Advanced Engineering Mathematics
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Lecture – 01
Analytic Functions

Hello friends. Welcome to my course on advanced engineering mathematics. We will first have the lecture on analytic functions. In order to define an analytic function, we need the concept of limit and derivative.

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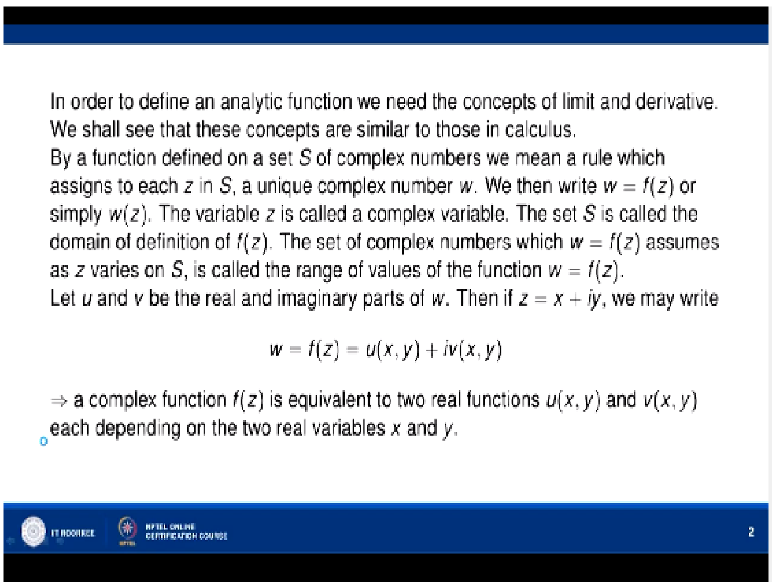

We shall see that these concepts are similar to those in real calculus. First we will define a function on a set S of complex numbers. By a function defined on a set S of complex numbers, we mean a rule which assigns to each z in S, a unique complex number say w. We then write w=fz or simply wz. The variable z is called a complex variable. The set z is called the domain of definition of fz.

The set of complex numbers which w=fz assumes as z fvaries on S, is called the range of values of the function w=fz. Now let us say that u and v be the real and imaginary parts of the complex number w, then if you take z=x=iy, we will have w=fz=uxy+ivxy because z depends on x and y and w is a function of z. So in general the real and imaginary parts of w that is u and v are functions of x and y and we therefore write w=uxy+ivxy.

Now by this we can say that a complex function fz is equivalent to 2 real functions uxy and vxy, each depending on 2 real variables x and y. For example, let us consider w=fz=z square+3z.

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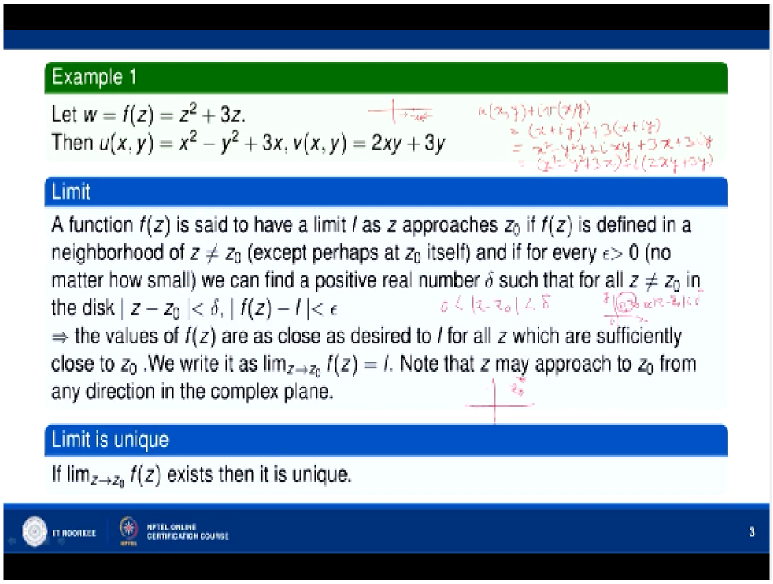

Then here you can see uxy+ivxy=x+iy whole square+3*x+iy. Now we can write it as, using iota square=-1, we have x square-y square+2ixy+3x+3iy and so we can write it as x square-y square+3x+iota*2xy+3y. So equating real and imaginary parts then uxy=x square-y square+3x and vxy=2xy+3y. So you can see that the real and imaginary parts of w, that is u and v are functions of x and y.

Now let us define a limit, a function fz is said to have a limit l as z approaches to z0 if fz is defined in a neighbourhood of z not equal to z0, except perhaps at z0 itself, when function need not be defined at the point z0 and then if for every epsilon>0, no matter how small, one can find a positive real number delta depending on epsilon such that for all z not equal to 0 in the disk mod of z-z0<delta mod of fz-l<epsilon.

This means that mod of fz must be less than epsilon in the deleted neighbourhood, 0<mod of z-z0<delta. See you take the complex plane, if z0 is a point here, then you construct an open circular disk with center at z0 of radius delta, then the open circular disk is mod of z-z0<delta. Now we are considering the set of all z such that mod of z-z0<delta but z is not equal to z0. So

we write 0<mod of z-z0<delta.

This is called a deleted neighbourhood of z0. So mod of fz must be less than mod of fz-l must be less than epsilon in the deleted neighbourhood of z0, that is 0<mod of z-z0<delta. Which means that the values of fz are as close as desired to l for all z which are sufficiently close to z0. Now mathematically we can write it as limit z tends to z0, fz=l. Now you can note here that z can approach to z0 from any direction in the complex plane.

In the real calculus, we have to do along line, when we say that x approaches to x0, then either we approach x0 from left or we approach x0 from right. But in the case of complex, z0 lies in the plane, so z can approach to z0 from any direction. So this has to be kept in mind while taking the limit. Then the limit of fz as z tends to z0, exists and it is unique. So let us prove this limit of fz as z tends to z0 exists if limit z tends to z0 fz exists.

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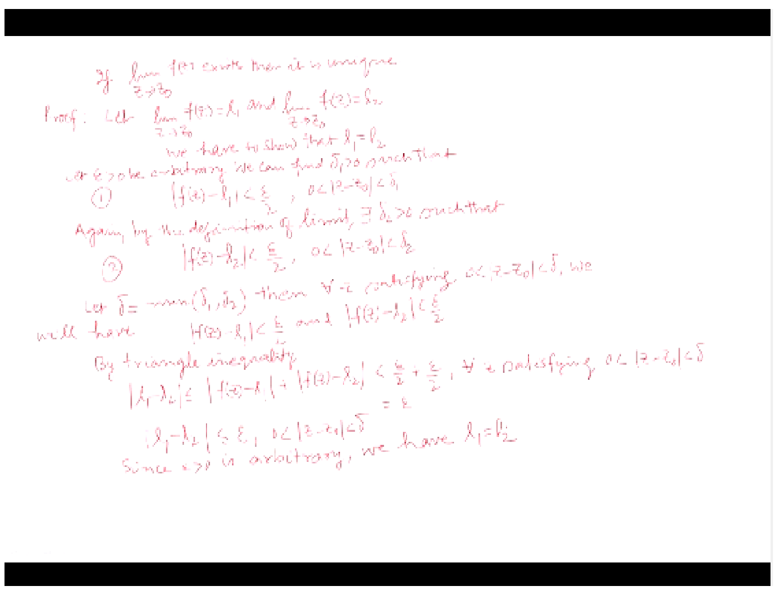

Then it is unique. So to prove this, let limit z tends to z0, fz be equal to l1 and limit z tends to z0, fz be equal to l2. Then we have to show that l1-l2. So let us take any epsilon>0, then by the definition of limit, for a given epsilon>0, we can find a positive number say delta 1>0 such that mod of fz-l1<epsilon y2 whenever 0<mod of z-z0<delta 1. This is because limit of fz, as z tends to z0, is equal to l1.

Now since limit z tends to z0, fz=2l, again by the definition of limit, for the same epsilon>0, there exists delta 2>0 such that mod of fz-l2<epsilon y2 whenever 0<mod of z-z0<delta 2, okay. Now let us say this is equation 1, this is equation 2. So now let us define, let delta be the minimum of the 2 positive numbers delta 1 and delta 2, then for all z satisfying 0<mod of z-z0<delta, we will have mod of fz-l1<epsilon y2 and mod of fz-l2<epsilon y2, okay.

Now let us apply triangle inequality. So by triangle inequality, fz-l2<epsilon y2, okay. Now let us apply triangle inequality. So by triangle inequality, mod of l1-l2 is less than or equal to mod of fz-l1+mod of fz-l2. And each one is less than epsilon y2 in the common neighbourhood, that is 0<mod of z-z0<delta. So for all z satisfying 0<mod of z-z0<delta, okay. So what we have? Mod of l1-l2 is less than or equal to epsilon, okay. in this neighbourhood.

Now since epsilon>0 is arbitrary, we can take it as a small as we please, okay. So we have l1=l2. So this is how we show that if the limit of fz as z tends to z0 exists, then it is unique, okay.

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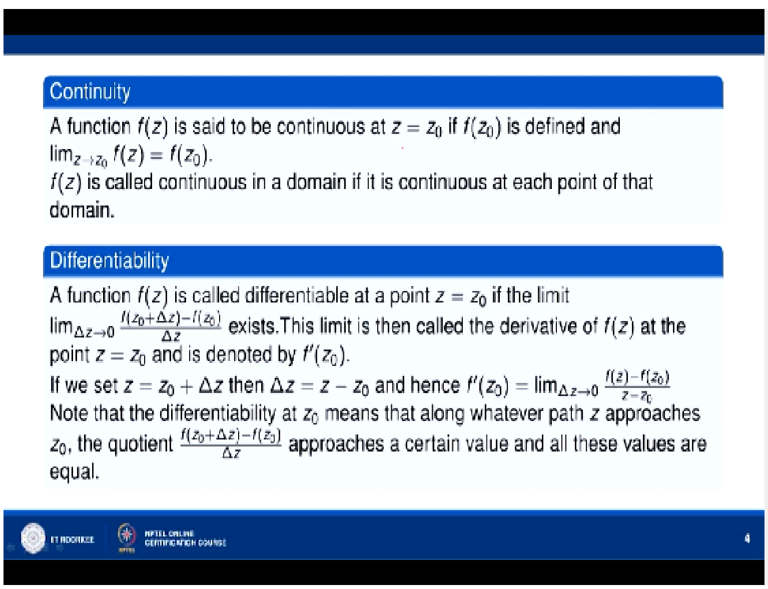

Now let us define the continuity of a complex function fz. A function fz is said to be continuous at z=z0 if fz0 is defined a limit of fz as z approaches z0=fz0. The function fz is said to be continuous in a domain if it is continuous at each point of that domain. Then we come to the concept of differentiability. A function fz will be called differentiable at a point z=z0 if the limit of fz0+delta z-fz0/delta z as delta z approaches 0 exists.

This limit is then called the derivative of fz at the point z=z0 and we denote it by f prime z0. Now if you put here z for z0+delta z, then you can see that delta z is z-z0. So this quotient will become fz-fz0/z-z0. And when z tends to z0, delta z goes to 0. So delta z goes to 0 will be replaced by z goes to z0. So in the alternative definition for the derivative of a complex function at the point z0 is f prime z0=limit z tends to z0, fz-fz0/z-z0.

Now let us note that the differentiability at z0 means that along whatever path, z approaches z0, the quotient fz0+delta z-fz0/delta z approaches a certain value and all these values are equal. This fact is very important here.

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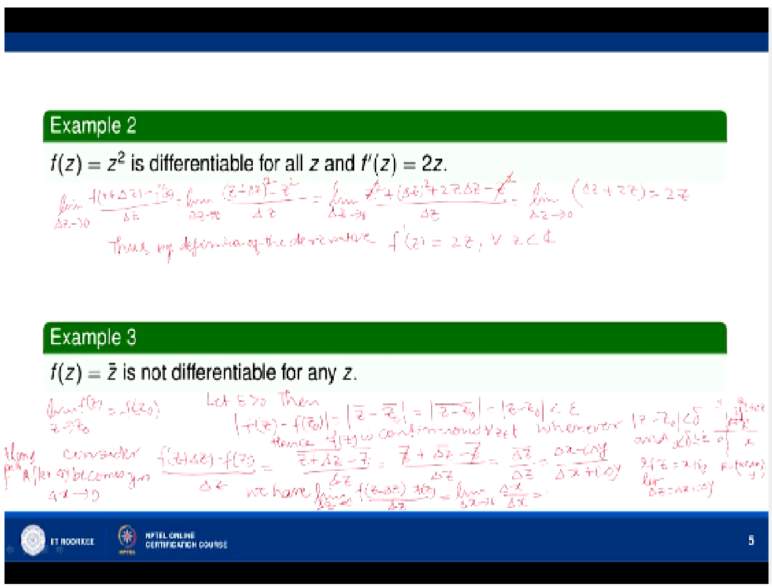

Now let us look at the function fz=z square and so that it is differentiable for all z and f prime z=2z, okay. So let us find the limit of fz+delta z-fz/delta z as delta z tends to 0. If this limit exists, then we shall say that the derivative of fz exists. And whatever is this value, that will be the derivative of fz at the point z. So we are finding the derivative of fz at the point z here. So this is limit delta z goes to 0, z+delta z whole square-z square/delta z, okay.

So this is equal to limit delta z tends to 0, z square+delta z whole square+2z delta z-z square/delta z, okay. So z square will cancel. We divide by delta z and get limit delta z tends to 0, delta z+2z, okay. When delta z goes to 0, delta z+2z goes to 2z. So we can say that the limit of fz+delta z-fz/delta z as delta z goes to 0 exists and is 2z. And thus by definition of the derivative, f prime z=this limit.

So f prime z=2z. So function is differentiable and f prime z=2z for all z belonging to the set of complex number c. Now let us come to another example, fz=z conjugate, okay. We shall show that this function is not differentiable for any z, okay, although it is continuous for all z. We can see its continuity also. Suppose you want to show the continuity of z conjugate, then what you do?

Consider, we can show that limit of fz as z goes to say some z0, okay. z goes to z0=fz0, okay. If you want to show the continuity of this function fz=z conjugate, then we have to show that limit of fz as z goes to z0=fz0. That is to, so for this to prove, what do we do? Let epsilon be greater than 0, okay. Then consider mod of fz-fz0=mod of z conjugate-z0 conjugate, okay. And this is equal to mod of z-z0 whole conjugate, okay.

But mod of z-z0 conjugate=mod of z-z0, okay. So mod of fz-fz0<epsilon whenever this is less than epsilon, whenever mod of z-z0<delta and 0<delta less than or equal to epsilon. So suppose epsilon>0, then you can take any delta which is positive but less than or equal to epsilon, we shall always have mod of fz-fz0<epsilon. Because mod of z-z0<delta and delta is less than or equal to epsilon.

So function fz is continuous at any z0, okay. So hence fz is continuous for all z, okay. Now let us show that it is not differentiable. So consider fz+delta z-fz/delta z, this is equal to by definition z+delta z conjugate-z conjugate/delta z. z+delta z conjugate is z conjugate + delta z conjugate-z conjugate, okay. This will cancel and you will get delta z conjugate/delta z. Now if z=x+iy, let delta z be delta x+idelta y, okay.

Delta z be equal to delta x+i delta y. So delta z conjugate will be delta x-i delta y/delta x+i delta y, okay. Now what we do is, suppose this is your complex plane. here is the point z and here is the point z+delta z, okay. So what we do? We first move down when we reach this z, okay. That is in this line parallel to x axis, okay. So we move down parallel to y axis till we reach this point here, okay.

So if this z is xy, this point is x+delta x y point, okay. So when we reach here, this is suppose P, this is Q, okay and this is R. So the coordinates of R are x+delta x and then y, okay. So when we reach this point R, we move towards the point P because delta z is tending to 0. So after delta y, this line is delta y. After delta y becomes 0, we reach the point x+delta x y and then we move towards P.

So delta x tends to 0. So after delta y becomes 0, delta x approaches to 0. If we move along path 1, let us say this is path 1. If we move along the path 1, okay, where we first move from the point Q to the point P along the line QR which is parallel to y axis and R is the point in the line, straight line parallel to x axis passing through P. So the coordinates of R are x+delta x y.

So when we reach R, delta y has become 0 and then we are moving towards the point P, that is in the direction RP. So delta x tends to 0. So when delta y becomes 0, then delta x tends to 0. So along path 1, the quotient approaches to, we have limit delta z tends to 0, fz+delta z-fz/delta z=, delta y has become 0, so limit delta x tends to 0, delta xy delta x and therefore 1. So along path 1, okay, along path 1, limit of fz+delta z-fz/delta z, as delta z goes to 0, is equal to 1.

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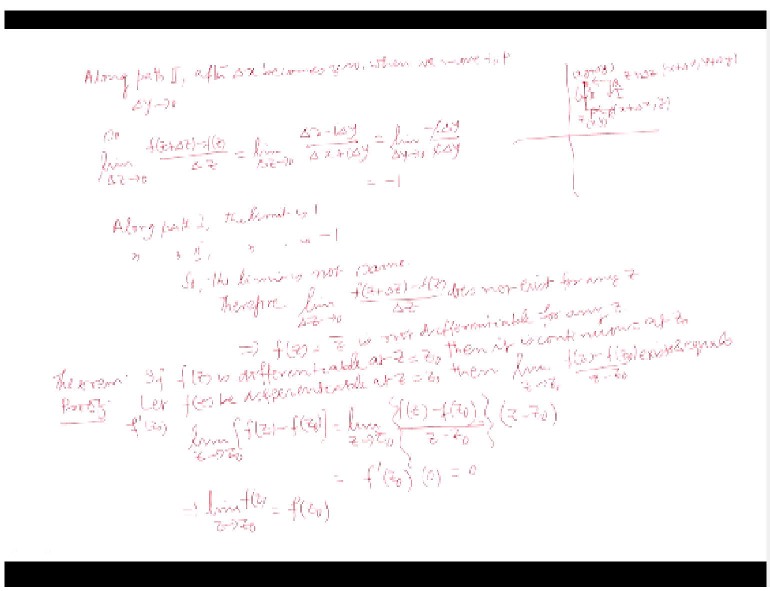

If we, here we had P point, here we have Q point and we are moving down parallel to y axis.

This was the point z+delta z. Here was the point z and so if this z is xy the Cartesian coordinates of P are xy, here the coordinates, Cartesian coordinates are x+delta x y+delta y. And this point R has coordinates x+delta x y. And this path we called as path 1. Along path 1, we have found the limit of fz+delta z-fz/delta z.

Now let us consider path 2, where we move parallel to y axis. From the point Q, we move parallel to y axis, then what will happen? Delta x will tend to 0. So when delta x has become 0, that means we reach this point. The Cartesian coordinates of this point are x y+delta y. So let us call this point as S. So we have, so after delta x has become 0, this path, let us call as 2. So along path 2 after delta x becomes 0, when we move to P, what happens?

Delta y goes to 0, okay. So limit delta z tends to 0, fz+delta z-fz/delta z which is equal to limit delta z tends to 0 delta x-i delta y/delta x+i delta y. This will be equal to limit delta y tends to 0-delta y/i delta y and what we will get? The limit will be -1. So along path 1, the limit is 1 while along path 2, the limit is -1, okay. So the limit is not same and therefore, limit z tends to 0, fz+delta z, limit delta z tends to 0, fz+delta z-fz/delta z does not exist for any z, okay.

And this means that fz=z conjugate is not differentiable for any z, okay. So we have seen that fz=z conjugate is continuous for every z but it is not differentiable for any z, okay. So there is a result which says that if fz is differentiable at a point z=z0, then it is continuous at that point. So that result also we can prove here. Of course, converse is not true as we have seen here. So let us prove that, we can prove this theorem.

The theorem is if fz is differentiable at z=z0, then it is continuous at z0. So let us prove this. So let fz be differentiable at z=z0, then limit z tends to z0, fz-fz0/z-z0 exist and equals f prime z0, okay. We have to show that fz is continuous at z=z0. So let us take, so we have to consider limit z tends to z0, fz-fz0. Let us consider this. If we can show that limit of fz-fz0 as z tends to z0 is 0, then limit of fz as z tends to z0 will be equal to fz0.

So this is equal to limit z tends to z0, fz-fz0/z-z0*z-z0. Now since limit of fz-fz0/z-z0 exist and equals f prime z0 and limit of z-z0 as z tends to z0 exists and is equal to 0, so limit of the product of this expression and this expression exists and we have f prime z0*0, okay. So this is equal to 0. This means that limit of fz, as z tends to z0, =fz0 which means that fz is continuous at z=z0. So any differentiable function is always continuous but the converse is not true as we have seen in the case of fz=z conjugate.

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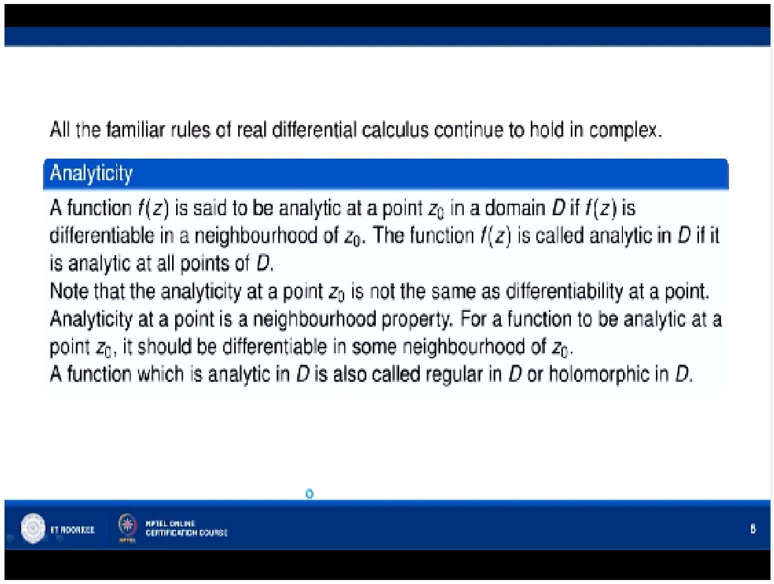

Now let us look at the rules of differential calculus. All the familiar rules of real differential calculus continue to hold in complex. Let us come to the concept of analyticity. A function fz is said to be analytic at a point z0 in a domain D if fz is differentiable in a neighbourhood of z0. So the function fz is called analytic in D if it is analytic at all points of D. Now you can note here that the analyticity of a complex function at a point z0 is not the same as the differentiability at a point. Analyticity at a point is a neighbourhood property.

For a function to be analytic at a point z=z0, the function has to be differentiable in a neighbourhood of z0. So that is the difference. A function which is analytic in D is also called regular in D or holomorphic in D.

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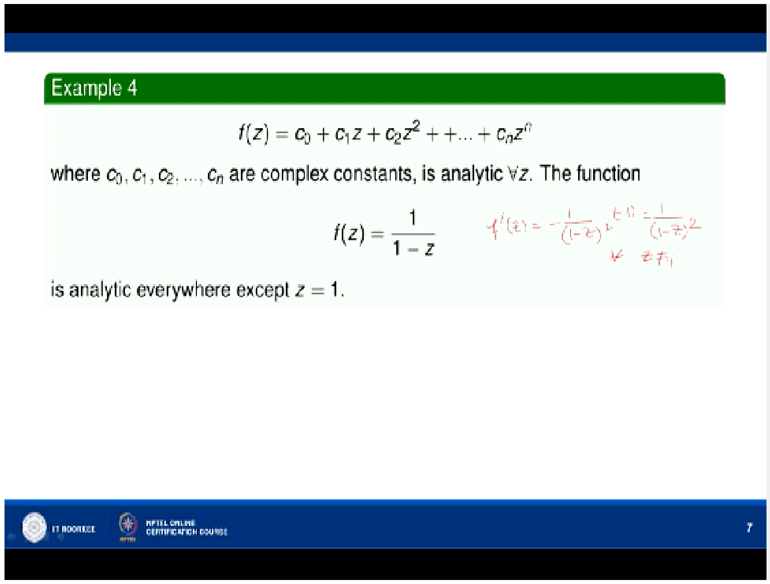

Let us say for example fz=c0+c1z+c2z square and so on, cnz to the power n where c0, c1, c2, cn are complex constant. This is a polynomial in z and so it is analytic for all z. It is differentiable for all z and therefore, it is analytic for all z. If you look at the function fz=1/1-z, we can see that it is not defined at the point z=1, okay. And then for all other points z, we can find its derivative f prime z=-1/1-z whole square*-1, that is 1/1-z whole square. So for all z not equal to 1, the function fz is differentiable and therefore, we can say that it is analytic for all z except z=1.

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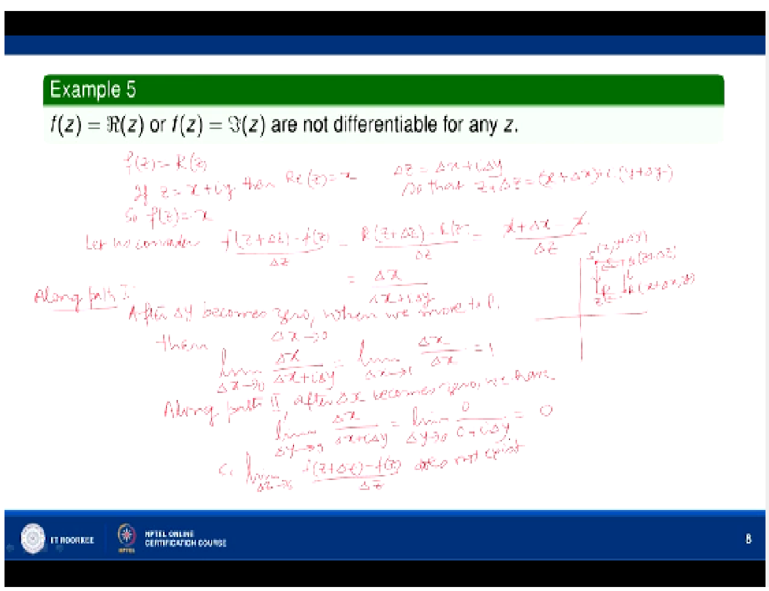

Let us look at the function fz=Rz, Rz means real part of z, Rfz=imaginary part of z. We can see that they are not differentiable for any z. For example, let us consider fz=Rz. So if z=x+iy, then real part of z=x, okay. So fz will be equal to x. We want to show that fz is not differentiable for any z. So let us consider fz+delta z-fz/delta z. We will take the limit of this as delta z goes to 0 and show that this limit does not exist.

So when z is x+iy, delta z, let us take as delta x+i delta y, so that z+delta z=x+delta x+i y+delta y. So real part of z+delta z, this is equal to real part of z+delta z, by definition, -real part of z/delta z. And real part of z+delta z is x+delta x, okay. -real part of z=x, /delta z. So this is equal to what? Delta x/delta x+i delta y, okay. Now again what we do? This is your point P and here is the point Q, okay.

So this is z point, this is z+delta z complex number. We move along the path 1. There is path l2 y axis, reach the point R. So this is your point x+delta x y, okay. So after delta y has become 0, when we move to the point P, delta y will tend to 0, okay. So after delta y becomes 0, when we move to P, delta x tends to 0. So then limit of delta x/delta x+i delta y as delta x goes to 0. This will be equal to limit delta x tends to 0, delta x/delta x which is equal to 1.

And when we move along this path, okay, so after delta x has become 0, we reach this point S, okay. Coordinates of this are x y+delta y. So then delta y will tend to 0. So along path 2, after delta x becomes 0, we have limit delta y tends to 0. Delta x/delta x+i delta y=limit 0/, delta x has become 0, 0+i delta y. This is equal to 0. So along path 1, the expression fz+delta z-fz/delta z goes to 1 while along path 2, fz+delta z-fz/delta z goes to 0.

So limit delta z tends to 0, fz+delta z-fz/delta z does not exist. And therefore, fz=real part of z is not differentiable for any z. Similarly, we can show that fz=imaginary part of z is not differentiable for any z.

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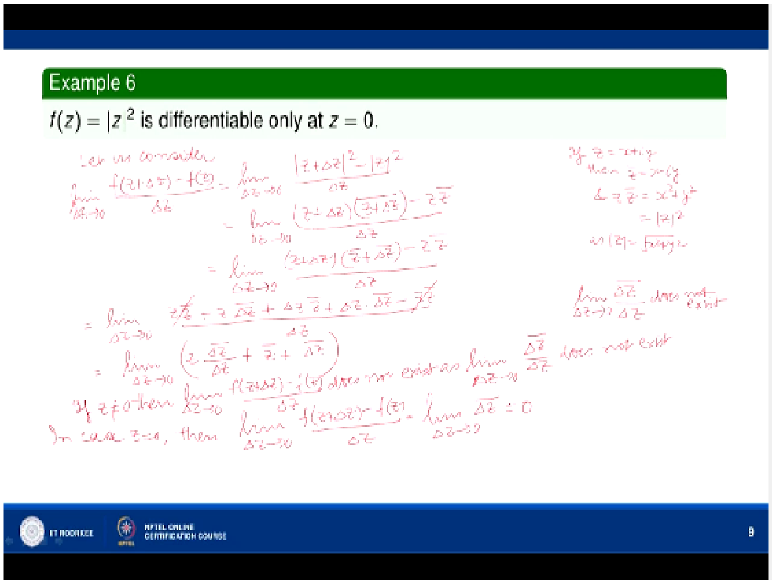

Now let us show that fz=mod of z square is a function which is differentiable only at z=0 and nowhere else. And this means that, we can conclude that, fz=mod of z square is not analytic at any point. Even at z=0, it is not analytic because it is not differentiable at any point other than z=0. For analyticity at z=0, we need a neighbourhood of z=0 in which it is differentiable, okay. So let us show that it is differentiable only at z=0.

So let us consider fz+delta z-fz/delta z and take the limit of this as delta z goes to 0. So when fz=mod of z square, fz+delta z will be mod of z+delta z whole square, so we will have, okay. Now we know that if z is any complex number, if z=x+iy, then z conjugate is x-iy. And z*z conjugate=x square+y square which is equal to mod of z square because of mod of z=under root x square+y square, okay.

So zz conjugate=mod of z square. So we can write here limit delta z tends to 0, z+delta z*z+delta z conjugate-z*z conjugate/delta z which is equal to limit delta z tends to 0, z+delta z*z conjugate+delta j conjugate-zz conjugate/delta z. Now this is equal to z*z conjugate, z*delta z conjugate, then delta z*z conjugate, then delta z*delta z conjugate-zz conjugate/delta z. Now this term will cancel with this and we will have then, you see dividing by delta z, we have z*delta z conjugate/delta z+z conjugate+delta z conjugate, okay.

Now here we can see that, we have already seen that limit of delta z conjugate/delta z when delta z goes to 0; we have already seen that limit of delta z conjugate/delta z as delta z goes to 0 does not exist. While proving that fz=z conjugate is not differentiable for any z, we have shown that limit of delta z conjugate/delta z as delta z goes to 0 does not exist. So if z is not equal to 0; if z is not equal to 0, this limit will not exist, okay. Limit of delta z conjugate/delta z; if z=0, this term will become 0. So this term will not disappear, okay.

But if z is not equal to 0, then limit of delta z goes to 0, fz+delta z-fz/delta z does not exist as the limit of delta z conjugate/delta z as delta z goes to 0 does not exist, okay. See when delta z goes to 0, if z is not equal to 0, this term, this does not have the limit. This term is independent of delta z. So this will remain z conjugate. This term will go to 0 when delta z goes to 0, delta z conjugate also goes to 0.

So with this term and this term, there is no problem but with this term, there is a problem if z is not equal to 0. And if z is not equal to 0, therefore, the limit of fz+delta z-fz/delta z as delta z goes to 0 does not exist. So in case z=0, then we have f limit delta z goes to 0, fz+delta z-fz/delta z=limit delta z goes to 0, fz=0, this term becomes 0, we have z=0. So z conjugate is also 0. So we have simply delta z conjugate, okay.

So when delta z goes to 0, delta z conjugate also goes to 0. So we have the limit 0. And so the function fz=mod of z square is differentiable only at 0 and nowhere else. So it is not analytic at any point. With this, I would like to end my lecture. Thank you very much for your attention.