Advanced Engineering Mathematics
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Lecture – 10
Morera's Theorem, Liouville's Theorem and Fundamental Theorem of Algebra

Hello friends welcome to my lecture on Morera's theorem, Liouville's theorem, fundamental theorem of algebra. We are going to discuss the consequences of Cauchy integral formula and higher order Cauchy integral formula. So suppose we first discuss the Morera's theorem.

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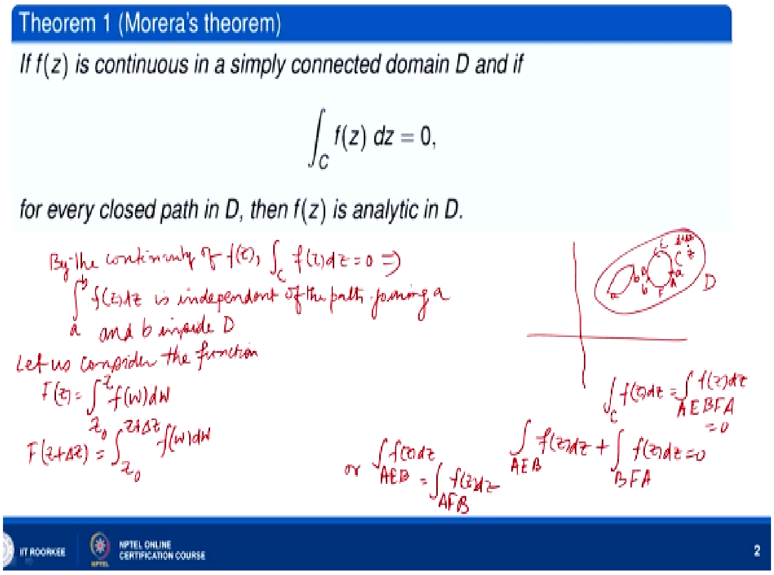

If fz is continuous in a simply connected domain D and if integral over c fz dz = 0 for every closed path in D, then fz is analytic in D. So let us prove this Morera's theorem, it is the converse of the Cauchy integral theorem. So we are assuming that fz is continuous in a simply connected domain D and integral over c fz dz = 0, so let us say this is your domain D okay and c is any simple close curve in D okay.

We are given that integral over c fz dz = 0. So by the continuity of fz integral over c, fz dz = 0 implies that the integral over c fz dz is independent of the path, we can write integral over a to b, fz dz is independent of the path joining a and b inside D. So you take any 2 points a and b in the domain D, you join them by any path okay. The integral over a to b fz dz will always be same.

This follows by the continuity of fz and integral over c fz dz = 0 why because if you take any point here okay, then integral over c, let us say this point is a, this point is b okay. So integral over c fz dz we can write as, let me call it as A point, this is B, this is some E point, this is F okay. So integral over A, E, B, F, A fz dz = 0 okay, we can write it as integral over A E B fz dz + integral over B F A, fz dz = 0.

Or we can say this follows because when we assume fz to be continuous integral over C can be written as integral over C1 + integral over C2, where C1 and C2 are the 2 paths of the curve C. So integral over AEB + integral over BFA and this is = integral over AEB, fz dz = integral over, when we take this term to the other side it will become – integral over BFA fz dz so it will be AFB okay.

So integral from A to B in the anticlockwise direction AEB is same as integral from A to B by the curve, through the curve AFB okay. So integral of fz from A to B does not depend on the path which joints the point A to the point B. It depends only on the end points. So by the continuity of fz when integral over c fz dz = 0. Integral over A to B fz dz does not depend on the path which join the point A to B, it only depends on the end points A and B provided they lie inside D okay.

Now let us consider the function fz = integral over z0 to z, fw dw okay. We can write it as from this we can, if z + delta z is the point z is any fixed point let me say z is the point here in D okay and z + delta z is another point in D, okay. So fz + delta z will be = z0 to z + delta z fw dw and then what we will have.

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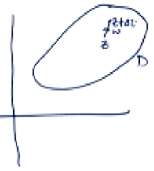
$$F(z+\Delta z)-F(z) = \int_{z_1}^{z+\Delta z} f(w)\,dw - \int_{z_1}^{z} f(w)\,dw = \int_{z}^{z+\Delta z} f(w)\,dw$$

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} \big(f(w)-f(z)\big)\,dw$$

By the continuity of $f(z)$, for a given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$|f(w)-f(z)| < \varepsilon$ whenever $|w-z| < \delta$



Since $\int_{z}^{z+\Delta z} \big(f(w)-f(z)\big)\,dw$ is independent of the path joining z to $z+\Delta z$. We can consider the line segment joining z to $z+\Delta z$

If we take $|\Delta z| < \delta$, then

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} \big(f(w)-f(z)\big)\,dw\right| < \frac{1}{|\Delta z|}\,\varepsilon\,|\Delta z| = \varepsilon$$

$$\Rightarrow \left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| < \varepsilon \Rightarrow \lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

$\Rightarrow F'(z) = f(z) \Rightarrow F'(z)$ exists at any z in D

$\Rightarrow F(z)$ is analytic function in D

$\Rightarrow F'(z)$ is an analytic function in D but $F'(z) = f(z)$ so $f(z)$ is analytic in D.

fz + delta z – fz = integral over z0 to z + delta z fw dw – integral over z0 to z fw dw, okay, so fz + delta z – fz will be = integral over z0 to z + delta z, fw dw – integral over z0 to z, fw dw and this will be = integral over z to z + delta z of fw dw okay. Now what we will do, let us divide fz + delta z – fz/delta z okay and subtract fz. Then what will happen, this will be = to integral over z0 to z + delta z fw – fz dw 1/delta z.

Because z is fixed, z is a constant, so integral over z to z + delta z fz dw will be = fz will come outside and we will have z to z + delta z D w = delta z, which will cancel with this delta z and we will get fz. So this can be written like this okay. Now by the continuity of fz okay, it follow that z is the point inside D. So by the continuity of fz it follow that for a given epsilon > 0 there adjust or delta greater than 0 such that mod of fw – fz is < epsilon whenever mod of w – z is < delta.

Now let us draw the figure again, suppose this is our domain d okay. Here is the point z z+ delta z is another point in the neighbourhood of z okay. We are given that mod of fw – fz is < epsilon whenever mod of z –w, w – z is < delta. Now we have already seen that integral over A to B fz dz is independent of the path joining A to B. So this integral okay, integral from z to z + delta z is independent of the path which joins z to z + delta z.

So since integral over z to z + delta z f w – fz dz dw is independent of the path joining z to z + delta z. So we can consider the line segment joining z to z + delta z okay. So let us consider this line segment which joins z to z + delta z okay, then if we take delta z to be sufficiently small that is mod of delta z < delta okay, then what will happen integral over then mod of fw

– fz for all w belonging to the line segment z to z + delta z will be having the absolute value < mod of 1/mod of integral z to z + delta z okay.

Fw – fz dw okay, we can take delta z also here. This will be < 1/mod of delta z, this 1/delta z and then mod of fw – fz will be < epsilon because this is valid for all w which satisfy mod of w – z < delta. So if you consider point w which lie on the line segment joining z to z+ delta z and you are taking mod of delta z < w < delta then this will be < epsilon * mod of delta and we will get epsilon okay.

So this will imply that mod of, this is < epsilon when delta z is sufficiently small, which implies that limit delta z tends to 0 f z + delta z – f z/delta z = fz. So by the definition of the derivative this is left hand side is f prime z. So f prime z = f z. So this means that f prime z exist at any z belonging to D and which mean that fz is an analytic function in D. Now we have already seen that if fz is an analytic function then all other derivatives of fz are also analytic.

So this implies that f prime z is an analytic function in D. Now f prime z = fz okay, but f prime z = fz, so fz is analytic in D. So this is the proof of the Morera's theorem.

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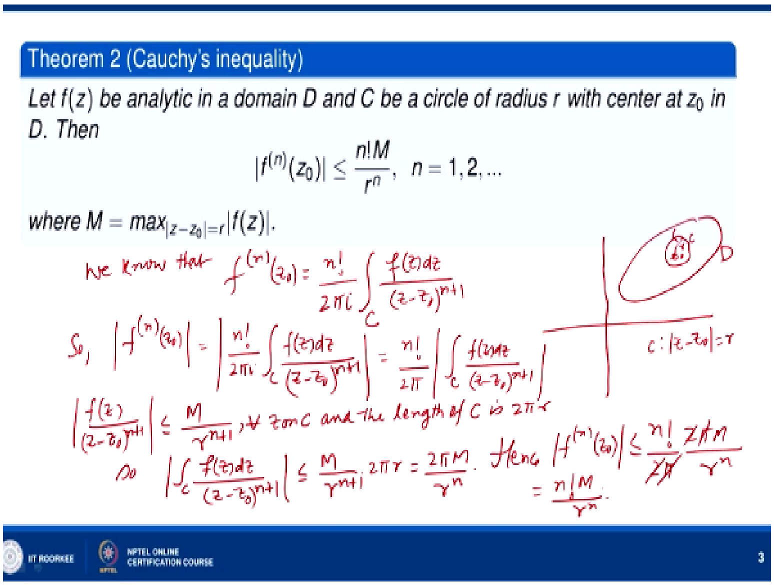

Okay, now let us go to Cauchy's inequality. So this is a consequence of higher order Cauchy's integral formula, let fz be analytic in domain D okay, and C be a circle of radius r with center at z0 okay, in D then mod of f and z0 is <= n factorial * M over r to the power n where M is the maximum value of mod of fz over the circle mod of z – z0 = r. The equation of the circle this one let me call it as C.

So then C is mod of z-z0 = r because it is center is at z0 and radius is r okay. Now we know that the f and z0 = by Cauchy's integral formula, higher order Cauchy's integral formula n factorial/2 pi i integral over C fz dz/z – z0 to the power n+1 okay. So mod of f and z0 will be = mod of n factorial/2 pi i integral over C fz dz/z-z0 to the power n+1 okay. Now mod of zi = 1 so this is mod of this is = mod of n factorial/2 pi and then mod of integral over c fz dz/z – z0 to the power n+1 okay.

Now mod of fz is <= M okay, for all z lying on the circle C okay. So mod of fz/z-z0 to the power n+1 is <= M/mod of z-z0 to the power n+1 means r to the power n+1 for all z on c okay and the length of c is 2 pi r. It is a circle of radius r. So mod of integral over C fz dz/z-z0 to the power n + 1 is <= M/r to the power n+1 * 2 pi r okay or we can say 2 pi M/r to the power n.

And hence mod of f and z0 is <= n factorial/2 pi * 2 pi M/r to the power n and we get the result, n factorial * M/r to the power n. So this is the proof of the Cauchy's inequality. We shall prove Liouville's theorem using this Cauchy's inequality.

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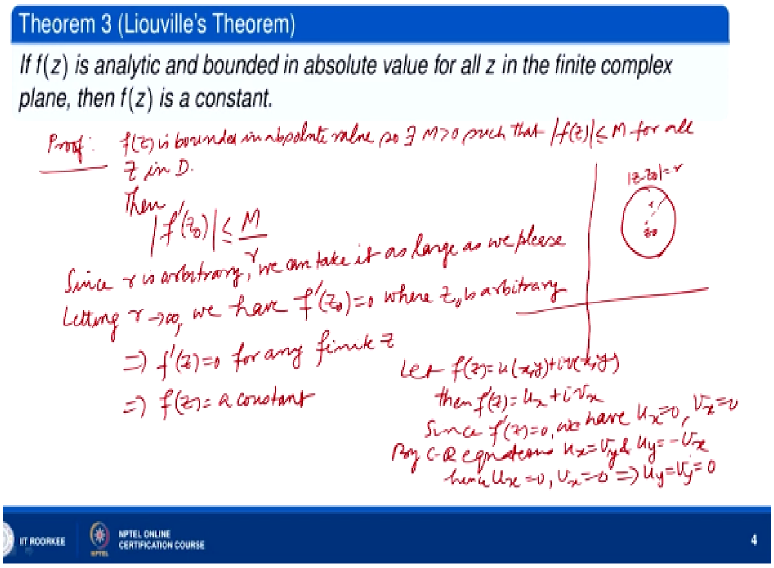

So let us prove the Liouville's theorem if fz is analytic and bounded in absolute value for all z in the finite complex plane, then fz must be a constant. It is a very important theorem, if function which is analytic in the whole finite z plane is entire function so this theorem tells us that bounded entire function is always a constant. Okay, now let us prove this. So we are given that fz is bounded and fz value.

So there are just a constant M greater than 0 such that mod of f z is <= M for all z in D okay. Now let us take the circle presented at z0. So mod of z – z0 = r okay. This is r, this is z 0, then by higher order Cauchy's inequality formula we have f prime z 0 or by the Cauchy's inequality mod of f prime z0 is <= n factorial * M/r to the power n. So <= n = 1 okay. So m1 factorial is 1 and so m/r okay.

So mod of f prime z 0 is <= m/r, z 0 is an arbitrary point in the domain, in the whole complex plane, you can take any point z0 and draw a circle with z0 as center and of radius r. so then mod of f prime z0 is <= r. now since r can be taken as large as we please okay since r is completely arbitrary we can take it as large we please. So letting r go to infinity we have f prime z0 = 0, okay.

Where z0 is arbitrary, which means that f prime z = 0 for any z, for any finite z. okay, which implies that f z is the constant. So in order to prove that fz is constant let us say let fz be = uxy + ibxy then we know that f prime z = ux + ibx okay. Since f prime z = 0. We have ux = 0 or bx =0, but by CR equations ux = vy and uy = -vx okay, so ux = 0 vx = 0 gives uy and vy are also 0s, which means that u and v are functions of x y such that their partial derivatives with respect to x and y are 0s and therefore u and v are independent of x and y.

So u is the constant and v is the constant and hence fz = u + 1v is the constant.

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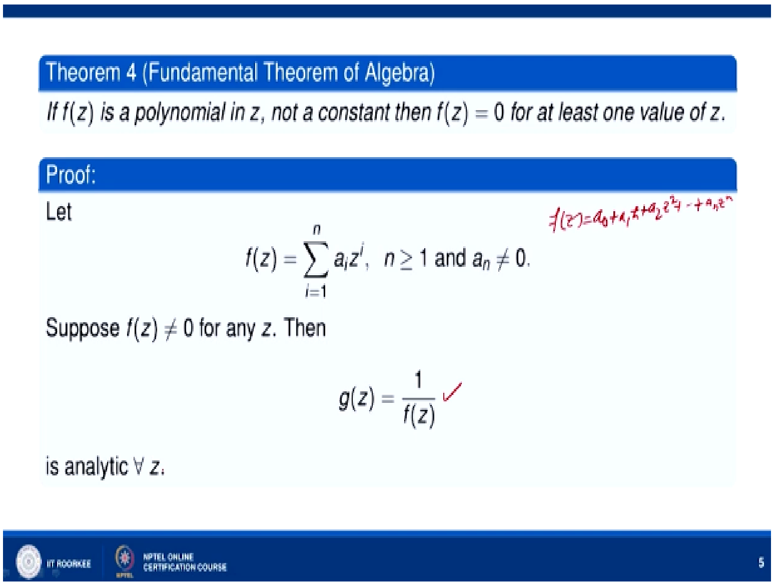

Let us now prove the fundamental theorem of algebra, if fz is the polynomial in z, not a constant okay, then fz = 0 for at least one value of z, fz will have at least 1 root which will make it 0, okay, so fz = 0 for at least 1 value of z. so let us say fz = sigma i = 0 2n ai z to the power i, that is we are writing fz as a0 + a1z + a2z square and so on an z to the power n okay. So let fz be sigma = 0 to n, ai z to the power i which is same as a0 + a1z + a2z square and so on an z to the power n.

And we are taking n to be >= 1 we do not allow n to be = 0 because if n is allowed as 0 then f z will become a constant. So n is >= 1 and more over that an is not = 0 so that it is a polynomial of degree n okay. Now suppose fz is not = 0. So we are going to prove this fundamental theorem of algebra by contradiction method. So let us assume that it does not vanish for any value of z.

Then we can consider the reciprocal of fz that is 1/fz as g z function okay. So let us consider gz, gz = 1/fz since fz is analytic and the fz is polynomial so it is analytic for all z and 1/fz is not 0 for any z. So gz is analytic for all z okay. Now fz is analytic for all z, g z is analytic for all z.

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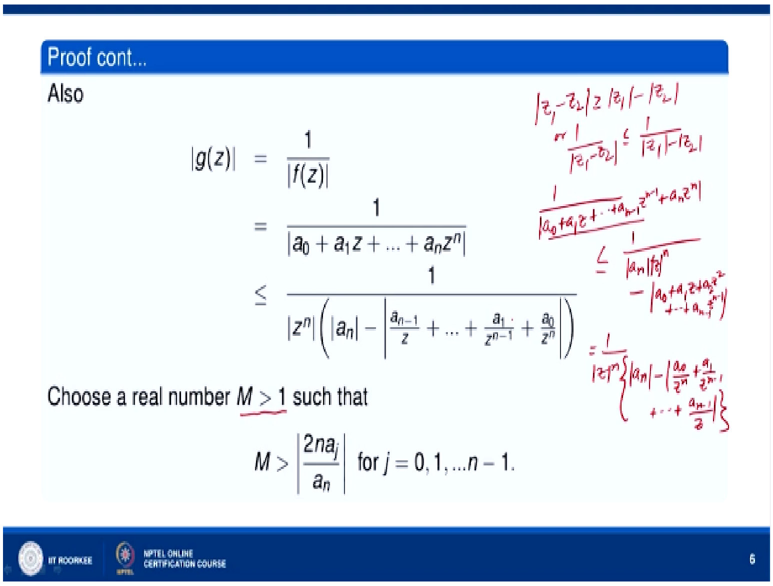

And also mod of g z = 1/ mod of fz which is = 1/ mod of a0 + a1 z and so on, an z to the power n, we can write it as mod of z to the power n and then mod of an – mod of because mod of z1 – z2 is >= mod of z1 – mod of z2 or we can say 1/ mod of z1 – z2 is <= 1/mod of z1 – mod of z2. So we are making use of this triangle inequality 1/mod of a0 + a1 z and so on an – 1 z to the power n-1.

This we are considering as 1 complex number and the other is an z to the power n. so that I can write it as <= 1/mod of an * mod of z to the power n, okay – mod of a0 + a1 z+ a2 z square and so on. An-1 z to the power n-1 okay and we can also write it like this 1/mod of z to the power n, mod of an – mod of z of power n we divide here. So it will become a0/z to the power n, a1/z to the power n-1 and so on, an-1 upon z okay.

So this is what we get, an-1 upon z, an-2 upon z square, a1 upon z -1 a0 upon z to the power n okay. So this is mod of gz is <= to this. Now let us choose a real number M to be > 1, okay and such that M is > mod of 2n aj/an okay for j = 0, 1, 2 and so on up to n-1, okay. So when we do this what we have is the following.

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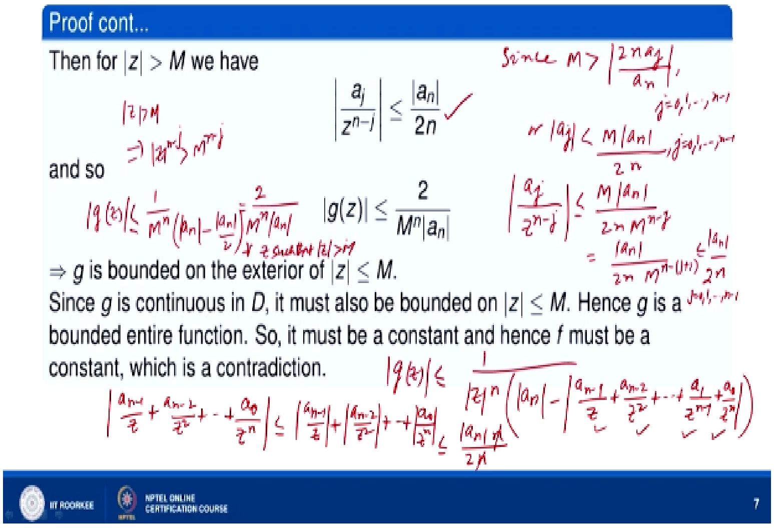

Then for mod of z > M we have mod of aj/z to the power n-j <= mod of an over 2 to the power n, how do we get that, let us look at this M is > mod of 2n aj/an, where j takes values 0, 1 and so on upto n-1 okay. So this I can say mod of aj < m times mod of an/2n okay. For j = 0, 1 and so on upto n-1 okay. Now if I consider this mod of aj over z to the power n – j okay so then this will be <= m mod of an over 2n and mod of z is > M okay.

M is >1 so mod of z greater than M implies that mod of z to the power n-j okay is greater than M to the power n-j okay. So this is M to the power n-j okay and M/n or you can say mod of an upon 2n m to the power n-j + 1 okay. So this is always <= mod of an/2 * n, where j takes values 0, 1 and so on up to n-1. Even when j = n-1, n-1 +1 will be = n and we will have m to the power 0, so we will have 1.

So mod of aj/z to the power n-j <= mod of an/2n for all j from 0 to n-1 okay. Now mod of gz okay, this is mod of gz. Mod of gz is <= 1/mod of z to the power n and then we have mod of an – mod of an-1/z, an -2/z square and so on a1/z to the power n-1 + a0 over z to the power n okay. So mod of these are n terms, so mod of each term here is <= mod of an/2n okay. So mod of an-1/z + mod of an-2/z and so on.

Mod of a0/z to the power n is <= mod of an-1/z + mod of an-2/z square, and so on mod of a0/z to the power n okay. This is <=, these are n terms, each term is <= mod of an/2n. So mod of an/2n * n okay. So this will cancel and we get mod of an/2 okay. So what do we get, we have mod of gz, thus mod of gz is <= 1 upon because mod of z to the power n is here. So M to the power n and mod of an – mod of an/2.

So this will become 2 times M to the power n * mod of an okay. Now this means that now n is a fixed number because it is the degree of the polynomial, so mod of gz is <= 2/M to the power n, mod of n for all z such that mod z is greater than M okay.

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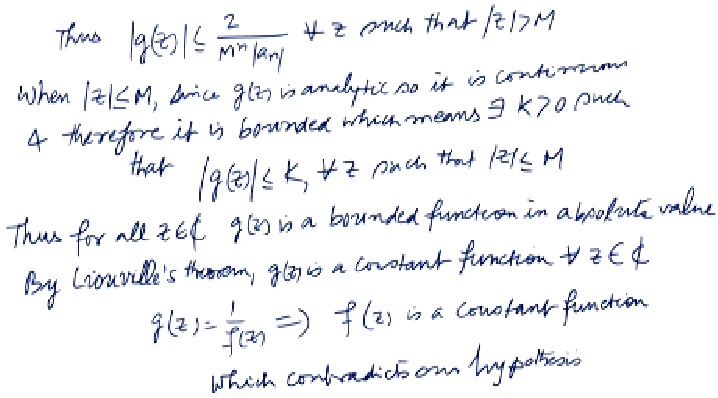

Now we show thus mod of gz is <= 2/M to the power n, mod of an okay for all z such that mod of z is > M. Now what happens for mod of z <= M. so when mod of z is <= to M okay, since fz is analytics since gz is analytic so it is continuous and therefore it is bounded, which means that they are just a constant k > 0 such that mod of gz is <= k for all z such that a mod of z is <= M, okay.

So when mod of z is <= M, mod of gz is <= k and when mod of z is > M, mod of gz is <= 2 over M to the power n, mod of an okay. So thus for all values of z, thus for all z belonging to c gz is bounded, is a bounded function, bounded and absolute value and therefore by Liouville's theorem gz is a constant function for all z belonging to c. now let us recall that gz = 1/fz okay.

So this implies that fz is a constant function, which is a contradiction to our hypothesis, okay we assumed that fz is not a constant function. So this proves the fundamental theorem of algebra okay. So this proves the fundamental theorem of algebra.

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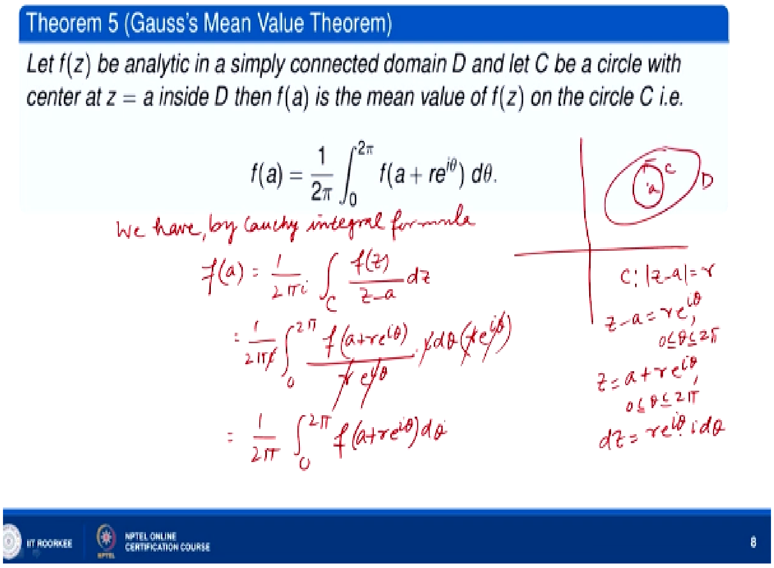

Now let us prove the Gauss's mean value theorem, let fz be analytic in a simply connected domain D, C is a circle with center at z = a inside D. So this is our circle C, the circle C is given by mod of z – a = say r, okay, it is radius is r, then this term says that the mean value of fz on the circle c okay, mean value of fz on the circle c is the value of f at the center that is a, fa is mean value of fz on the circle c.

So let us prove this, we have by Cauchy's integral formula fa = integral/c fz/z-a dz. Now c is mod of z-a = r and we are moving around c in the anticlockwise direction so we can write parametric form of c, r e to the power i theta, where 0 is <= theta <= 2 pi, so z will be = a + r e to the power i theta, 0 <= theta, <= 2 pi. So this will be = 1/2 pi i, integral/0 to 2 pi f, a + r e to the power i theta divided by z – a is r e to the power i theta.

And dz = d0 because a is constant. So r e to the power i theta * id theta we have, so id theta * r e to the power i theta. So this cancels with this and we get i also cancel with i here and we get 1/2 pi integral/0-2 pi f a + re i theta d theta, so this is how we prove Gauss's mean value theorem.

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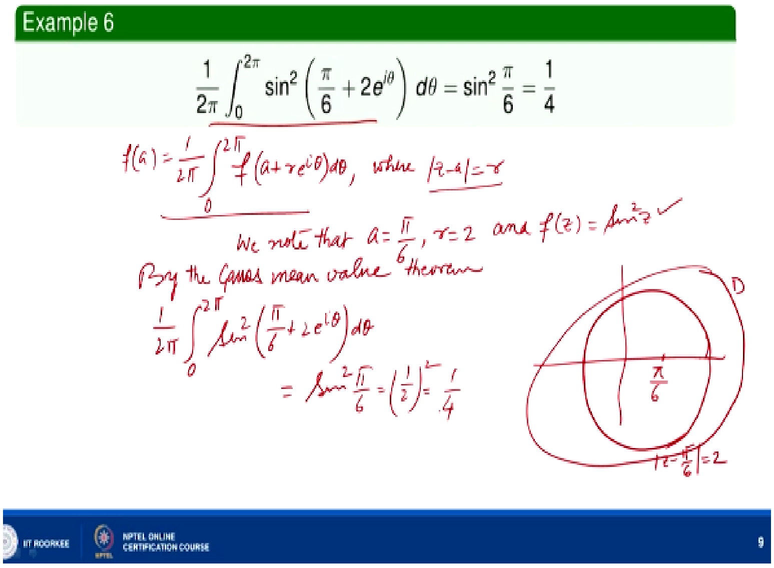

Now let us consider this problem, 1/2 pi we have to valuate this integral, 1/2 pi integral 0 to 2 pi sin square pi i/6 + 2 e to the power i theta d theta okay. So let us compare it with the Gauss's mean value theorem, in the Gauss's mean value theorem we have fa = 1/2 pi integral/0 to 2 pi f a + r e to the power i theta d theta, where mod of z – a = r okay. So here we are given sin square a + r e to the power i theta.

So if you compare this integral with this integral okay. We notice that a = pi/6 r = 2, okay, and fz function okay, fz function is sin square z okay. So that fa + r e i theta is sin square a + re i theta okay. So by the Gauss mean value theorem sin square z is analytic in the whole complex plane so you can take any domain D okay, which encloses this circle, mod of z – a = r that is mod of z – pi/6 = 2 okay.

So take any domain z = pi/6 is here okay, pi/6 is here and you are drawing a circle with radius 2 okay, so this is mod of z – pi/6 = 2 okay and take any domain which contains this circle okay. So fz is analytic in this simply connected domain D and it contains the simple close curve, the current circle mod of z – pi/6 =2. So by Gauss's mean value theorem we have 1/2 pi integral 0 to 2 pi sin square pi/6 + 2 e to the power i theta d theta = fa.

That is sin square pi/6 and sin pi/6 is 1/2, so we have 1/2 square and we get 1/4. So this is how we evaluate the value of this integral by using Gauss's mean value theorem.

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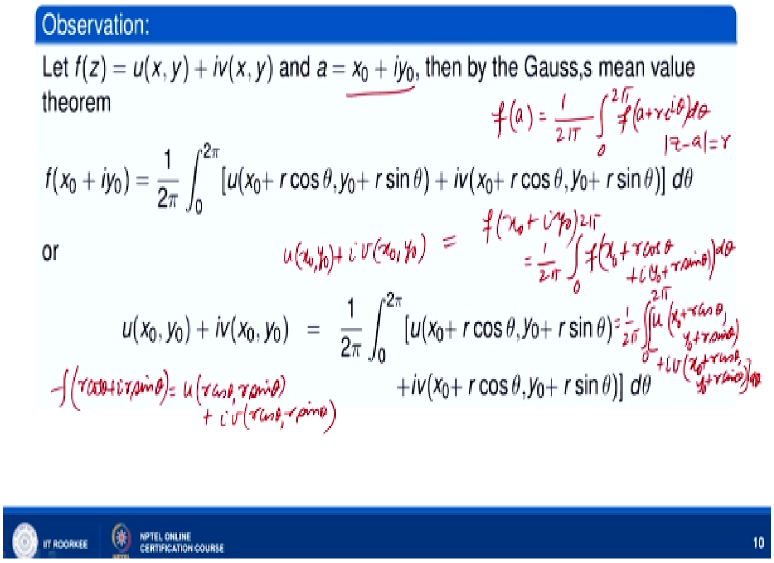

Now let us consider the function fz = uxy + iv xy and take a to b x0 + i y0 in the Gauss mean value theorem, okay and the Gauss mean value theorem gives fa = 1/2 pi integral/0 to 2 pi f a + r e to the power i theta d theta okay. So a is any complex number and we are integrating here along the circle mod of z – a = r okay. So let us take a = x0 + i y0, then f x0 + i y0 okay, f x0 + i y0 = 1/2 pi 0 to 2 pi okay.

Here we have e to the power i theta is cos theta + i sin theta. So we get f of a + r cos theta + i r sin theta, d theta okay. This means that fz is u x y + iv xy where z is r cos theta so f of z, z is r e to the power i theta let us say. So r cos theta + ir sin theta, this is = u r cos theta, r sin theta and then we have i v r cos theta r sin theta. So here also this will be 1/2 pi integral 0 to 2 pi u a + r cos theta and then r sin theta.

We are adding a 2, a means we are adding, a is x0 + i y0 okay. So this will be x0, x0 + r cos theta and here we will have i times y0 + r sin theta, okay. So a is x0 + i y0. So we will have xf x0 to r cos theta + i times y0 + r sin theta, so this will be u x0 + r cos theta and then y0 + r sin theta and we have +iv x0 + r cos theta y0 + r sin theta, D theta okay and left hand side is what f x0 + i y0 gives you ux0 y0 + i v okay.

So equating real and imaginary parts, what we will get ux0 y0 = 1/2 pi integral over 0 to 2 pi, u x0 + r cos theta y0 + r sin theta and b x0, y0 will be = 1/2 pi, integral 0 to 2 pi v x0 + r cos theta y0 + r sin theta d theta okay.

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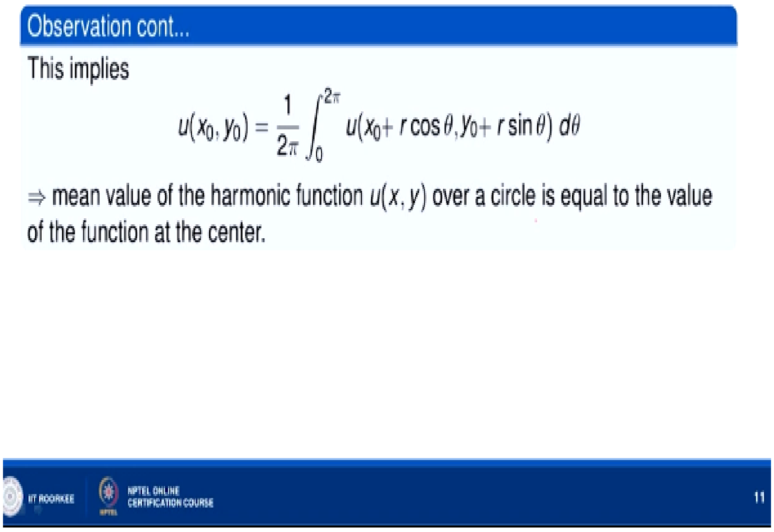

So we get u x0 y0 = 1/2 pi integral 0 to 2 pi u x0 + r cos theta y0 + r sin theta d theta and we know that uxy vxy harmonic functions because they are real imaginary parts of the analytic function fz. So mean value of the harmonic function uxy over the circle mod of z-a =r = the value of the function uxy at the center of the circle.

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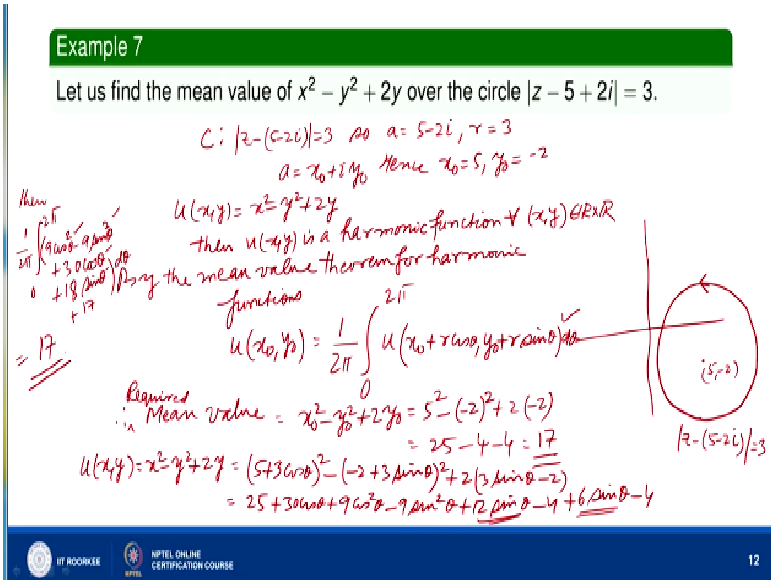

Now let us do one problem on this, let us find the mean value of x square – y square + 2y over the circle mod of z – 5 + 2 i = 3. So here we are given the circle C as mod of z – 5-2i = 3. So a = 5-2i and r = 3. We have taken a = x0 + i y0, okay. So hence we have x0 = 5 and y0 = -2 okay, it is easy to check that if you call uxy as x square – y square + 2y then uxy is the harmonic function for all xy belonging to r cross r okay.

Now so in the complex z plane you consider the circle having center at 5-2, so 5-2 means here okay. So 5-2 and draw the circle with radius 3 okay. So we will have some circle like this, mod of z-5- 2i okay. So by the mean value theorem harmonic functions u at x0 y0 = 1/2 pi integral/0 to 2 pi u x0 + r cos theta y0 + r sin theta d theta. So the mean value of, required mean value okay.

Required mean value = ux0 y0, u x0 y0 means x0 square – y0 square + 2 y0 and this means that 5 square – 2 square + 2 times -2, so it comes out to be 17. One can directly verify okay by evaluating this integral okay. If you want to evaluate this integral you need to write uxy in terms of x0 y0 and r theta okay. So if you want to do that then u xy = x square – y square + 2 y okay.

x = x0 + r cos theta, x0 is 5, r = 2. So we write 5 + 2 cos theta for x, square – y0, y0 = -2. So -2 + r = 3, this r is 3 here. So here what we will have, y0 + r sin theta for y okay. So y0 = -2 + 3 sin theta whole square okay + 2 times y that is 3 sin theta – 2 okay. So what we will get, 25 + 30 cos theta + 9 cos square theta and then we get – 9 sin square theta and then we get + 12 sin theta and what we get is -4 + 6 sin theta -4 okay.

This expression we integrate over the interval 0 to 2 pi and then divide by 2 pi okay. So then this gives you, so this gives you then 1/2 pi integral/0 to 2 pi okay. This is how much, 9 cos square theta – 9 sin square theta okay, 9 cos square theta – 9 sin square theta okay, 12 sin theta + 6 sin theta is a 18 sin theta and we have 30 cos theta, + 18 sin theta and we have 25 – 4 -4 so 17 + D theta okay.

So the integral this this this this, the all integrals becomes 0 and we get 17 * 2 pi/2 pi and we get 17 okay. So by putting the values of x and y in terms of x0 y0 and r in the expression for u x y and an integrating with respect to theta over the integral 0 to 2 pi and then dividing by 2 pi we get the same value 17. So this is how we can verify the mean value theorem in this case for harmonic function. With this I would like to end my lecture, thank you very much for your attention.