps around the point a in counterclockwise direction. Now we have the analytic definition here of the winding number, let gamma be any closed contour in C, such that a does not belong to gamma, that is a belongs to C – gamma then the index or binding number of gamma about a denoted by n gamma a is given by the integral n gamma a = 1/2 pi i integral over gamma dz/z-a.

So this n gamma a represents the index of the closed contour gamma about the point a or the bounding number about the point a, bounding number or gamma.

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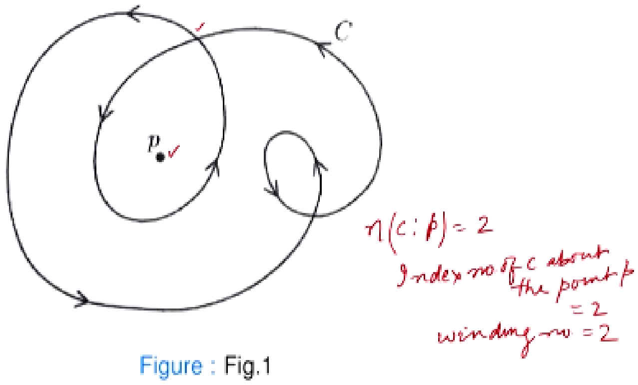


Figure : Fig.1

Now you can see here this is our point P and we are moving like this. Suppose we start from here okay. So we are moving anticlockwise okay about the point P okay twice okay. So this means that n, if this is your curve gamma okay, here we are taking it as curve C, so n c p okay = 2. We are moving about this point P in the anticlockwise direction okay twice. So index number of C okay about the point a, here we have P, about the point P = t or winding number = 2.

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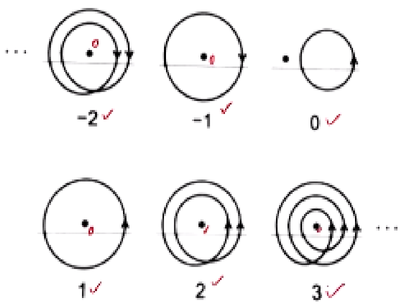


Figure : Fig.2

Now let us look at this point here, this is your origin suppose okay, then we are moving twice in the clockwise direction okay about the origin. So index number is = -2 here, here again we are moving about this point origin let us say, so we are moving clockwise okay, once and therefore the index number is -1, here we are moving anticlockwise but the point this 0 is outside. We are not moving in the anticlockwise direction about the point 0 okay.

It is lying outside, so index number is 0 and here we are moving about the point 0 in the anticlockwise direction 1, so binding number is 1, here we are moving about this point 0 twice in the anticlockwise direction. So winding number is 2. Here we are moving 3 times okay in the anticlockwise direction about this point, let us say origin, so the binding number is 3.

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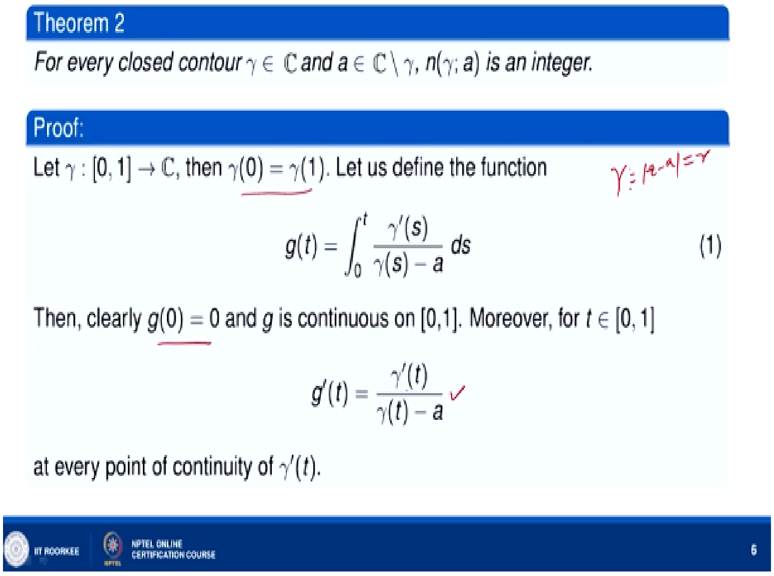


Okay now let us show that for every closed curve, we have earlier seen that in the case of gamma being circle, mod of z – a = r okay and gamma a is an integer okay. When we take anticlockwise direction it is positive integer, when we take clockwise direction it is a negative integer. Now let us prove that for every simple closed curve okay gamma in C where a is any complex number in C – gamma, that means it does not belong to gamma okay.

And gamma is an integral, so let us start with the proof, let gamma be a mapping from 0, 1 * C then since gamma is a closed curve okay gamma 0 will be = gamma 1 okay. The value of gamma will co-inside at 0 and 1. Now let us define the function gt = integral over 0 to t, gamma dash s over gamma s – a ds, then you can see that g0 = 0, because when you put t = 0 the lower and upper limit of this integral will be 0.

So g0 = 0 and furthermore from this definition of g we can see that g is continuous on 0, 1 interval. Further for t belonging to 0 on interval g dash t = gamma dash t/gamma t –a okay by differentiation under the sign of integration. Now at every point of continuity of gamma dash.

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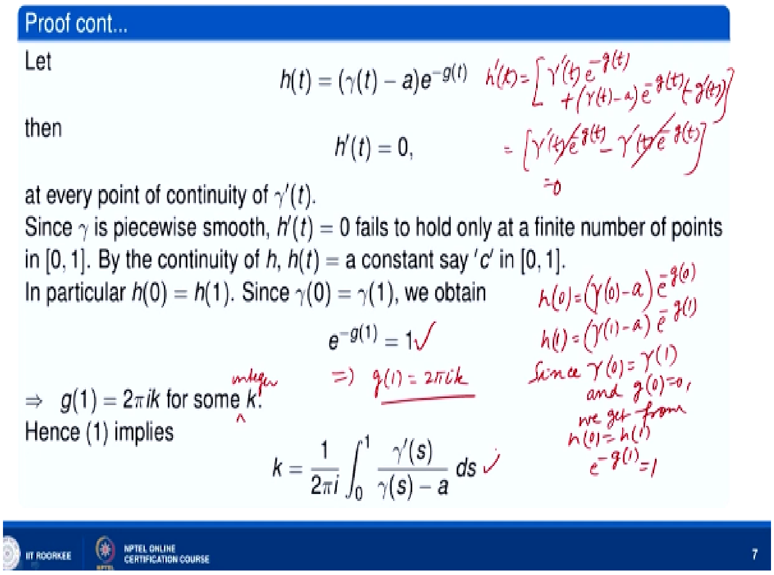


Let us define function ht = gamma t – a e to the power – g t okay. Then h dash t. let us find h dash t here, h dash t = it is product of 2 functions of t, so gamma dash t * e to the power – gt + gamma t – a * e to the power – gt * - g dash t, okay, and let us go to the previous slide g dash t = gamma dash t/gamma t –a. So g dash t * gamma t – a = gamma dash t okay. So we have g dash t * gamma t – a = gamma dash t.

So this will be = gamma dash t e to the power – gt – gamma dash t e to the power – gt, okay, so this cancels with this and we get 0. So h dash t = 0 at every point of continuity of gamma dash. Now since gamma is piecewise smooth h dash t = 0 fails to hold only at a finite number of points in 0, 1 interval. By the continuity of h therefore ht is a constant, let us say C okay, some constant C in the interval 0, 1.

In particular, we can say that h0 is same as h1 okay. Now h0 = gamma 0 – a e to the power – g0, okay and h1 = gamma 1 – a e to the power – g1 okay. We know that gamma 0 = gamma 1 okay and g0 = 0 okay, so what do we get. Since gamma 0 = gamma 1 and g0 = 0, we get h1 here okay. So we get from h0 = h1, what we get, gamma 0 – a will cancel with gamma 1 – a okay. What we will get e to the power –g0.

E to the power –g0 means 1 okay, so e to the power – g1 = 1 okay, that is we get this equation and this we will mean that g1 = 2 pi i k, okay for some integral k, for some integer k okay. So now what is g1? Let us see g1 = integral 0 to 1, gamma dash s/gamma s – a ds, so k = 1/2 pi i integral over 0 to 1 gamma dash s/gamma s-a ds okay. So n gamma a where n gamma is this okay 1/2 pi i integral over gamma dz/z-a, okay it is an integer okay we are taking gamma as gamma t here okay or gamma s.

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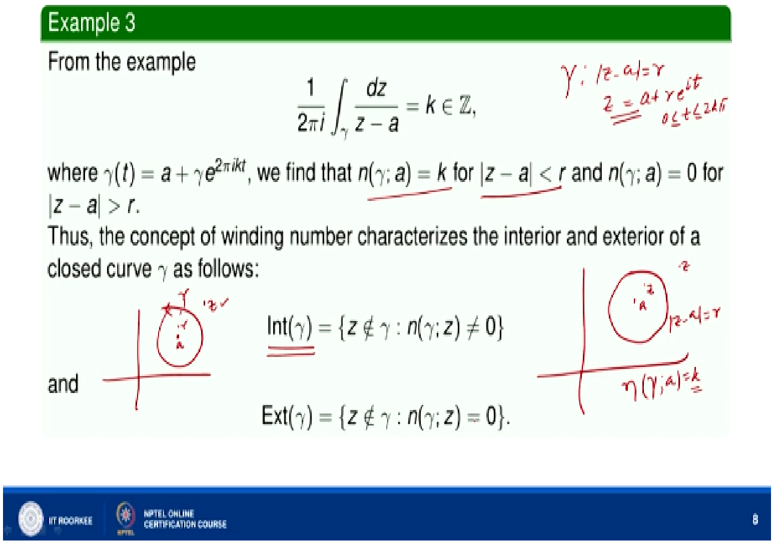


So from the example 1/2 pi i integral over gamma dz upon z – a = k where k is an integer, okay let us go back to the example with which we started, we had considered gamma as a circle okay, at the point z = a of radius r. So in that example we had taken gamma as mod of z – a = r that is gamma bar given by z = a + r e to the power it okay. So and t was varying from 0 to 2 k pi okay.

So gamma t was a + gamma, a + r e raise to the power 2 pi i kt okay. We find that n gamma a = k, so this means what, suppose this is 0.a okay, this 0.a and we are getting this circle mod of z – a = r okay, then from here it follows that n gamma a, from this example it follow that gamma a was = k, so if z is inside this circle, mod of z – a = r then and gamma a = k, then for that means for any z inside the circle and gamma a = k and when z is outside okay.

So when mod of z – a is < r and gamma a = k, suppose z is outside okay, it is not inside let us draw another figure, z is here suppose then and gamma a = 0 if z lies outside okay. Mod of z – a > r. So thus the concept of binding number characterizes the interior and exterior of the closed curve gamma s follow. For the interior of gamma okay, interior of gamma by interior of gamma what do we get it is the set of all those points that do not belong to gamma, and for which n gamma z is never 0.

Exterior of gamma is those points z which do not belong to gamma and n gamma z = 0 because you take any z outside this circle okay, then for that n gamma z will be = 0. So if n gamma a = k for mod of z – a < r and n gamma a = 0 for mod of z –a > r, so interior of gamma is the set of all those points which do not belong to z such that n gamma z. So we can see the interior and exterior of a closed curve by this. So this should be n gamma a not 0 and n gamma a = 0.

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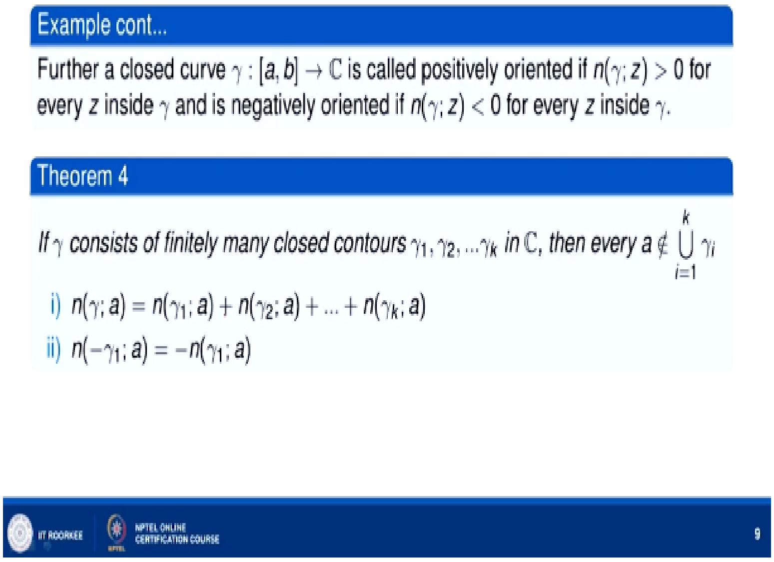


Now let us consider a closed curve gamma from a, b to C, it is called positively oriented. If n gamma z is > 0, for every z inside gamma and it is negatively oriented if n gamma z is <, so we can say that if gamma is a closed curve it will be positively oriented if n gamma z is > 0 for every z inside gamma and it is negatively oriented if n gamma z is < 0 for every z inside gamma.

Now if gamma consists of finitely many closed contours gamma 1, gamma 2, gamma k and C then for every a which do not belong to union of gamma i okay, n gamma a = n gamma 1a + n gamma 2 a and so on and gamma k a okay, that is the binding number of gamma about the point a is some of the binding numbers of the curves to gamma 1, gamma 2, gamma k about the point a.

This we can easily see from the figures here, you can see here this here the gamma consists of 2 curves gamma 1 and gamma 2, so n gamma about this point, let us say this is origin, so n gamma about the point origin is n gamma 1 about the point origin and n gamma 2 about the origin. Here n gamma about the point origin is n gamma 1, you can call it as gamma 1, this as gamma 2, this as gamma 3 okay and similarly for the other curves okay.

And this identity tells us that the binding number of n okay about the curve – gamma 1, so gamma 1 we are taking in the anticlockwise direction – gamma 1 means we are going in the clockwise direction. So binding number about the point a of the curve – gamma 1 okay is same as – of binding number of gamma 1 about the point a okay. So this is made clear.

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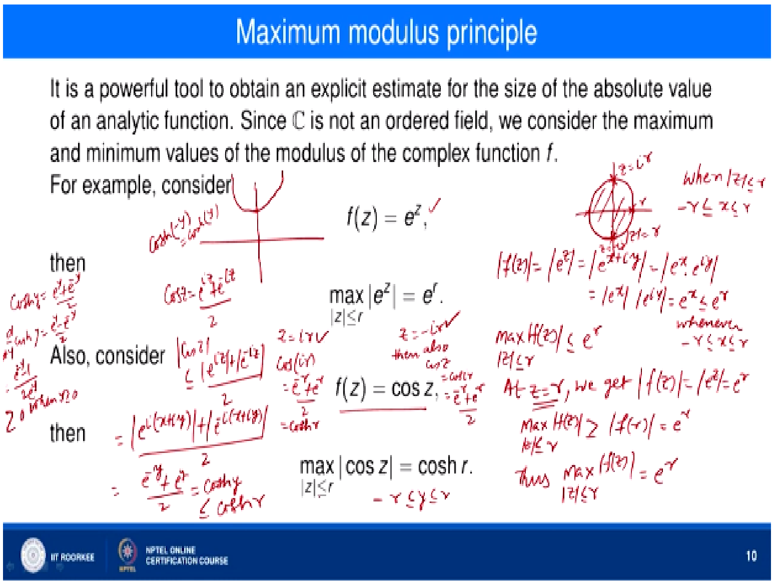


Now let us discuss maximum modulus principle. Maximum modulus principle is the powerful tool to obtain an explicit estimate for the size of the absolute value of an analytic function. Since C is not an ordered field we can consider the maximum and minimum values of the modulus of the complex function okay. So let us consider say for example fz = e to the power z okay.

Then if you take the close circular disc mod z <= r okay, let us consider the closed circular disc mod of z <= r, then maximum value of mod of e to the power z okay this closed circular disc, mod of z <= r is e to the power r and it is attained at the point z = r, this okay. Here z = r, so it is attained on the boundary of the closed circular disc. Okay now let us us see how we get this maximum value, mod of fz = mod of e to the power z which is = mod of e to the power x + i y.

So this is mod of e to the power x * e to the power i y and then mod of e to the power x * mod of e to the power i y gives us, e to the power x is always positive because x is real. So this is e to the power x and mod of e to the power i y is mod of cos/+ i sin/ so that is = 1. So mod of fz = e to the power x. Now for this circle mod of z = r okay, x varies from – r to +r okay.

So when mod of z is <= r, -r is <= x, <= r okay. So and e to the power x is an increasing function of x, so this is <= e to the power x or whenever –r is <= x, <= r. So this is valid for all z okay, such that mod of z is <= r. So maximum value of mod of fz when mod of z is <= r is <= e to the power r. Now let us show that there is a point satisfying mod of z <= r where mod of fz = exactly e to the power r.

And that is the point z = r. So at z = r we get mod of fz = mod of e to the power z = e to the power r. Mod of e to the power r is e to the power r because e to the power r is always positive. So there is a point z = r where mod of fz becomes e to the power r. So maximum value of mod of fz, when mod of z is <= r okay, is definitely > mod of fr, okay and mod of fr is e to the power r.

So what do we notice, maximum value of mod of fz, when mod of z is <= e to the power r and here maximum value of mod of fz when mod of z is <= r is > r = e to the power r and thus maximum value of mod of fz when mod of z is <= r is e to the power r and we can see that e to the power r, the maximum value of mod of fz is attained on the boundary of mod of z <= r.

Now if you take fz = cos z okay, then cos z = e to the power iz + e to the power – iz/2 okay. So mod of cos z is <= mod of e to the power iz + mod of e to the power – iz/2 and this is = mod of e to the power i times x + iy + mod of e to the power – i times x + iy/2. So here what do we get, this is e to the power ix * e to the power i square y, i square y means –y, mod of e to the power ix = 1 and so this is e to the power –y.

And here we get e to the power – ix whose modulus is 1 and then we get e to the power –i square y which is e to the power y. So e to the power –y + e to the power y/2 which is cos hyperbolic y, okay, and cos hyperbolic y is an increasing function of y and here what is happening is that for mod of z <= r, y also varies from –r to +r okay. So –r <= y <= r. So this is <= cos hyperbolic r okay.

You can easily see that cos hyperbolic y is an increasing function because cos hyperbolic y is e to the power y + e to the power – y/2 okay and when you differentiate cos hyperbolic y you get e to the power y – e to the power –y/2 which is nothing but e to the power 2y – 1/2, this okay, cos hyperbolic y is this function okay, it is an even function right. So at y = 0 cos hyperbolic y takes value 1, so it is like this okay.

And so it is a symmetric function about this y okay, cos hyperbolic – y = cos hyperbolic y. So when we consider cos hyperbolic y, we need to consider the values of y, positive values of y, for the indicative values of y, it is graph is the same because it is symmetric. So it takes the same values as it takes for positive values of y. so e to the power 2 pi – 1 over 2 e to the power y is always >= 0 when y is >= 0 okay.

So it is an increasing function of y, so e to the power –y + e to the power y/2 is greater than, is an increasing function for all values of y. So cos hyperbolic y <= cos hyperbolic r. So what we have maximum value of mod of cos z when mod z is <= r is <= cos hyperbolic r and then what do you do, you take. Now you want to show that maximum value of mod of cos z when mod z is <= r is >= cos hyperbolic r okay.

So for that you need to consider z = ir, if you take z = ir then cos ir = e to the power –r + e to the power r/2 that is cos hyperbolic r. You can also consider z = -ir. If z = -ir then also cos z = cos ir = e to the power i square r, so e to the power –r + e to the power r/2 and both these points –ir okay and ir okay. Whether this point you take or you take this point, for both the points cos z takes the maximum value e to the power r + e to the power –r/2 and these are the points here. This is z = ir, this is z = -ir okay.

So maximum value of mod of cos z for mod z <=, mod of z <= r will always be >= the value of mod of cos z at z = ir or z = -ir which is cos hyperbolic r. So maximum value of mod of cos z when mod z is <= r = cos hyperbolic r okay.

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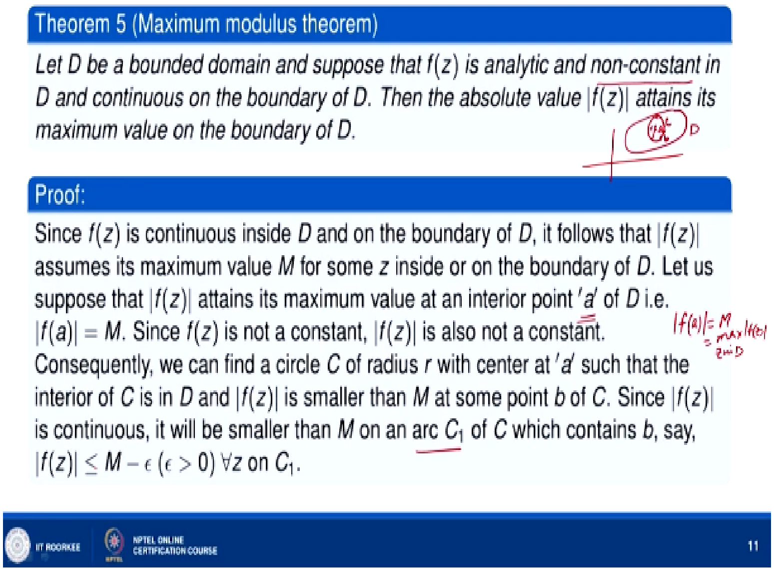


So we go to now this maximum modulus theorem, let D be a bounded domain and suppose that fz is analytic and non-constant in D and continuous on the boundary of D, then the absolute value of mod of fz attains it is maximum value on the boundary of D, as we have seen in the case of fz = e to the power z and fz = cos z.

Then their maximum values in the case of e to the power z is e to the power r it is attained at the boundary point z = r while in the case of cos z the maximum value of mod of cos z is attained at the points z = ir and z = -ir which lie on the boundary of mod of z <= r. So let us prove this, since fz is continuous inside D and on the boundary of D, on the boundary of D we are given that it is continuous inside D, it follows from the analyticity okay.

So since fz is continuous inside D and on the boundary of D it follows that mod of fz must assume it is maximum value for some z inside r on the boundary of D, by the continuity. Now let mod of fz attains it is maximum value at an interior point okay. So we are assuming that, actually have to prove that it assumes it is maximum value on the boundary of D, but here we want to prove it by contradiction method.

So we are assuming that, it assumes it is maximum value at a point interior to D okay and that point let us say be = a. So that is mod of fa = M. Since fz is not constant we are assuming that fz is non-constant okay, so mod of fz is also not constant okay and consequently we can find a circle C of radius r, so let us say suppose this is your domain okay D, so a is a point here. we can construct a circle.

Now mod of fa we are assuming to be = maximum value M, M is the maximum value of mod of fz for all z in D okay. So maximum value of mod of fz we are assuming that, it occurs at the point a okay. So mod of fa = M and since fz is not constant mod of fz also will not be constant so we can find a circle c, let us cost up to circle c here okay, constant circle c of radius r.

Let us say this radius is r okay, a bit center at a such that the interior of C is in D, the curve lies C, curve lies completely inside D and mod of fz is smaller than M at some point b of C, okay, so there is some point b on the curve C such that mod of fz is smaller than M because M this is the maximum value and it is not constant, so it will have to happen that there will be some point on the curve C at which mod of fz is smaller than M.

So since mod of fz is continuous it will be smaller than M, fz is continuous so mod of fz is continuous. So there will be an arc of the circle C okay, let us call that arc as C1, which contains B and such that mod of fz is <= M – epsilon for all z on C1. So there will be some part of this curve C, which we can call as C1 which contains the point B and for which mod of fz is <= M – epsilon okay.

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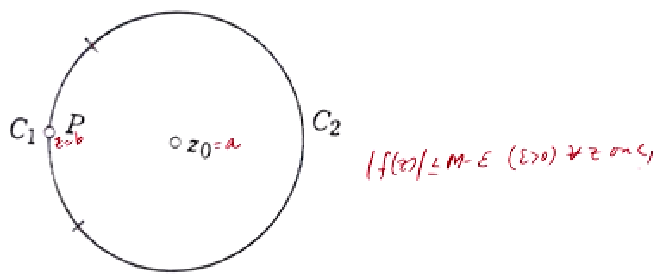


Figure : Fig.3

So this is the curve C okay, this is the curve C and we have taken this arc okay, this is my point B okay, z = b here okay, and this is z0, z0 is a actually, z0 is a we are taking the curve C, we center at a. So z0 in this figure is actually a and the arc C1 of the circle C contains the point B and mod of fz is <= M- epsilon okay, where epsilon is > 0 okay. For all z on C1 okay.

So let us say C1 has length 1, the complementary arc C2 okay, the complementary arc C2 then will have length 2 pi r – l1 okay.

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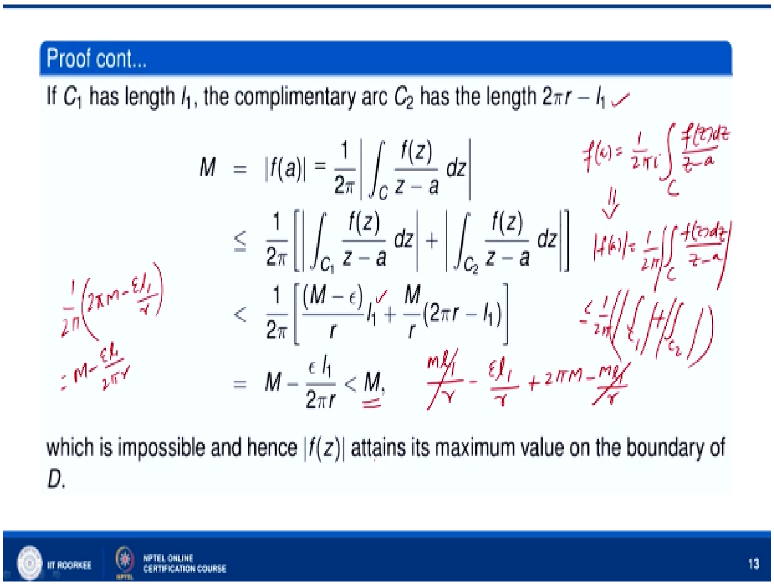


So M = mod of fa then < f because fa is given by Cauchy's integral formula 1/2 pi i integral over C, fz over z-a dz okay. So mod of fa will be <= this will imply mod of fa, this is = 1/2 pi integral over C mod of integral over C fz dz/z- here. Okay so and then we can break it into 2 parts, this is <= 1/2 pi mod of integral over C1 + integral over C2 okay. So with the integrant being fz/z-8 okay. So integral over C1 fz/z –a + integral over C2 fz/z –a okay, like this.

So 1/2 pi, now the mod of fz/z –a on the curve C1, okay, on the curve C1 is <= M – epsilon over r, because the circle C is having center at a and radius r. So mod of z – a = r on C1 okay and on C1 mod of fz is <= M – epsilon. So M- epsilon/r * length of C1. So length of C1 is l1 and then here mod of fz is <= M okay, because M is the maximum value okay. So M and then mod of z – a = r length of C2 is 2 pi r-l1.

So when you simplify this what you get is M- epsilon * l1/2 pi R okay. Because this is if you multiply what you get Ml1 okay/r – epsilon l1/r okay and here what do we get here, M 2 pi M okay – M l1/r, so this cancels with this okay and we get 1/2 pi times – epsilon l1/r. So this is M- epsilon l1/2 pi r. Now definitely this quantity is < M because epsilon l1/2 pi r is a positive quantity. So what we get M is < M which okay, which is impossible and therefore mod of attains it is maximum value on the boundary of D.

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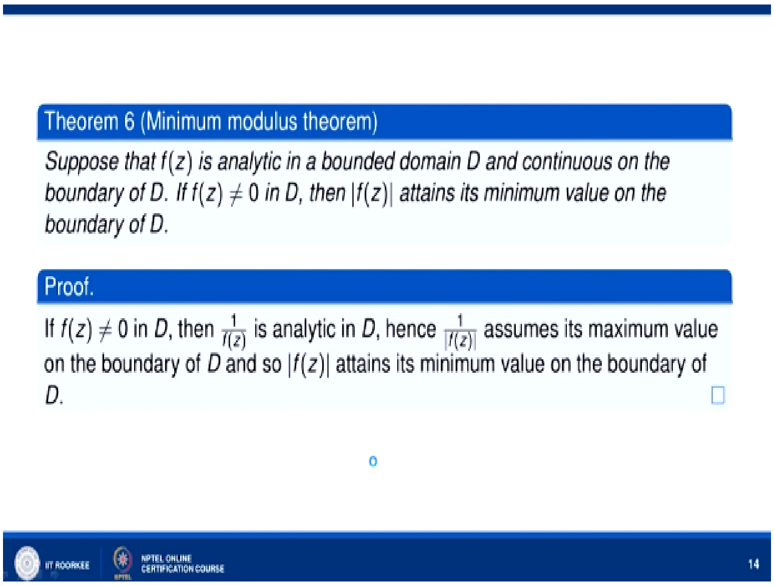


Now we can easily prove the minimum modulus theorem from the maximum modulus theorem. Suppose that fz is analytic in a bounded domain D and continuous on the boundary of D if fz is not equal to 0 in D then mod of fz attains it is minimum value on the boundary of D. So suppose that if fz is not equal to 0 and D then let us suppose that, we consider the function 1/fz okay.

So then 1/fz is analytic in D and hence by the maximum modulus theorem 1/fz will assume it is maximum value on the boundary of D and when 1/fz is maximum on the boundary of D mod of fz is attaining it is minimum value on the boundary of D.

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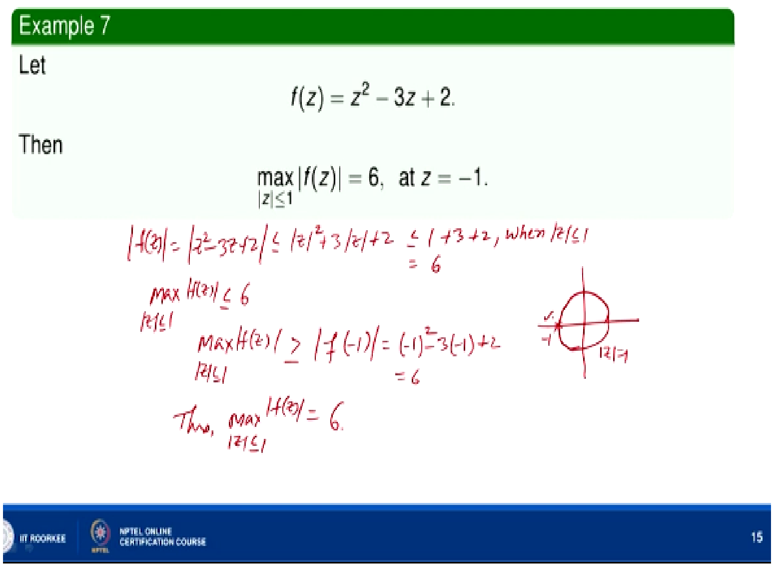


Let us say for example fz = z square – 3 z +2 okay. We need to find the maximum value of fz okay, mod of fz, when mod of z is <= 1. So mod of fz = mod of z square – 3 z + 2 which is

<= mod of z square + 3 times mod of z + 2 okay and which is <= 1 + 3 + 2 okay, whenever mod of z is <= 1. So this is = 6 okay. So maximum of mod of fz, when mod of z is <= 1 is <= 6.

Now we notice that at z = 1, z = -1 okay z = -1 fz becomes equal to 6. So maximum value of mod of fz when mod of z is <= 1 is greater than or = mod of f -1 okay. Because -1 lies on the unit circle, okay, this is mod of z = 1, so this is -1 here okay. So when you take z = -1 it satisfies the inequality mod of z <= 1 and so the maximum value of mod of fz when mod z get <= 1 will always be greater than mod of f-1, but this gives you -1 square -3 * -1 + 2 which is = 6.

So maximum value of mod of z when mod z is <= 1 is >= 6 and thus this is = 6 okay. This is = 6 which is attained, maximum value is attained at the boundary, that is at the point z = -1 of mod z = 1 okay. With that I would like to end my lecture. Thank you very much for your attention.