Dynamical Systems and Control

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Lecture – 09
Solution of Linear Systems – III

Hello friends, welcome to this lecture in this lecture we will continue our study of finding the n linearly independent solution of x dash = Ax where a is n cross and n matrix. And in previous class we have seen some examples where we have n distinct eigen values of the coefficient matrix a and there we have find out n linearly independent eigen vectors. And hence we can write down the general solution in the form of n linearly independent solution.

And we have seen that in the case when we do not have a distinct eigen values then we may or may not get n linearly independent eigen factors. And hence we may not get n linearly independent solution of the form of e power lambda t * vector v. So, we generalize and then we try to generalize we try to get the motivation from the scalar case that in the scalar case of solving x dash = ax let me write it here.

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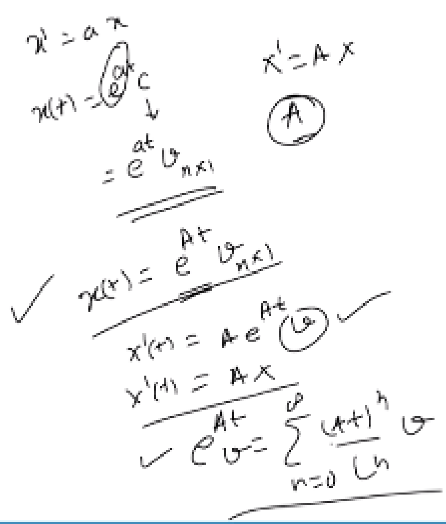

That x dash = ax our solution is coming out to be xt = e to the power at * sum constant c now we have done this case that the first place of this constant. We have written that e to the power at and let us say v is n cross 1 and we have seen that it is very nice if a coefficient matrix A has distinct eigen values for distinct eigen values. This can be verified it will give you solutions n linearly independent solution.

But in case when A has repeated eigen values then this may not work good and then we thought to generalize this candidate that this candidate here. And we thought that it is possible to find out a formula for e to the power at v n as cross . We have seen in previous class that this will work as a solution in fact we have shown that x dash t = A e to the power At * v and this implies that this is AX so X dash t will be A of X.

For any constant vector v here but e to the power At we have shown that it is nothing but power series then it is At n/ factorial n n is from 0 to infinity. So, in this sense we simply say that this is an infinity series in the powers of matrix A, and it is quite difficult to find out the sum of the matrix sum of this infinite series. Then we will focus our attention on this vector v such that this infinite matrix infinite series will be say truncated into a final many terms.

That is what we want to discuss in this lecture so what we have done in previous class.

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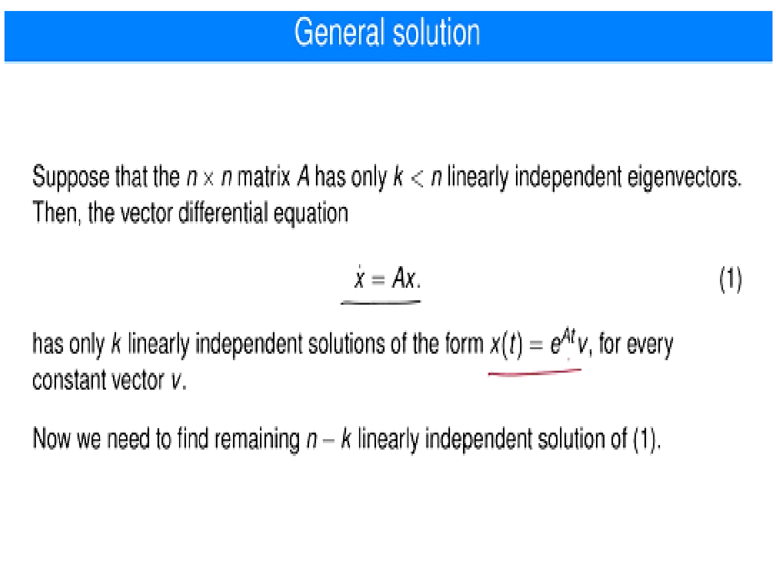

That in the case when we have say a k a number of linearly independent eigen vector which is strictly less than n and then we have to look at the solution of the form e to the power At v. And with the help of this we try to find out the remaining n-k linearly independent solution.

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Similar to the scaler case we may try $x(t) = e^{At}v$ as a solution. Let the matrix A be an $n \times n$ matrix then the function e^{At} may be defined as the limit of the following series

$$I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^nt^n}{n!} + \ldots, \tag{2}$$

provided the series of matrices converges.

It can be shown that the infinite series (2) converges uniformly for all t, in fact the following inequality is true

$$\|e^{At}\| \leq e^{\|A\|t}, \text{ for each fixed but arbitrary } t.$$

And we define e to the power At as a limit of this infinite series and we have shown that this series is converse uniformly for all t norm of e to the power At is < = e of norm of At for each fixed but arbitrary t.

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Since the convergence of the series (2) is uniform and hence series can be differentiated term by term. In particular

$$\begin{aligned}\frac{d}{dt}e^{At} &= A + A^2t + \ldots + \frac{A^{n+1}}{n!}t^n + \ldots \\ &= A\left[I + At + \ldots + \frac{A^n}{n!}t^n + \ldots\right] \\ &= Ae^{At}.\end{aligned}$$

That is $e^{At}v$ is a solution of (1) for every constant vector v, since

$$\frac{d}{dt}e^{At}v = Ae^{At}v = A(e^{At}v).$$

And we have shown that d/dt of e to the power At v = A e to the power At which shows that this is actually e to the power At v is actually a solution for constant vector v here.

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Now we want to find n linearly independent vectors v for which the infinite series $e^{At}v$ can be turned down to be a finite series and can be summed exactly Observe that

$$e^{At}v = e^{(A-\lambda I)t}e^{\lambda It}v$$

for any constant λ, as $(A-\lambda I)(\lambda I) = (\lambda I)(A-\lambda I)$.

It can be proved that $e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$, when $AB = BA$. Moreover,

$$e^{\lambda It}v = \left[I + \lambda It + \frac{\lambda^2 I^2 t^2}{2!} + \ldots\right]v = \left[1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots\right]v = e^{\lambda t}v.$$

Hence, $e^{At}v = e^{\lambda t}e^{(A-\lambda I)t}v$.

And to find out to simplify this e to the power At because using the infinite series formulation of e to the power At is quite difficult to calculate. So, we look at this observation that e to the power At can be written as e to the power A – lambda it* e to the power lambda it *v. So, here this we are able to do because these two matrix commute. So, here we have shown that here we have observe that e to the power A=B = e to the power A* e to the power B.

Or e to the power B* e to the power A. When this A and B are 2 commutative matrix it means if AB = BA then I can write this in general it may not be true for arbitrary matrix A and B. So, here since A – lambda I n lambda i commute then I can write e to the power At v as e to the power A - lambda it e to the power lambda itv and also, we have shown that e to the power lambda it e is nothing.

But lambda tv it means that e to the power Atv = e to the power lambda t * e to the power A- lambda I t * v. So now we will focus on this and we try to find out the vector v such that this matrix in fact e to the power A – lambda I t * v is basically I + A – lambda I t + so on A- lambda I n t to the power n/ factorial n and so on * v. So, we will focus, and we will find v such that infinite series can be truncated in to a finite terms.

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General Solution

While finding the general solution of (1) or equivalently finding n linearly independent solutions of (1), we may have the following two possibilties.

1. In first case, A has n linearly independent eigenvectors, and hence the differential equation (1) has n linearly independent solutions of the form $e^{\lambda t}v$.
2. In the second case A has less than n linearly independent eigenvectors that is if A has only k ($k < n$) linearly independent eigenvectors. Then we have only k linearly independent solutions of the form $e^{\lambda t}v$. This may happen when all eigenvalues are not distinct i.e. some of the eigenvalues are repeated eigenvalues.

So, our strategy will be like this that will finding the general solution of 1 that is x dash = A of x or equally finding n linearly independent solution of 1.In fact if we are able to find out any n linearly independent solution of 1 then we can write down the general solution as well. Then we have the following 2 possibilities 1 possibility is that A has n linearly independent eigen vectors and hence we can write down n linearly independent solution.

Of the form e to the power lambda tv and in the 2nd case A has less than n linearly independent eigen vectors then A has only k linearly independent eigen vectors. Then we have only k linearly independent eigen solution of the form e to the power lambda t v this may happen when eigen values are not distinct. So, it means that here the case is corresponding to the case when eigen values are repeated eigen values, so we know focus on this case 2.

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Here we may recall a result of linear algebra that the number of linearly independent eigenvectors (geometric multiplicity(GM)) can not exceed the number of repetition of the corresponding eigenvalue(algebraic multiplicity (AM)) that is $AM \geq GM$.

So to find remaining $n-k$ solutions we consider only those eigenvalues of A for which $AM > GM$.

Let λ is such an eigenvalue of A and find all vectors v for which

$$(A-\lambda I)v \neq 0, \text{ but } (A-\lambda I)^2 v = 0.$$

For each such vector v (known as generalized eigenvector)

$$e^{At}v = e^{\lambda t}e^{(A-\lambda I)t}v = e^{\lambda t}[v + t(A-\lambda I)v]$$

is an additional solution of (1). We do this for all such eigenvalues λ of A for which $AM > GM$. The aim is to find $AM - GM$ number of remaining solutions.

So, here we may recall a result of linear algebra that says that the linearly independent the number of linearly independent eigen vectors correspond which we call as geometric multiplicity cannot exceed the number of repetition of the corresponding eigen value that is a commonly known as algebraic multiplicity that is AM > = GM. Let us try to understand this that we have a matrix A then we have some eigen value lambda.

Suppose it is a repeated say k times suppose it is a lambda is the root which is repeated k times then we call the algebraic multiplicity of lambda I is k so algebraic multiplicity of lambda i is given as k. The number of reputation of this root is your algebraic multiplicity now corresponding to lambda i just find out your number of linearly independent eigen vectors and the number of linearly independent eigen vectors corresponding to lambda i is known as geometric multiplicity of lambda i.

So, lambda i so geometric multiplicity is what the number of linearly independent eigen vectors corresponding to this lambda i and that we call as G of M geometric multiplicity. Now if geometric multiplicity is = algebraic multiplicity then for each lambda i then we have n linearly independent eigen vectors, but we have shown that in this. In the case when algebraic multiplicity is strictly > geometry multiplicity and the result says that this algebraic multiplicity is > = geometric multiplicity.

In fact, if you remember this a matrix 1 1 0 0 1 1 0 0 1 here you have shown that algebraic multiplicity of 1lambda 1 = 1 is 3. But geometric multiplicity corresponding to lambda1 = 1 is only 1 so here algebraic multiplicity is 3 and geometric multiplicity is 1. Here we have seen that there is only 1 linearly independent solution is available that is I think 0 0 1 that we can verify again so here we have only 1 linearly independent solution.

So, geometry multiplicity is 1 algebraic multiplicity is 3. So, here this is the result of linear algebra this you can find out in any standard textbook of linear algebra then. So, it means that we are in trouble when we have the case then AM is strictly > GM. So, to find remaining n-k solutions we consider only those eigen values of A for which AM is strictly > GM. So, let us say that lambda is such an eigen value of A and find all vectors v for which A- lambda I v is != 0 it means v is not an eigen vector of A corresponding to lambda but A – lambda i whole square * v = 0.

So, it means that it is not an eigen vector but that it is a kind of a solution of A – lambda square x = 0. So, for each this of vector v which is commonly known as generalized eigen vector we can write down the expression for e to the power At v as e to the power lambda t e to the power A lambda I to the v. This we have already shown now focus on this e to the power A- lambda i t so this e to the power lambda t have written as it is now this is what this is written as let me write it here.

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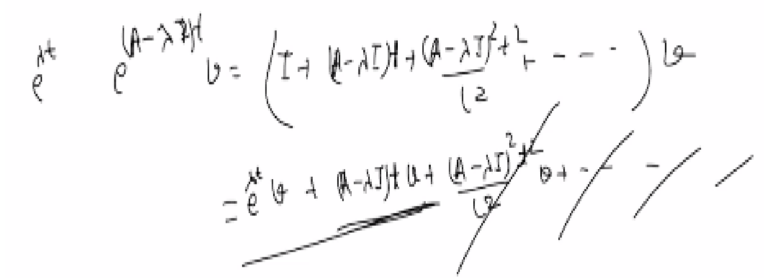

It is what e to the power A – lambda i t v so it is I + A – lambda I t + A – lambda I square t square/factorial 2 and so on then if we write it this as that is v + A – lambda I tv + A – lambda I whole square t square factorial 2 v and so on. So, it is basically e to the power lambda t so if it is e to the power lambda t v. So, here if you look at if this is non 0 then we make that this term is 0 and if we make this term as 0 so remaining term will also be 10 to 0 simply goes to 0.

So, it means that this infinite series is now reduce to only 1st 2 term so that is what is the idea here that here we simply find out e to the power A-lambda I t that is I* v + A – lambda t*v. And since A-lambda square v e = 0 then we can verify that A- lambda I cube * v is also 0 in fact it can be written as A-lambda I * A- lambda I whole square v now this is 0 so A – lambda applied on 0 is gives you 0.

And hence we can say that higher powers of A - lambda I 3 4 operating on v will give you value 0 so it means that this infinite series is now reduced to only 2 term that is v +t A – lambda i v = 0. Now if you say that if v is such that A- lambda iv = 0 then it will reduce to only 1 term that is e to the power lambda t *v and that will shows us e to the power At is nothing but e to the power lambda tv and in that case v is known as eigen vector of A corresponding to the eigen value lambda.

So, here we have shown that if v is an eigen vector then it is reduces to e to the power lambda t * c if v is not an eigenvector and we call it as a generalized eigen vector than it will contain a 1 at least 1 more term to e to the power lambda t * v. So, here what we have shown that if we choose v in a way. In this way then e to the power At v can be written as e to the power lambda t v+t A – lambda I v.

And this will work as an addition solution of 1 and we do this for all such eigen value lambda of a for which AM is strictly > GM and the idea is to find out the remaining AM- GM number of solutions.

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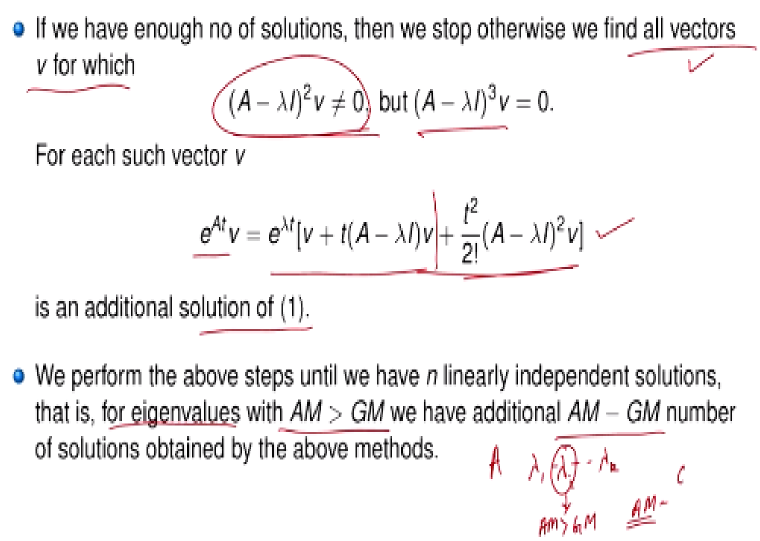

And if we have enough number of solution by the previous step then we stop, and we write the general solution in terms of n linearly independent solution. But if we do not have enough number of solution then we find out all vectors v for which this A - lambda I square v is non 0. But the next higher power that is A – lambda I q v is = 0 and in this case your e to the power At be will contain 1 more term that is t square / factorial 2 A – lambda I whole square v .

So, if this term is 0 then it will be reduce up to only these 2 term but if this term is non 0. And next term is higher term is 0 then we have additional 1 additional term so e to the power At v is now used to e to the power lambda t v + t times A – lambda I v + t square/ factorial 2 A – lambda I square v. And we call this is as an additional solution of fun and we keep on repeating this until we get n linearly independent solution that is for eigen value which have AM is > GM AM – GM number of solution obtained by the given method.

So this is the criteria, let me summarize this you have a matrix a. And there we have some eigen values say lambda 1 to lambda k and suppose there exists some lambda I for which your AM is > GM then for this lambda I you perform the given a step and you find out the generalized eigen vectors. In fact, this vector is also known as generalized eigen vectors and you find out generalized eigen vectors till that we have the GM number AM number of solution available.

We already have AM -; we already have GM number of solutions the remaining AM – GM number of solutions we can obtain by the given method. It means that we need to find out a generalize eigen vector AM- GM number of generalize eigen vector.

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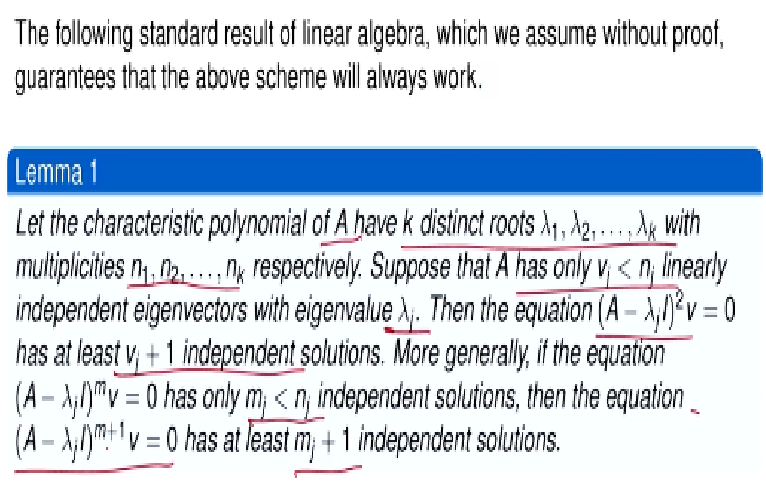

Now the guarantee that this method will work is given by the following a Lemma which we assume without any proof. And this basically related to finding the Jordan canonical form of a given matrix now what this Lemma says let us try to understand. Let the characteristic polynomial of A have k distinct roots that is lambda 1 to lambda k with multiplicities n1 to nk commercial respectively.

So, it means algebraic multiplicity of lambda 1 is n1 algebraic multiplicity of lambda 2 is n2 and so on suppose that S has vj that is < nj linearly independent eigen vector with eigen value lambda j. So, here algebraic multiplicity is nj but geometric multiplicity is only vj then the equation e-lambda I whole square v = 0 at least 1 more independent solutions. So, it means that if we are having only vj number of linearly independent eigen vectors.

Where vj is strictly > nj then the said equation A – lambda ji whole square v = 0 has at least 1 new independent solution that is vj+ 1 independent solution. Earlier we have only vj independent solution now we have 1 that is vj+ 1 independent solution of A – lambda jI whole square v = 0. So, more generally if the equation A – lambda j I m v = 0 has only mj which is strictly < independent solution.

Then the equation A – lambda j I e to the power m + 1 v = 0 has at least mj+1 independent solution. So, it means that here this system will have mj+1 independent solution out of which you can say that is already obtained. So, we will chose 1 new solution which is linearly independent solution to the previous 1. So, this Lemma is the guarantee that the above what we have defined A will going to work.

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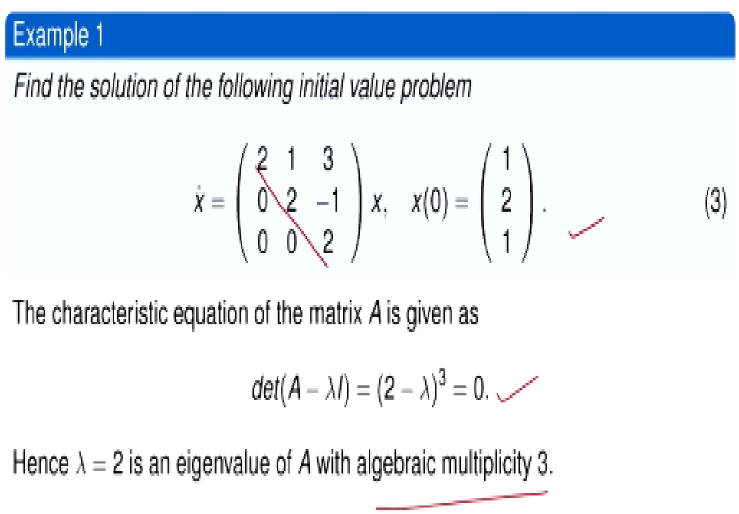

So, now let us take 1 example based on this procedure so example 1 find the solution of the following initial value problem x dash = 2 1 3 0 2 -1 0 0 2 x and initial condition is given as at t

= 0 that is 1 2 . So, here 1st thing we need to find out since we need to find out the initial value problem, so we need to look at the general solution and then we are to find out the values of constant and with the help of initial conditions.

So, 1st first we find out the 3 linearly independent solution so far that we need to look at the eigen values. So, if you will look at the eigen values since it is upper triangular matrix so eigen values are nothing but the diagonal entries and if you look at diagonal entries are 2 2 2. So, here eigen values are 2 with a algebraic multiplicity 3 so eigen values of this matrix is 2 with algebraic multiplicity 3.

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In this example A has only one linearly independent eigenvector for repeated eigenvalue $\lambda = 2$ i.e. for $\lambda = 2$, the algebraic multiplicity AM is 3 and the geometric multiplicity GM is 1 that is $AM > GM$.

So we have to apply the above given method to find the remaining $3 - 1 = 2$ solutions. Consider the equation

$$(A-2I)^2U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now look at the eigen vector so to find out an eigen vector let us find out the solution of A – 2I V = 0 so we simply say that we solve this A – 2I V = 0. And when you solve this, we will get that v2 an v3 = 0 in fact we can solve this, and it is v2+ v3= 0 and v3 = 0 so v3 = 0 implies that v2 = 0 so we have only 1 arbitrary value that is v1. So, here v1 may take arbitrary value and we can write down that v1 may be 1 0 0 and 1 solution.

We can obtain of the form of e to the power lambda tv that is given as x1t = e to the power 2 1 0 0. So, by the previous method we have only 1 solution of the form e to the power lambda t v, so we need to find out the remaining 2, linearly independent solution for that we have to find out they generalize eigen vector.

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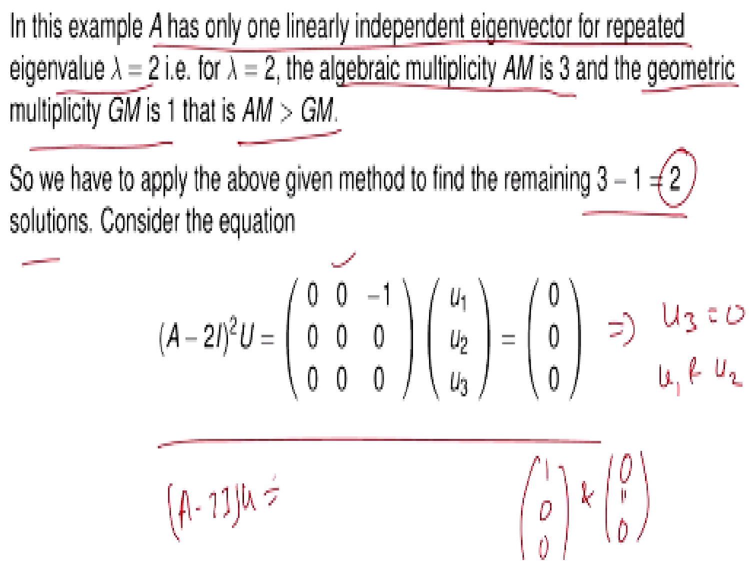

So, here in this example A has only 1 linearly, independent eigen vector for repeated eigen value that is lambda = 2. So, algebraic multiplicity is 3 and geometric multiplicity is 1 so AM is strictly > GM so one thing we observed like this that AM > GM in this example. So, here we need to find out the discuss method to find out the remaining 3 - 1 that is 2 linearly independent solutions so for that look at the solution of A – 2I square U = 0.

So, if A – 2I U = 0 has say 1 solution than A – 2I whole square U must have 2 solution at least so just calculate A – 2I whole square and it is coming out to be 0 0 -1 0 0 0 0 0 0. And u1 u2 u3 here I am taking the liberty that A – 2I is square calculated in this way you please calculate so when you look at the new 3 is = 0 u1 and u2 R between so there are possibility like 1 0 0 and 0 1 0 so here we have 2 linearly independent solution of A- 2I square U = 0.

And here we choose a value which is not which is independent to your previous eigen vector because here we had to put 1 more condition that is A – 2I u is non 0 so it means that the earlier eigen vector that is 1 0 0 if we take than A – 2I u has to be 0 so we drop this and write down our solution as 0 1 0.

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This implies $u_3 = 0$ and both u_1 and u_2 are arbitrary. Now, we choose the vector

$$U = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

such that $(A - 2I)U \neq 0$ that is LI to the eigenvector V and satisfies $(A - 2I)^2 U = 0$. Hence

$$x^2(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t}e^{(A-2I)t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \qquad (5)$$

is a second linearly independent solution of $\dot{x} = Ax$.

So, here we take the vector u as 0 1 0 so that A – 2I u is not = 0 and of course that is linearly independent eigen vector and it satisfies A – 2I square u = 0. And with the help of this you know solution 2nd solution next to t is written as e to the power At 0 1 0 now e to the power At has this expression e to the power 2t e to the power A- 2I t 0 1 0 and now A – 2I is basically what this is I + A – 2I t + A -2I square factorial 2 t square and so on.

When you operate on v then this will give you v and this will give you some value and this there onward it is 0. So, it will have only few term that is identity means 1 0 0 0 1 0 0 01 + t times A – 2I t basically is what you have already calculated.

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Let us find an eigenvector of A corresponding to the eigenvalue $\lambda = 2$:

$$(A - 2I)V = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

and hence $v_2 = v_3 = 0$ and v_1 may take arbitrary value. Therefore

$$x^1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (4)$$

is the only one nonzero solution of the form $e^{\lambda t}v$ of (3).

So, A – 2I is 0 1 3 0 0 -1 so 0 1 3 0 0 -1 0 0 0 and this* v when you simplify what you will get you will get here the value e to the power 2 t1 0 so here this I is basically 1 0 0 0 1 0 0 0 1 + t times 0 1 3 0 0 -1 0 0 0 * v and * e to the power 2t. Now let us calculate this for this will let us write it t e to the power 2t and this is I + t times A – 2I * 0 1 0.

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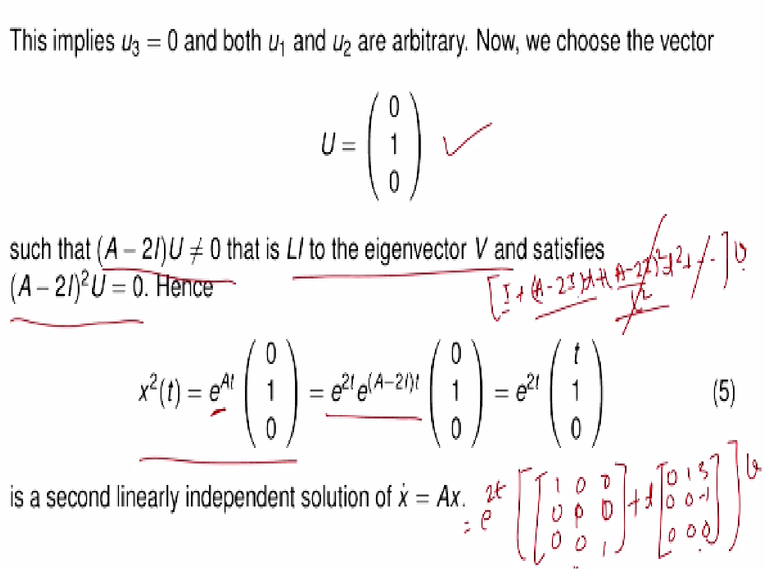

When you simplify it is written as e to the power 2t as it is this is written as 1 t 3t 0 1 -t 0 0 1 and operating 1 is 0 1 0 to solve this you will get e to the power 2t * t 1 0. And that what we have written as the 2nd solution x to t as e to the power 2t t1 0 and this is linearly independent solution of x = AX. So, how you can check this, and this is a linearly independent solution that you can easily check at a point t = 2 0.

If you look at the 1 st solution will be what e to the power 2t 1 0 0 and it is e to the power 2t t 1 0 if you put t = 0 you can see that x 1 0 x 2 0 are linearly independent. What is x1 0 here x10 is basically 1 0 0 and x2 0 is basically is coming out to be 0 1 0 and you can see that this 2 are linearly independent so x2t and x1t are also linearly independent now.

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But still we have only two linearly independent solutions, therefore we perform one more step of the method and find all the 3 solutions of (3). Now

$$(A-2I)^3 W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Obviously every vector w is a solution of this equation. The vector

$$w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is a vector which satisfy $(A-2I)^3 w = 0$, but $(A-2I)^2 w \neq 0$ and $(A-2I)w \neq 0$.

But still we have only 2 solution because by the previous method we are having only 1 new solution we have obtained. But we need 2 linearly independent solutions so we perform 1 more iteration of the method and find all the 3 solution of 3 so look at A – 2I cube * W = 0 now A-2 I cube is basically nothing but a 0 matrix * W1 W2 W3 =0 so here you can simply say that this will have 1 more solution of the previous method.

That is in previous method that is in previous step we have 2 solution so here we will have 3 solutions so here we have solutions we can have 1 0 0 0 1 0. And 0 0 since we need to look at that A-2I W is non 0 and A-2I square W is non 0 so we have to leave this 2 options. So, here we take the solution as W as 0 0 1 so that A – 2I cube w is 0 but A- 2I square w is non 0 and A – 2I w is also non 0 so here we take w as 0 0 1 and corresponding to w the solution is what.

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Hence

$$x^3(t) = e^{2t} e^{(A-2I)t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (6)$$

$$= e^{2t}\left[I + t(A-2I) + \frac{t^2}{2}(A-2I)^2\right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (7)$$

$$= e^{2t} \begin{pmatrix} 3t - \frac{1}{2}t^2 \\ -t \\ 1 \end{pmatrix}. \quad (8)$$

is the third solution corresponding to $\lambda = 2$.

Solution is given as x3 t = e to the power 2t * e to the power A – 2I t * 0 0 1 and if you expand this e to the power A – 2I t then it is nothing but I + t A -2I + t square / factorial 2 A -2I square operating on 0 0 1. And when you simplify, I am not simplifying here I am observing that if you simplify you will get e to the power 2t 3t – 1/2t square -t 1 so this will be your 3rd solution corresponding to lambda= 2.

Now we have obtained 3 linearly independent solution corresponding to lambda = 2. So, we stop our method and we can write down the general solution as.

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Hence the general solution of the equation (3) is given by

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t),$$

where $x_i(t)$, $i = 1, 2, 3$ are linearly independent solutions of (refamgm) and given by the equations (4),(5) and (6) respectively. By using initial conditions, we have $c_1 = 1, c_2 = 2$ and $c_3 = 1$.

$x_1(0), x_2(0), x_3(0)$

Hence

$$x(t) = e^{2t} \begin{pmatrix} 1 + 5t - \frac{1}{2}t^2 \\ 2 - t \\ 1 \end{pmatrix}.$$

is the solution of the initial value problem (3).

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 3t - \frac{1}{2}t^2 \\ -t \\ 1 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

C1x1 t +c2x2 t + c3x3 t where x1 t is the form of e to the power lambda t * v and x2 and x3 t are solution obtain with the help of generalize eigen vector. And here we can check that these are linearly independent solution for that you simply put t = 0 and calculate x1 0x20 x30 and you can see that these are linearly independent eigen vectors. We can write that xit are linearly independent solution of equation 9 and given by the equation 4 5 and 6 respectively.

And using an initial condition we can fix our say constant c1 and c2 and c3 and in this case our solution is conscience on coming out to be 1 2 and 3. In fact we can write it a solution as follows same so x of t = c1 now e to the power 2t you can take it out and this 1 0 0 + c2 e to the power 2t and here it is t10 + c2 +c3 e to the power 2t.And here solution is basically 3t-1/2 t square 3t – 1/2 t square and 2 –t it is –t. and 1

So in initial condition is given as what initial condition that we look at initial condition is 1 2 1 so x of 0 is 1 2 1 so when you put t = 0 what you will get 1 2 1 – c1 1 0 0 +c2 0 1 0 + c3 0 0 . And you can easily check that c1 is coming out to be 1 c2 is coming out to be 2 and c3 is coming out to be 1 so putting c1 c2 and c3 values in the general solution. You can write down the solution of the initial value problem as e to the power 2 t and 1+5t-1/2tsquare 2-t and 1.

So, this is the solution of the initial value problem 3 so here by this example we have shown that in case of a repeated eigen value how to find out the n linearly independent solutions here. Now we to concept fundamental matrix solution so once we have n linearly independent solutions then we can write down the general solution. And we can write down the general solution as this x of t as.

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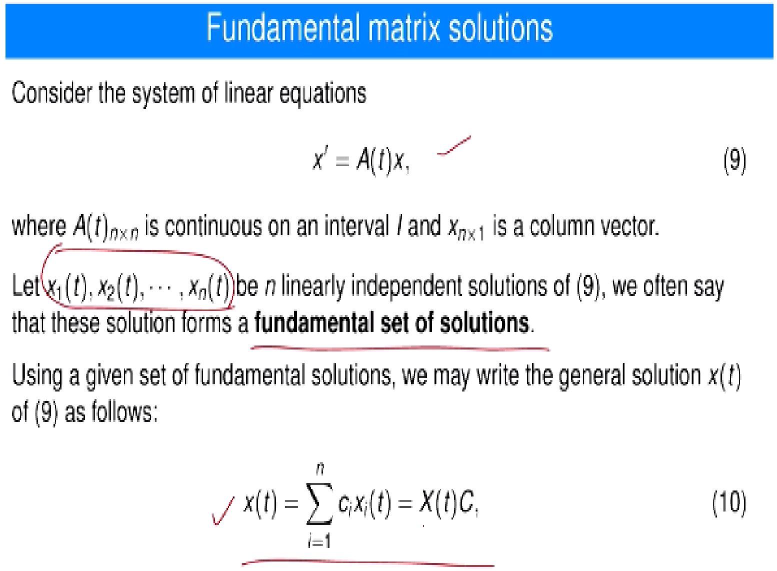

A linear combination of x dash AT where x of At are linearly independent solution of the equation x dash = AT x then when we have n linearly independent solutions x1 to xn t. We commonly called this linearly independent solution as a fundamental set of solutions. Now here we may have different choices of n linearly independent solution of equation 9 so we may have different fundamental set of solutions.

So, fundamental set of solutions basically gives you a set off n linearly independent solution of x dash = Atx. And once we have fundamental set of solutions then you can write down the general solution in the form of fundamental set of solution and it is given as xt * c where xt is basically what.

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where $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is a column vector and $X(t) = [x_1, x_2, \cdots, x_n]$ is a square matrix whose columns are $x_1, x_2, \cdots, x_n$.

Definition 2

A matrix of n rows whose columns are solutions of (9) is called a solution matrix of (9).

Definition 3

A matrix $X(t)$ is called a fundamental matrix solution of (9) on I if its columns form a set of n linearly independent solutions of $\dot{x} = A(t)x$ on I.

Xt can be considered as a matrix whose columns are n linearly independent solutions off x dash = Atx. And c is a column vectors c1 to cn it means that your any solution x of t is given as x of t* c where x of t is basically n cross n matrix and c is n cross 1 where xt is a matrix whose columns are x1t x2t and so on. It means that with the help of these n linearly independent solution we form a matrix which we call as a matrix solution.

With the help of matrix solution we can write down the general solution as this xt = at*c so we have a new definition. We say that a matrix of n rows whose columns are solutions of x = At I is called a solution matrix this xt is a solution matrix because columns of this matrix forms a columns of this matrix is basically solutions of x dash = At x then we define fundamental matrix solution.

The only difference between the matrix solution and fundamental matrix solution is that the columns are now linearly independent solutions of x dash = At x. So, it means a matrix x t is called a fundamental matrix solution of 9 on I if its columns form a set of n linearly independent solutions of x dash = A tx. So, it means that if columns form a fundamental set of solutions then this matrix is a fundamental matrix solution. Is it okay?

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Example 2

Find a fundamental matrix solution of the system of differential equations

$$\dot{x} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} x. \qquad (11)$$

Observe that the eigenvalues of the matrix are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = -2$. Also, we can calculate the respective eigenvalues of the matrix which are given as

$$v_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

So, now let us consider the example here in this example we need to find out a fundamental matrix solution of the system of differential equation x dash = 1 -1 4 3 2 -1 2 1 -1* x. So, here 1st thing we need to absorb that what are the eigen values of this matrix here I am leaving it to you to see that the eigen values are given as1 3 and -2. And please observe that these are all distinct, so it means that corresponding eigen vectors are also linearly independent.

And we can find out like 1 as -1 4 1 v2 as 1 2 1 and v3 as -1 1 1 as linearly independent eigen vectors corresponding to1 3 and -2.

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Here we have 3 linearly independent eigenvectors so 3 linearly independent solutions of (11) are given as

$$e^{t}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \; e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

$x_1(t)$ $\quad$ $x_2(t)$ $\quad$ $x_3(t)$

$$X(t) = \left[x_1(t),\; x_2(t),\; x_3(t) \right]$$

So, here we can find out the linearly independent solutions as e to the power t * -1 4 1 * e to the power 3 t * 1 2 1 and e to the power -2t -1 1 1. So, it means that here this is your x1t and this is x2t and this is your x3t so once you have x1 x2 x3 t these are linearly independent matrix having this x1 t x2t and x3t as columns is not only a matrix solution. But it is also a fundamental matrix solution because these 3 are linearly independent solution.

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Example 2

Find a fundamental matrix solution of the system of differential equations

$$\dot{x} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} x. \qquad (11)$$

Observe that the eigenvalues of the matrix are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = -2$. Also, we can calculate the respective eigenvalues of the matrix which are given as

$$v_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

So, here this xt = - e to the power t e to the power 3t -e to the power -2t 4 e to the power t 2 e to the power 3t e to the power –2t e to the power t e to the power 3t e to the power -2t is fundamental matrix solution of the equation x dash = Ax where A is given by 1 -1 4 3 2 -1 2 1 -1. So, here we obtained the fundamental matrix solution like this so with this we stop discussing of this lecture.

And we will continue discussion of about the fundamental matrix solution in the next class, thank you very much for listening us. Thank you.