Dynamical Systems and Control
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Lecture – 59
Relation between Continuous and Discrete Systems - I

Dear students. Welcome to this lecture on the relation between the continuous and discrete systems. So in this lecture, we will see the controllability property of the continuous and discrete system and their relations.

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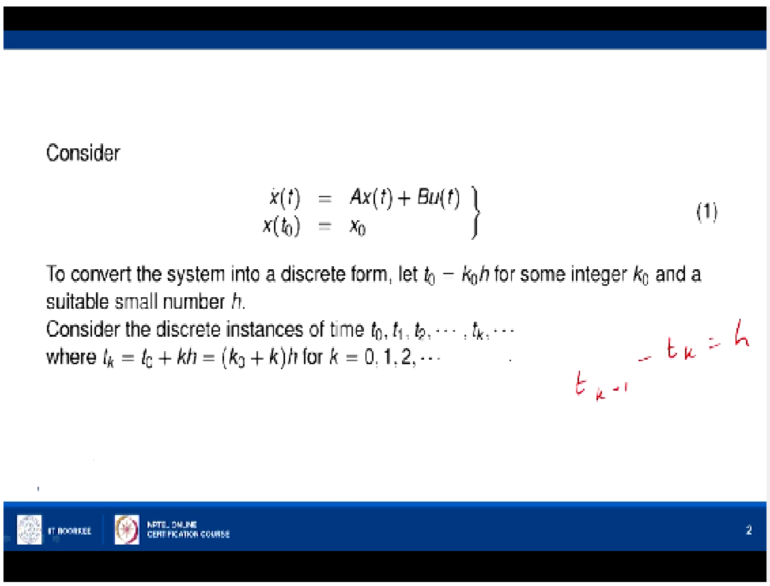

That is in many practical problems, a continuous control system of this form x dot t=Ax+Bu with initial condition x of t0=x0 needs to be converted into a discrete control system because when we solve a continuous system using a digital computer, then it is required that it has to be discretized and then solved. So when we discretize a continuous system, then the important property such as the controllability, observability, stability should not be lost in the equivalent discrete system.

So we will see that under what condition the continuous system and the corresponding discrete system behave in a same manner. So in this lecture, we will consider the controllability properties of these 2 systems. So we consider the system I and let us take the initial time instant as t0 and we can write it in the form of k0*h where k0 is a suitable integer and h is small number, positive number.

Then we consider the discrete time instances t1, t2, tk, etc., where it is given by the increment in each is h that is tk+1-tk is always equal to h, the discrete. We consider tk to be equal to k0+k*h where k is 0, 1, 2, 3, etc., the time instances.

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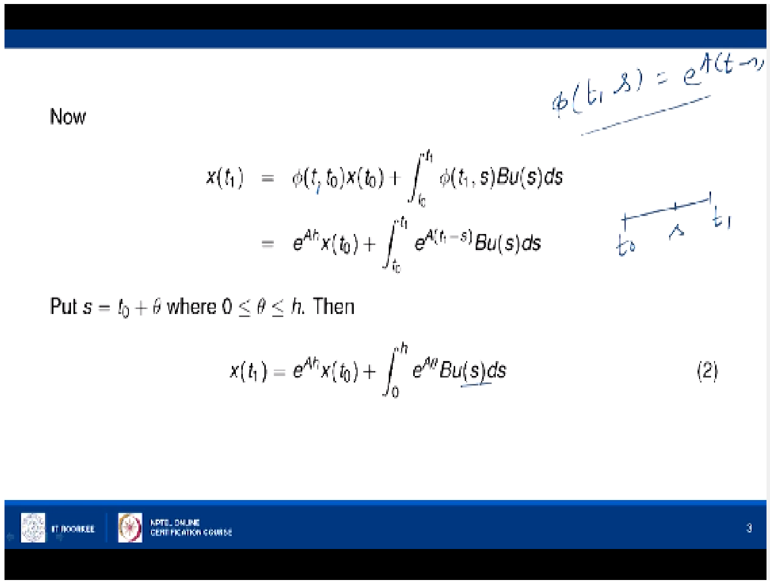

Now we can find the value of the state x of t1, x of t2, etc. at the time instances, discrete time instances. So x of t1 is given by phi of t1, t0*x of t0 t0 to t1 of phi of t1, sBusds. So that is nothing but e to the power, because phi of t, s=e power At-s, for the constant matrix A, autonomous system. So we get e to the power A, t1-t0 is h*x of t0+integral t0 to t1 e to the power At1-sBusds.

So here we are considering with t0 to t1, s is in between. So we can write s as t dot + some theta, the theta is from 0 to h. So substituting this value of s in the integral, we get the expression x of t1 is e power Ah*x of t0+integral 0 to the power A theta Busds.

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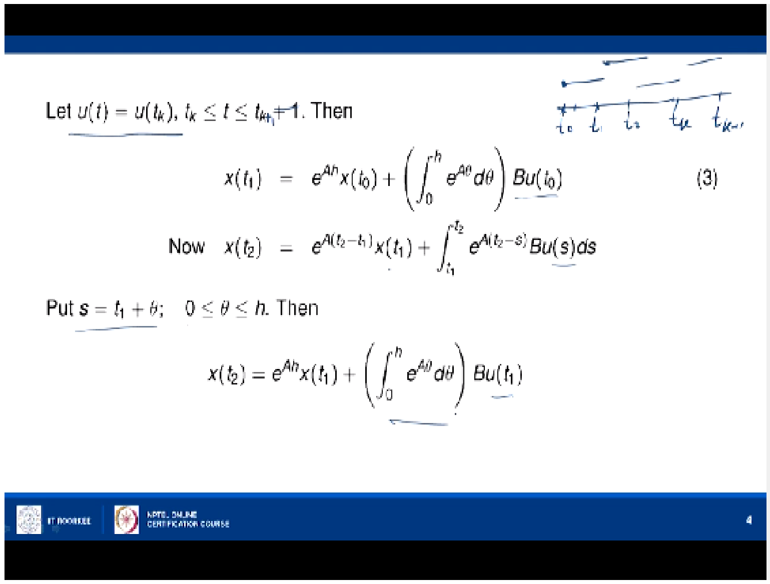

So same way we can also calculate x of t2 as given in this expression. So if we define the control function u of t as a discrete type of function u of tk in the interval tk to tk+1. So in this interval, we have t0, t1, t2, tk, tk+1. So we take the u function as a step function type of thing. That is its value u of t in the interval t0 to t1 is u of t0. And the interval t1 to t2 is the value u of t1, etc. So we can get many, the discrete values for the control function u of t which takes the value only in the left boundary of each interval.

So substituting this u of t step function inside the integral in the integral here. So in the first integral, t0 to t1 u of s is u of t0. So that can be taken out of the integral, we get the expression to be like this. Similarly, x of t2 is e to the power A*t2-t1x of t1+integral t1 to t2 of this u of s. So u of s is now u of t1 because it is in the second interval and s is equal to t1+theta where theta varies from 0 to h and substituting that, we will get the integral like this.

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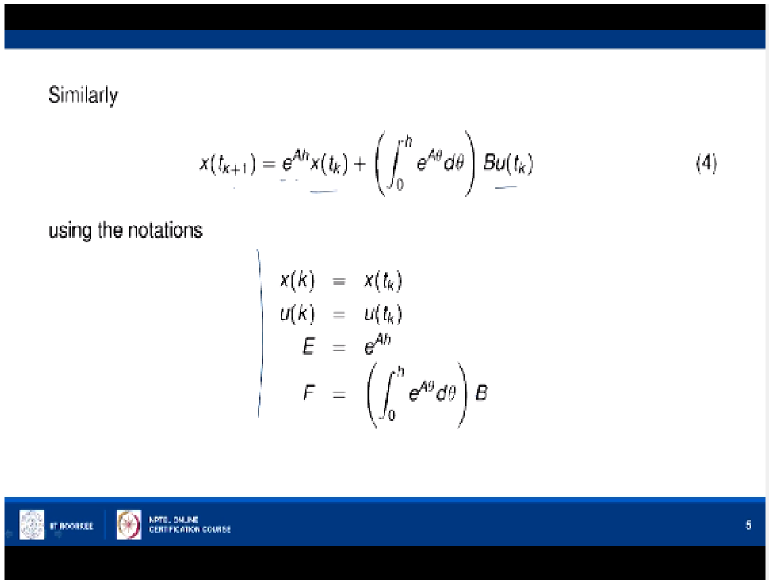

So now if we proceed in the similar way, we get x of tk+1 is e power th*x of tk + integral 0 to the power A theta d theta*this. Now if we use this notation x of k is x of t suffix k and u of k is u of t suffix k, e is the matrix given here, e to the power Ah and F is the matrix given in this second term, integral 0 to the power A theta d theta*the matrix B.

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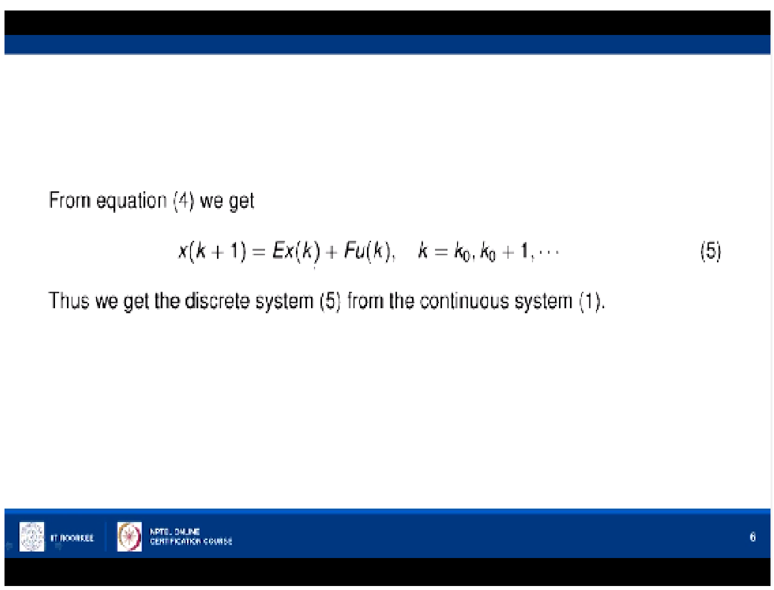

So if this matrix is F, then by substituting these in the equation 4, we get the corresponding discrete system that is x of k+1 is ex of k+Fu of k, k is ranging from k0, k0+1, etc. So from the continuous system 1 getting the discrete system 5 here. So now can we use this discrete system for all purpose that is the controllability, observability, stability of system I, can we apply it to the system 5 as it is?

So that is what the question is. So we will see here in the following theorems that in some cases, the controllability is lost for the discrete system. If the continuous system is controllable, always there is no guarantee that the discrete system is also controllable. So we have to be careful about selecting the discrete points so that the controllability property is preserved between the continuous system and the discrete system.

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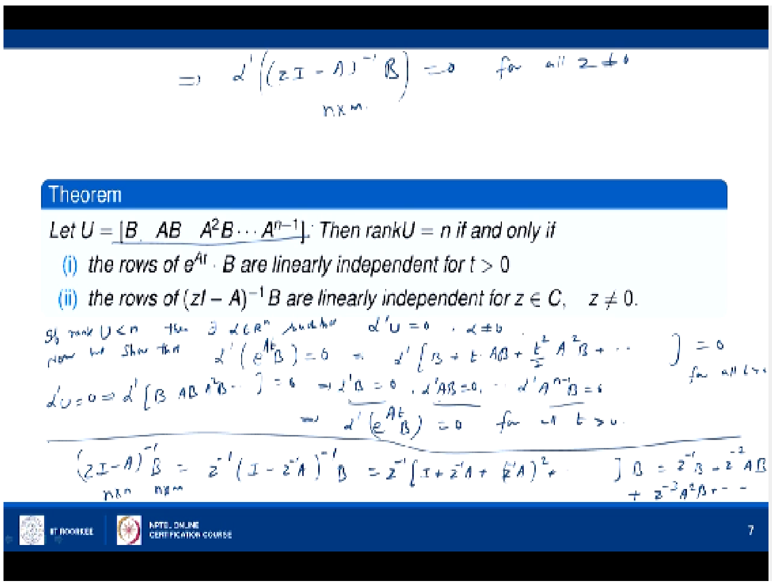

If we consider the matrix U is B, AB, A square B, etc., A power n-1 B and the rank is equal to n if and only if the rows of the matrix e power At*B are linearly independent for all t>0. Similarly, the rank of U is equal to n if and only if the rows of the matrix zI-A inverse*B are linearly independent for z belonging to the complex plane and z is non-0. So any one of this condition, these conditions are now equivalent, the rank of U is equal to n.

It is equivalent to the rows of e power At*B are linearly independent or that is also equivalent to the rows of zI-A inverse*B are linearly independent as given in this theorem. So we can briefly see one of the proofs. So if the rank of U<n, we can prove that the rows of e power At*B are linearly dependent. And if the rows of e power At*B are linearly dependent, then rank of U is less than n.

So these 2 conditions become equivalent. Similarly, we can also prove the condition for the first and third which are equivalent conditions. So if rank of U is strictly less than n, then there exist a vector alpha belongs to Rn such that alpha dashed*U=0 and alpha is not 0. Non-0 vector alpha will exist, so that this is satisfied, it is a usual property of matrix.

So now if we also prove that to show that the rows of e power At*B are not linearly independent, again if you are able to show that, now we show that alpha dashed*e power At*B also will be equal to 0 for the same non-0 vector alpha. So if this condition is satisfied, then it is clear that the rows of e power AT*B are not linearly independent in the case because there exist an alpha non-0 satisfying this condition.

So we can write it directly that is we want to show that alpha dashed*this, if we expand it, we will get B+t*AB+t square*A square B + etc. So if you show that this is equal to 0 for all t>0, then we way that the rows of this matrix is, they are linearly dependent in that case. But that is very obvious because alpha dashed*U-0, it implies alpha dashed*, the matrix U is B AB A square B etc., that is equal to 0.

Or separately if you take the blocks, this implies alpha dashed B is 0, alpha dashed AB is 0, etc. Alpha dashed a power n-1B=0. So then it is also obvious that alpha dashed A power n*B A power n+1*B etc., they also will be equal to 0. So from the previous step, we can see that, we can just multiply alpha dashed*B, first term that is 0. Alpha dashed*A*B that is also equal to 0 where t is a variable here but alpha dashed*AB is 0 alpha dashed A square B is 0, etc.

So all the terms are equal to 0. So this implies that alpha dashed*e power At*B=0 for all t>0. So this implies that the rows of the matrix e power At*B are linearly dependent. So in the similar manner, we can show the converse also. If alpha dashed e to the power At*B=0 for all t, then if you go in the reverse process, we can show that separately, they are all 0 that is alpha dashed*B is 0 alpha dashed*AB is 0, etc.

So that can be shown. With little bit more effort, we can show the converse part of this one. Similarly, we can show that if the rank of U<n, then the rows of zI-A inverse B is also, they are linearly dependent. So the second part is here. We can write zI-A inverse B, so this can be written

as, if you take z outside from the bracket, we get z inverse*I-z inverse A inverse*B from this one.

And if you expand I-z inverse A power -1, we will get I+z inverse A+z inverse A whole square, etc., infinite series*B. So if you multiply it again, we will get z inverse*B, the first term, +z power -2*AB+z power -3*A square B+etc. So we can see that it is similar to this e to the power AtB. Here if they are all in terms of t, a function t, t square/2, etc. is the variable coming in this equation.

Here the variable is z in a slightly different way. But the structure is the same. We have B, AB, A square B as the coefficient of this variables. So the same argument if rank of U<n, there exist an alpha dashed such that alpha dashed*U=0, that implies the same. Alpha dashed of B is 0, alpha dashed of AB is 0, etc. for the non-0 vector alpha. So if you multiply alpha dashed with this one, so this implies that alpha dashed of zI-A inverse*B=0 because each term is separately 0.

Alpha dashed*B is 0. Alpha dashed*AB is 0. This is true for all z not equal to 0. So we can easily see that the size of the matrix is n*n because A is a n*n matrix and I is n*n, so the inverse is also n*n. And B is n*m, so the matrix is n*m matrix. There will be n rows here and they are all function of, they are not constant rows here because z is a variable, so there are n rows which are functions of z here.

So we are considering the linearly independence and dependence of functions, the vector functions here. So it indicates alpha dashed*that matrix=0 for all z, implies that the vectors are linearly dependent here. So the conclusion is if rank of U is strictly less than n, then the rank of zI-A inverse B is also, they are linearly dependent. Similarly, if the rows of zI-A inverse*B are linearly dependent, then the rank of the matrix U is strictly less than n, that can be also proved here.

So it automatically implies that the rank of U=n if and only if these conditions are satisfied in both sides. So now we can observe that the U matrix, rank of U=n implies the controllability of the system x dot=Ax+Bu. So if the continuous dynamical system is controllable. Now we want to see under what condition the discrete, corresponding discrete system is also controllable.

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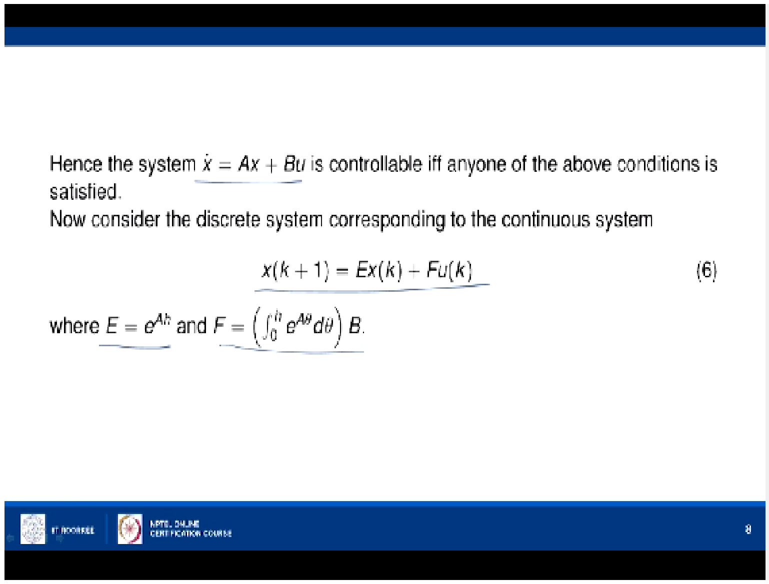

So the system, continuous system is x dot=Ax+Bu and the corresponding discrete system is x of k+1=E*x of k+F*u of k where e is given by this expression as we have seen earlier and F is given by this. The A and Bare constant matrices.

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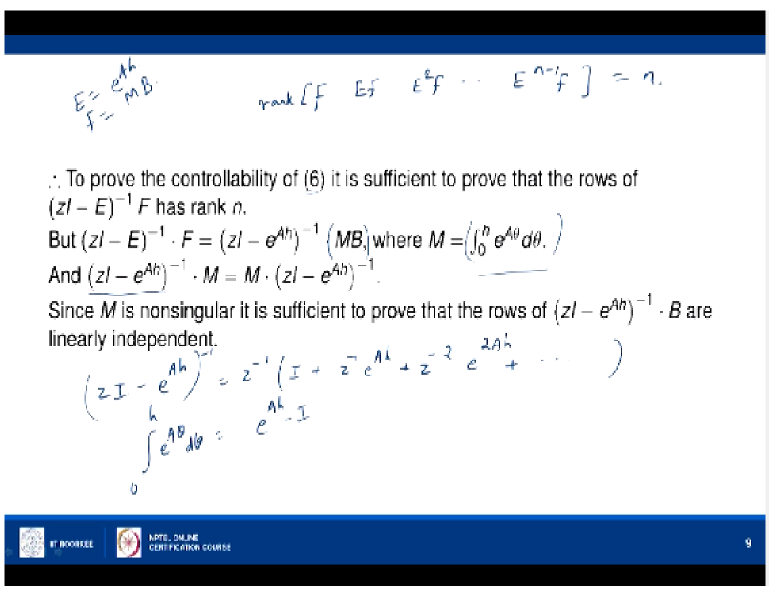

Now we want to show the controllability of the system 6. 6 is the discrete system. So if the continuous system is controllable when the discrete system is also controllable, so just we have seen in the previous theorem that the system 6 is controllable if and only if, either we have to show that the rank of F EF E square F, etc., E power n-1F, so that should be equal to n. This is the standard condition for the controllability of the autonomous system.

But here E is the matrix e to the power Ah and F is the matrix given by the previous expression where F is given by M*B where M is given by this e to the power A theta*d theta integral 0 to h. So F is given by M*B. So in order to prove the controllability of the system 6, we have to show that the rows of the matrix zI-E inverse F, they are all linearly independent by the previous result.

So now F is given by M*B where M is given by this, so we want to prove that the rows of the matrix zI-e power Ah inverse*MB are linearly independent. Now we can show that the matrix zI-e power Ah inverse and M, these are commutative. Because M is given by this expression and the matrix zI-e power Ah its inverse, as we have seen earlier, if you take z outside this integral, it is I-z inverse etc.

So its value is I+z inverse e power Ah+z power -2e power 2Ah+etc. infinite series. And similarly, the matrix M is also made up of the matrix e to the power A type of thing. We can do the integration of this. We will get the similar result. This is, so that is also equal to e power Ah-I. Is not it?

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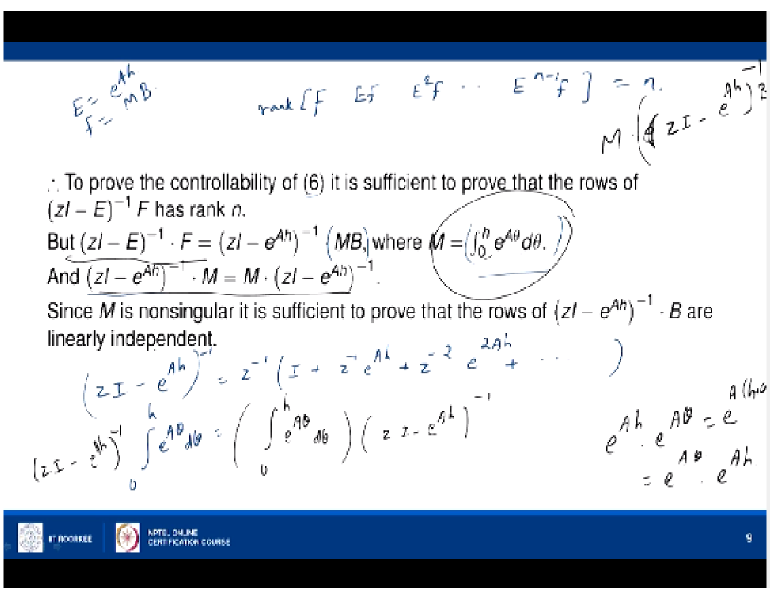

This expression we will get the solution of this one, this matrix*; if you observe the matrix M, M=integral 0 to h e power A theta d theta, we can see that the M matrix and zI-e to the power Ah, they are commuting. Because we can see that zI-e power Ah inverse*0 to the power A theta d theta will be obtained by multiplying the terms like e power Ah*e power A theta. But they are commutative because it is equal to e power A theta*e power Ah.

So in each term, e power A*h and e to the power A*theta or e power 2*Ah e power A theta, these are all always commuting, so we get finally the commutativity property of this expression. So in order to prove the controllability of the discrete system, we have to prove that the rows of the matrix zI-E inverse*F are linearly independent.

So it, instead of doing that, if you substitute F=M*B and M is taken to the left side of zI-e to the power Ah, so we will get, we have to only check the linearly independence of the rows of the matrix zI-e power Ah inverse*B. But M is a non-singular matrix, this one is a non-singular matrix, that can be verified because for any matrix A, e to the power A theta is non-singular and by after putting the limits, we can see we will arrive at a non-singular matrix.

So if M is non-singular, then the rank of M*this matrix is same as the rank of the remaining matrix. So we have to only see that the rows of zI-e to the power Ah inverse*B are linearly independent. So this theorem is useful. This result is useful in deciding the controllability property of the discrete system.

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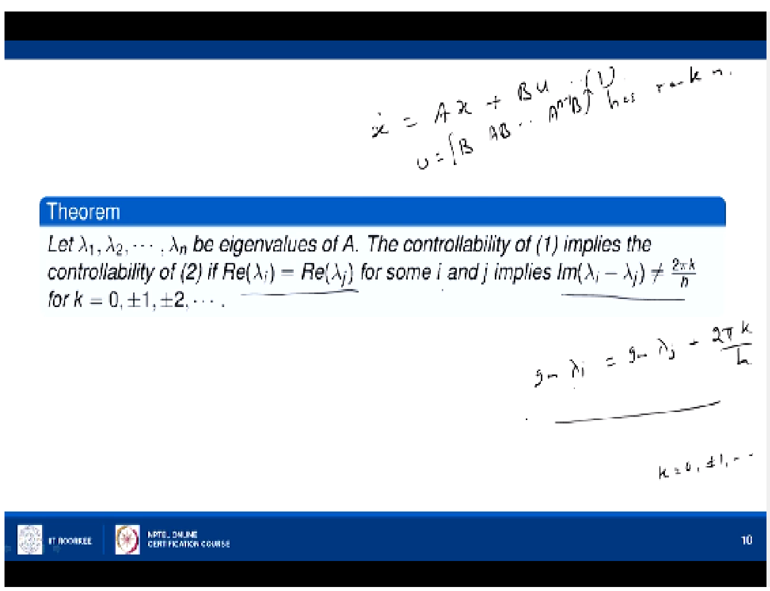

So we will not give the elaborate proof of this expression because it is quite lengthy and we have to consider the Jordan canonical form and so many other lemmas behind this. So what we will see is the result, we have to test the linearly independence of the rows of this matrix which is equivalent to the controllability of the discrete system. So using that previous result, we can come to this particular conclusion.

The theorem gives that if lambda 1 lambda 2 lambda n are the eigenvalues of the matrix A, so the matrix continuous system x dot=Ax+Bu. So in this system, A is the given matrix. If lambda 1 lambda 2 lambda n are the eigenvalues, then the controllability of the system 1 implies the controllability of the system 2 if the real part of lambda i=real part of lambda j for some i and j.

So any 2 eigenvalues, lambda i and lambda j have same real part, then their imaginary part should be like this that is imaginary part of lambda i should not be equal to imaginary part of lambda j+2pi k/h here, okay. So the imaginary part of lambda i, so if it is equal for example, 2pi k/h, then the system, the discrete system is not controllable. Even if the continuous system is controllable, if B AB etc.

A power nB has rank n and if the eigenvalues behave like this, the real parts are same and imaginary part are like this for k=0+-1 etc. So if k=0, that means imaginary part is also the same, real parts are the same, imaginary parts are the same, so the eigenvalues are repeated, lambda i=lambda j. But if they are distinct, lambda i is not equal to lambda j but the real parts are same, the imaginary parts are like this.

For example, for k=1, if you have imaginary part of lambda, i is imaginary part of lambda j+2pi/h where h is the time increment. Then the system, the discrete system will not be controllable even though the continuous system is controllable. So one conclusion is if the eigenvalues are distinct, completely distinct and real, then the system is, the controllability of the continuous system implies the controllability of the discrete system.

And if the real parts are the same and imaginary parts are behaving like this in this expression, then the controllability of the continuous system implies that of the discrete system. So that is the conclusion of this particular theorem. The proof of this theorem is based on the proof of this expression, that is zI-e to the power Ah inverse*B. Now we will see for some example, this particular concept.

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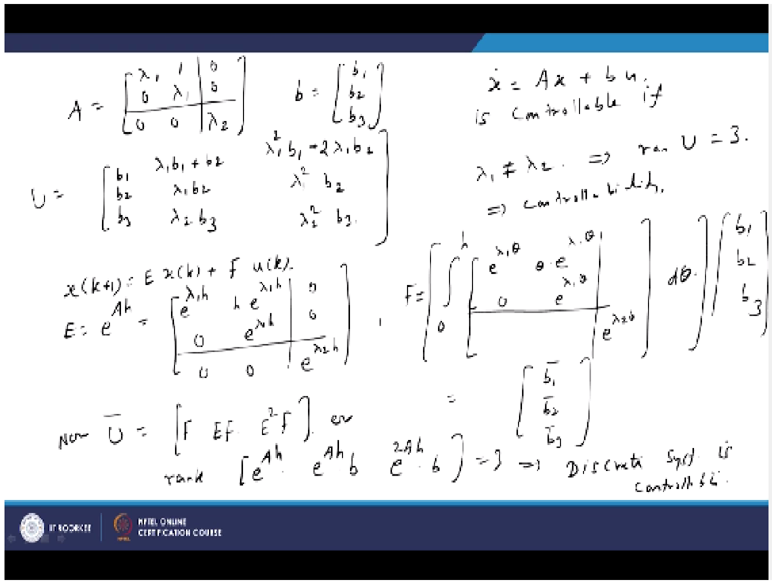

So if you assume that the matrix A is given by this lambda 1 is the eigenvalue and lambda 2, so let us say lambda 1 is not equal to lambda 2 and it is in the Jordan canonical form like this and the matrix b is a column vector b1 b2 b3. So in this case, the system x dot=Ax+Bu is controllable if we have b, b is the column vector, b1, b2, b3. A*b that gives lambda 1b1+b2 and second is lambda 1*b2, then lambda 2*b3.

Third column, third row*the column. Now A square b is the first row*this, we will get lambda 1 square b1+2lambda 1 b2. Then second row*this column will give 0lambda 1 square b2. Third row*this, will give lambda 2 square*b3. So this is the U matrix. So if this, we can observe that here if lambda 1 is not equal to lambda 2, then in all the cases, the system, the rank of the matrix is 3 and the system is controllable.

So this implies rank of U=3 that implies controllability. Now the equivalent discrete system, so that is given by x of k+1=Ex of k+F*U of k. So if we calculate here the matrix E=A*h where h is the time increment, some small real number. So if you calculate exponential for this Jordan canonical form, we will get e to the power lambda 1h, then h*e power lambda 1h o. Then e power lambda 2h and F matrix is given by integral 0 to h of e to the power A theta.

So in the place of h we have theta. So it is e power lambda 1 of theta and theta*e power lambda 1 theta 0 e power lambda 1 theta*d theta. So this whole thing is multiplied by b1 b2 b3. So this matrix is the F matrix here in this problem. So after evaluating, we may get some expression. Let us call it as b1 bar b2 bar b3 bar after calculating this integral. So here we will get the condition that this expression.

Now U bar is nothing but this expression F is in the place of b, the first column. The second column is E*F E square*F here. So we can see here in case of lambda 1=lambda, okay; so now in the previous theorem, we have seen that instead of checking the controllability for E and F matrix, it is sufficient to check the controllability for E matrix and b matrix because M is commutative with the zI-e power Ah inverse matrix.

So we have concluded that, it is enough to consider e power Ah and b matrix for the controllability. So we see here e to the power Ah is given by this and we need not consider this F matrix as product of this and this. It is enough to consider only the b matrix for the controllability. So we can check the rank of the matrix e power Ah e power Ah*b and e power 2Ah*b. That should be equal to 3.

If you are able to get this expression, then we say that this discrete system is controllable. Because if you calculate this F as given here and calculate b1 bar b2 bar b3 bar, that is one way of checking the controllability of the discrete system. But we have earlier shown the equivalence condition that it is enough to check with e power Ah and the b matrix itself the controllability. So we check this one. So if you calculate these 3, e power Ah is given here and b matrix is given, so what we will see here is?

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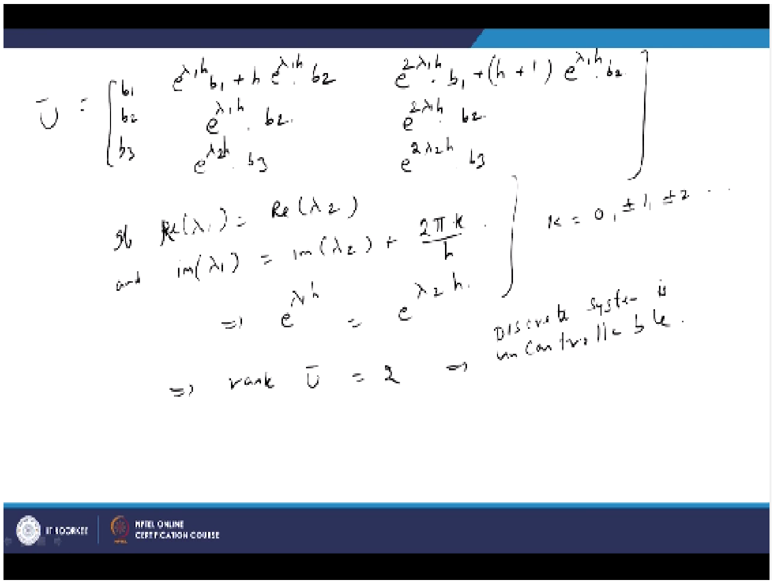

We have b1 b2 b3 and e to the power lambda 1h*b1+h*e power lambda 1h*b2 that is first element, e power lambda 1h*b2 second element and e power lambda 2*b3. The next is e power 2lambda 1h*b1+h+1*e to the power lambda 1h*b2, okay. So now we can see that in this expression, if real part of lambda 1=real part of lambda 2 and imaginary part of lambda 1 is imaginary part of lambda 2+2pi k/h.

So this will imply that e power lambda 1h will be equal to e to the power lambda 2*h that can be easily verified from this expression. So if e power lambda 1h and e power lambda 2h, both are same, so that will imply that the second row and third row will become the same here. So this will imply the rank of this matrix will be 2. So if you call this matrix as U bar, the rank of U bar=2 in this case. So this implies the system is uncontrollable. So this example illustrate that the condition given in the theorem statement, the imaginary part should be behaving like this.

So that implies the uncontrollability of the discrete system provided; if we select the imaginary part of lambda 1 and imaginary part of lambda 2 in this manner for any integer 0+ or -1, etc., any of these integer satisfies this condition, then we get e to the power lambda 1h=e power lambda 2h and substituting the values here, in the U bar matrix, we get the second row and third row to be the same and that implies rank of U bar=2.

So that implies the discrete system is uncontrollable. So we have seen through an example that the controllability of the continuous system need not imply the controllability of the discrete system. So it is essential to select the discrete points suitably so that the property of controllability is maintained between the continuous and the discrete system. So in the next lecture, we will consider the observability property and the stability property of the discrete system and the corresponding continuous control systems. Thank you.