Dynamical Systems and Control
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Lecture – 58
Stability for Discrete Systems

Dear students, welcome to this lecture on the stability of discrete systems, so in this lecture we will see some results on the stability of discrete system which are analogous to the stability of continuous dynamical systems.

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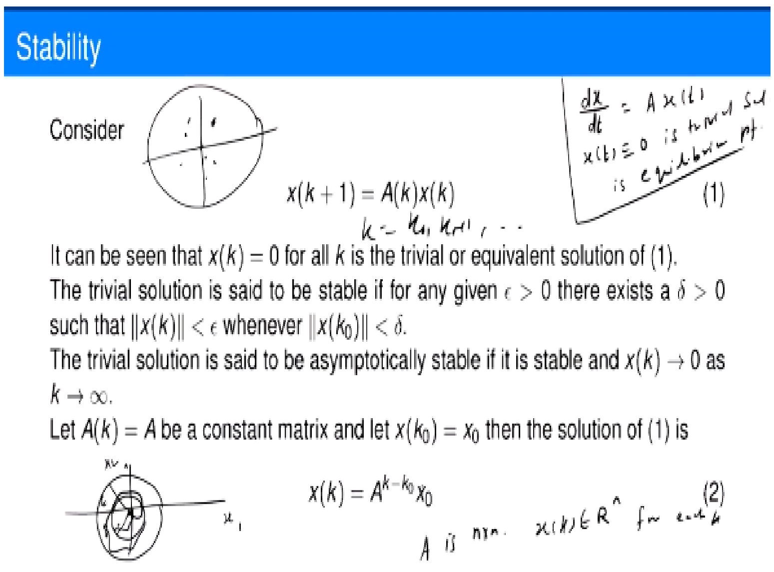

So, earlier we have seen the continuous dynamical system of the form dx/dt = A * x of t, where A is a constant n cross n matrix, so the result, the x of t identically = 0 is the trivial solution of this continuous system and it is also called the equilibrium point; equilibrium point of the dynamical system, so the stability of the system at this equilibrium point was analysed using the eigenvalues of the matrix A.

So, the result was if all the eigenvalues of A has negative real parts then, the system is asymptotically stable and if any of the Eigen value have negative real part, a positive real part then the system will be unstable and if or if the eigenvalues are purely imaginary in some cases, we can see that it will be stable, so there are some special condition on that particular case where the eigenvalues are imaginary.

So, depending on the algebraic and geometric multiplicities, we can decide the stability or instability of such systems, so here those results have been earlier seen while dealing with the continuous dynamical systems. So, analogous to those results we can see some result in the case of the discrete systems. So, let us consider the discrete system x of k +1 = A of k x of k for various time instances we can let say k = k0 k0 +1 etc.

And it can be easily seen, if x of of k = 0 for all k and if we substitute it in the right hand side, we will get x of k + 1 is also = 0, so we can conclude that the 0; x of k is identically = 0 for all k is a trivial solution of this system; system one or it can be called the equilibrium point, the origin is called the equilibrium point of this system. So, here A is a n cross n matrix and x of k belongs to Rn for each value of k is the state variable.

Now, we will state the definition of the stability of the system at this equilibrium point, so here the trivial solution is said to be stable if for any given epsilon that is > 0 there exist a delta positive such that whenever the initial condition x of k0 is < delta, then the solution x of k lies within the epsilon neighbourhood, so here we see that if this is the state space for example, in the case of 2 dimension, let us say x = x1, x2, 2 variables are there.

Then for every given epsilon, 0 is the trivial solution, x1 is 0, x2 is 0 is the trivial solution of the system then, for every given epsilon the radius of the circle is epsilon there exist a delta > 0, we can find a circle of radius delta such that whenever the initial condition x of k0 is inside the delta circle, then the solution x of k for all the values of k, afterwards so that will always lie within the epsilon circle.

When we have take the discrete points because the solution is not a continuous one, x of k0 is x0 is a point and then next point is x of k0 + 1 that will be a discrete point, so if you connect all the discrete points for each value of k, we will get a curve; we will get a discrete set of points all of them will lie within the epsilon circle, so that is the definition of the stability of the system. In addition to that stability, if you also have that x of k tends to 0 as, k tends to infinity.

So as, k becomes larger and larger, the initial point if you start, then for each value it will approach the origin, x of k0 is starting point and x of k0 + 1 k0+2 etc., these point is finally reach the origin, in the limiting case as, k tends to infinity then we say that the system is asymptotically stable, so it is a similar definition that is the analogous to the continuous case. Now, we will see the condition on the stability, condition on the matrix for the stability of this discrete system.

So, let us consider instead of the time varying system A of k, let us consider A of k is a constant matrix A, then we have already seen in the previous lecture, the solution x of k = A to the power k – k0 * x0.

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Let $\lambda_i : i = 1, 2, \cdots, n$ be given eigenvalues of A. If A is diagonalizable then we can find n linearly independent eigenvectors $x_i : i = 1, 2, \cdots, n$ such that

$$Ax_i = \lambda_i x_i, i = 1, 2, \cdots, n.$$

$$\text{and} \quad A^m x_i = \lambda_i^m x_i \tag{3}$$

for any $+ve$ integer m.

As the eigenvalues $\{x_1, x_2, \cdots, x_n\}$ forms a basis of R^n, the initial vector x_0 can be expressed as

$$x_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \tag{4}$$

So, let lambda i; i = 1 to n be the eigenvalues of the matrix A, so if A is diagonalisable, then we can find n linearly independent eigenvectors corresponding to these eigenvalues lambda i, so let us say x1, x2, xn are the n linearly independent eigenvectors which satisfy the equation like this, Axi = lambda i xi for i = 1 to n standard definition of eigenvalues and eigenvectors, then multiplying repeatedly, we can easily see that A power m xi = lambda i to the power m * xi for i = 1, 2 up to n.

So, this is for any positive integer m satisfies and because we have n linearly independent vectors in Rn, it will form a basis in Rn, so any vector can be written as a unique linear combination of the eigenvectors, so initial condition x0 is a vector in Rn, so it can be written as c1x1 + c2x2 etc. cnxn as given in equation 4.

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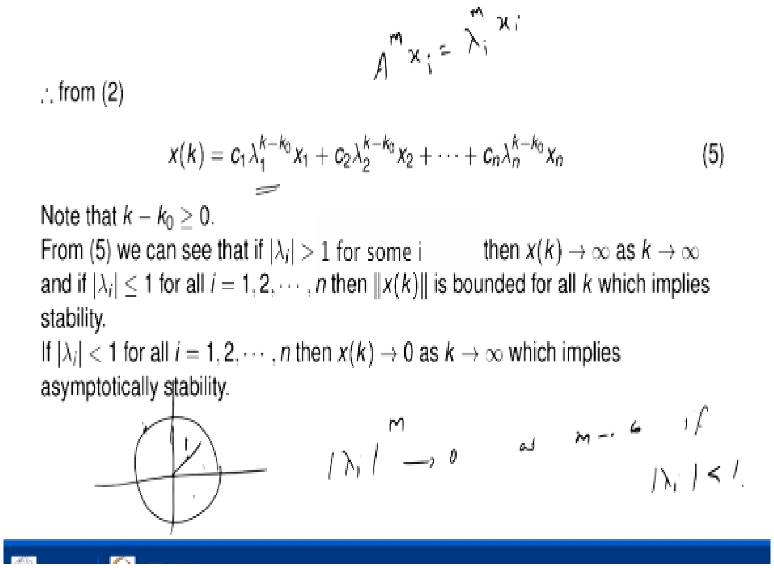

So, substituting that initial condition in the expression 2 that is x of k is A to the power k – k0 * x0, so in the place of x0, if you substitute equation 4, we get the expression like this because A power m xi = lambda i to the power m xi, so we can substitute using this, we get the expression x of k to be like, now we take k larger because we are interested in k tending to infinity, so we take k is > k0 and further it tends to infinity.

And when we have k – k0 always positive and if the eigenvalues are such that modulus of lambda i is > 1, even if one of the eigenvalue have modulus > 1, we can easily see that it will tend to infinity lambda for example, if lambda 1 is > 1, then its positive power will keep increasing and then it will tend to infinity, it implies that x of k will tend to infinity as, k tends to infinity.

So, for the stability condition we need that the modulus of the eigenvalue lambda i all of them has to be <= 1, so if the modulus is < 1, strictly < 1, we can easily see that the modulus of lambda i to the power any positive number, it will tend to 0 as m tends to infinity, if this is < 1, strictly < 1, so we can see that if all the eigenvalues have the modulus strictly < 1 or if they lie within the unit circle.

If we take the unit circle in the complex plane and if all the eigenvalues lie within the unit circle then we say that the system is asymptotically stable and if the modulus of the any of the eigenvalue = 1, we can see that this that particular term is always a constant, if we take modulus here and then if the modulus = 1, a particular term is constant and if the remaining eigenvalues are inside the unit circle, the remaining terms will tend to 0.

So, we can say that the modulus of x of k, it may not converges to 0 but it will remind a bounded value because C1 is; C1, C2, CN are bounded values, so modulus of x of k will be bounded provided the modulus of lambda i or any some of the modulus of the eigenvalues are 1, so in that case we say that it is stable here.

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In case A is not diagonalizable then with suitable modification in the above discussion the stability result can be proved.

So, similarly in the case that A is not diagonalisable we will get the Jordan canonical form, so in that case instead of n linearly independent eigenvectors here, in the diagonalisation case, we will get n linearly independent eigenvectors but in the case of the non- diagonalisable matrix, we will get the linearly independent eigenvectors, the number of linearly independent eigenvectors will be strictly < n.

So, we will be able to find the generalised eigenvectors and then we get a basis consisting of eigenvectors and generalised eigenvectors in the space Rn, then the initial condition can be written as the linear combination of eigenvectors and the generalised eigenvectors and the proof will be similar, once we write the solution x of k in the form of equation 5 using eigenvectors as well as generalised eigenvectors.

Then, we will get the similar result that if all the eigenvalues lie within the unit circle, it will be an asymptotically stable and if any of the eigenvalue lie outside the unit circle, it will be unstable etc. so this result is similar.

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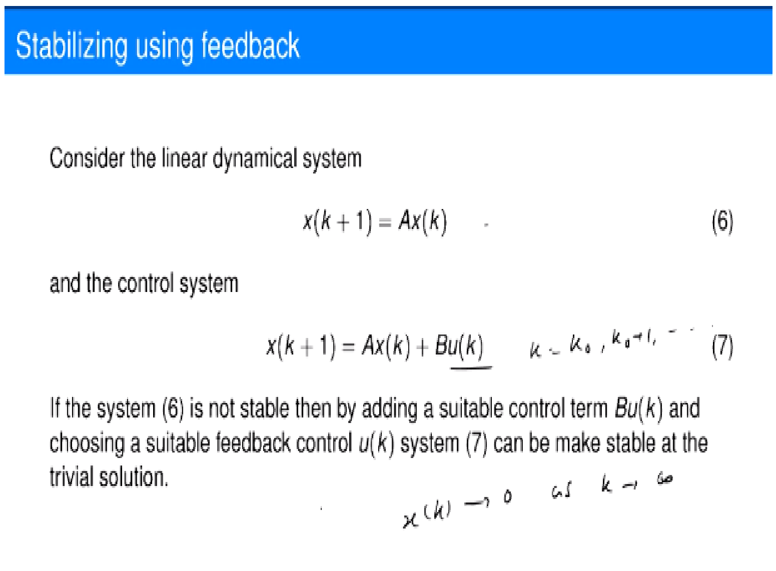

Now, stabilising using feedback, so let us consider the dynamical system given in equation 6 here, if the matrix A is such that the eigenvalue, some of the eigenvalue lie outside the unit circle, then the system is unstable now, in order to make the system stable by applying a control term, so we can consider the control system x of k +1 is A * x of k + B * u of k, where u of k is the control variable for various values of k, k is initial condition is k0 k0 + 1 etc.

Now, in order to make the system say asymptotically stable, so we want that x of k should tend to 0 as, k tends to infinity is the requirement for asymptotically stable system, we have to select the control u of k in a special manner, so that the system becomes stable or asymptotically stable.

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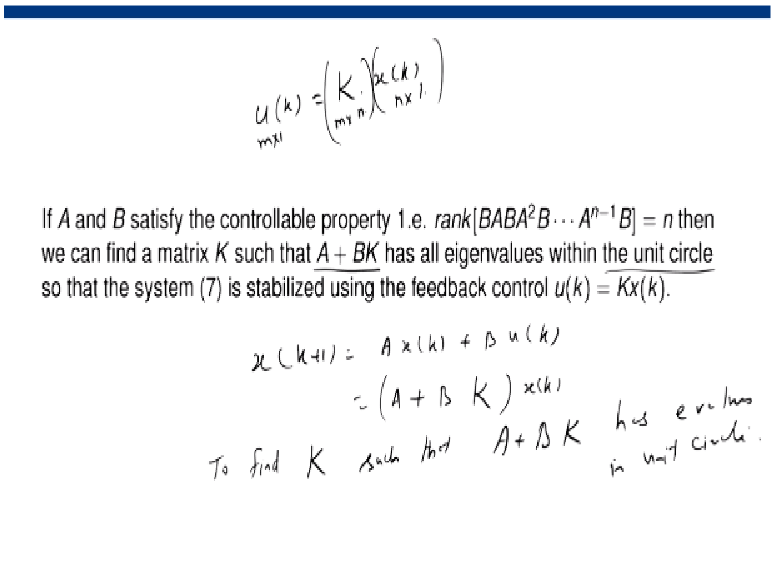

So, this procedure is again similar to that of the continuous case as we have seen in the feedback control lecture that we can find a feedback control u of k such that u of small k, it is some matrix K times x of k, where k; this is a m cross 1 matrix and this is n cross 1 matrix, so we will get m cross n matrix, the k matrix is m cross n and so we have to find a m cross n matrix K such that A+ BK has all the eigenvalues within the unit circle.

So that the resulting system x of k + 1 is; so, x of k + 1 is Ax of k + Bu of k and if you substitute u of k to be K times; capital K times x of k, so we get a simple system like this and if the all the eigenvalues of A+ BK has a modulus < 1 or they lie within the unit circle, then we can say that the system is stabilised, in that case x of k will converge to 0 as, k tends to infinity and the procedure is exactly similar to the case of continuous system which we have seen in the feedback control.

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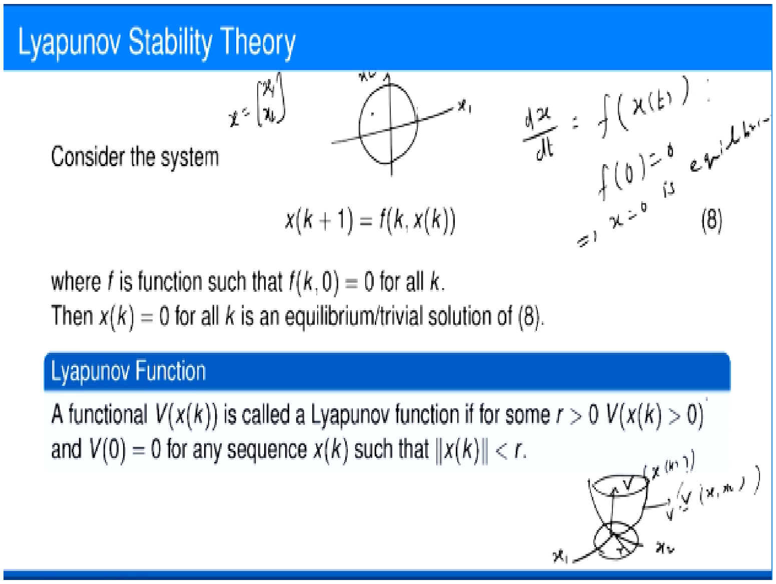

Now, another analogous theory that is we have seen the Lyapunov stability theory for the continuous system and if you have dx/ dt = f of x of t, where f of 0 = 0, f is a non-linear function such that f of 0 = 0, this will imply that x = 0 is equilibrium point of the given continuous system, so to analyse the stability of the system at this equilibrium point, we have seen the Lyapunov theory that is if there exist a Lyapunov function V that is v is a positive definite function that is V of x is strictly positive for all x non-zero and V of 0 = 0.

And the derivative of V with respect to is negative definite, then the system is asymptotically stable, so that was the Lyapunov stability theorem for asymptotically stability and if dV/dt is negative semi-definite, then the system is stable, so this result we have seen in the stability lectures. Now, we will see a similar result for the discrete system, so a function V of x of k is called the Lyapunov function, if for some r > 0, the radius r > 0, V of x of k is > 0.

And V of 0 = 0 for any sequence x of k within this circle of radius r, so let us for simplicity assume that the vector x is in r2, x is x1, x2, x of k is x1 of k, x2 of k, so these are the; this is a state space now, if you take your radius r, a circle of radius r and if x is within this circle x of k, then V of x of k, so we will get the surface, x1, x2 is this and then the value of the V; V of x of k, so if the radius of the circle is r in the x1 x2 plane.

Then corresponding to each value of x1 x2 within this circle, we will have a positive definite function so which represent the surface V of x1 x2, the surface is given by V = V of x1 x2.

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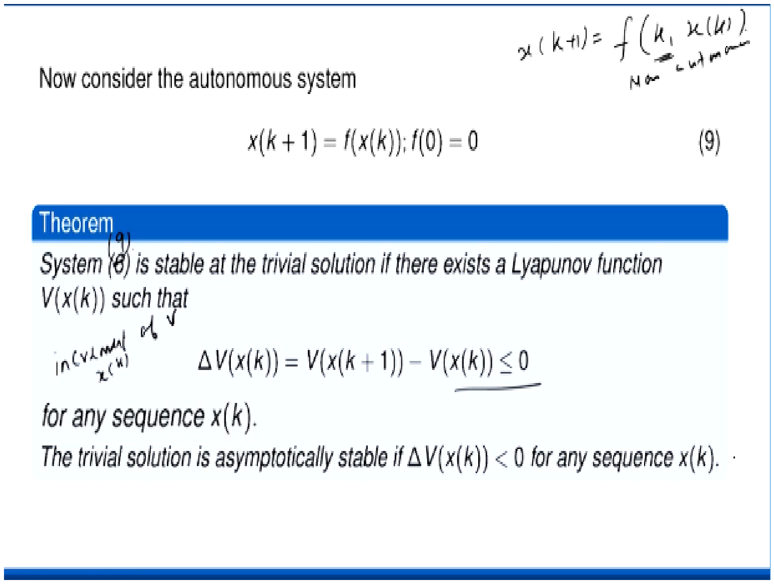

Within the circle of radius r, so if there exist a r satisfying this condition, we say that it is a Lyapunov function and will see the result on the stability of the system, autonomous system, so if we consider x of k + 1 is a non-linear function k and x of k, then it is called time varying system that is non-autonomous system and so, this is non-autonomous and because k is appearing explicitly in this equation.

And if you have the equation like 9, x of k + 1 is f of x of k, where k does not appear explicitly but it can appear in as a part of x of k, then it is called the autonomous system, so the system here 9, actually it is 9, the system 9 is stable if the trivial solution, the system 9 is stable at the trivial solution, if there exist a Lyapunov function as defined earlier and del V of x of k that is the increment; increment of the function V, okay at x of k.

So that is given by V at x of k + 1 – V at k, the increment if it is <= 0, in other words V is a decreasing function with respect to the variable k here, so if this happens then we say that the trivial solution is asymptotically stable and if it is <= 0 then, it is a system is stable, so it is analogous to the continuous case, in the place negative semi-definite, we have that the increment is <= 0.

And in the place of negative definite, negative definiteness of the function V, we have the increment is strictly < 0 for any sequence, so basically both of the conditions are very much similar to that of the continuous case so similarly, the proof is; proof of this Lyapunov theory is exactly similar to that of the continuous case, only thing is in the continuous case, we will have the continuous function x of t.

And in the discrete case, we have the discrete sequence, the points x of k at discrete points, so the result is similar, the proof is also similar here.

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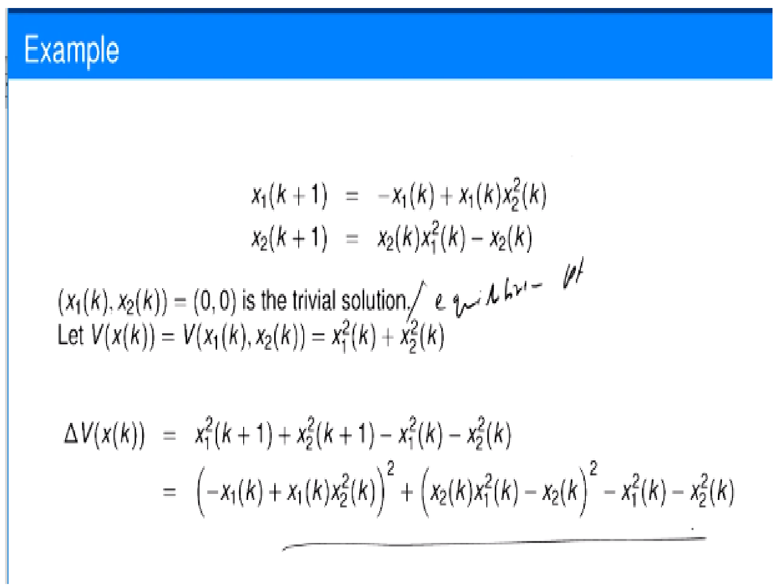

So, now let us see a simple example to illustrate the theory, consider this system x1 k + 1 = - x1 of k + x1 of k x2 square of k and x2 of k + 1 is x2 of k x1 square – x2 of k for various values of k, integer values of k and it can easily seen that when x1 and x2 both are 0, then it satisfies this equation, therefore 0 0 is the trivial solution or the equilibrium point of the system. So, if we consider the Lyapunov function, V of x of k that is V of x1 x2 is x1 square + x2 square for each value of k.

Then it is easy to see that it is positive definite strictly > 0 if x1, x2 is not the origin and at the origin, it is 0 value and the increment del V x of k, if you substitute directly that is V of x of k + 1 is +x1 square k +1 + x2 square k + 1 -V of x of k is x1 square k – x2 square k, so this expression from the given equation if you substitute for x1 k +1 and x2 k + 1, we get this expression.

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$$= -2x_1^2(k)x_2^2(k) + x_1^2(k)x_2^4(k) - 2x_1^2(k)x_2^2(k) + x_1^4(k)x_2^2(k)$$

$$= -x_1^2(k)x_2^2(k)\left(2 - x_2^2(k)\right) - x_1^2(k)x_2^2(k)\left(2 - x_1^2(k)\right)$$

$\therefore \Delta V(x(k)) < 0$ if $x_1^2(k) + x_2^2(k) < 2.$

$\Longrightarrow$ asymptotic stable.

And further by simplifying, we get del V of x1 x2 is strictly < 0, provided these quantities are positive that is x2 square is < 2 and x1 square, if it is < 2, then we will get strictly < 0 sign or in other words, if x1 and x2 lies within the circle of radius 2, then also we can say that the right hand side is strictly < 0, so according to the definition, the value r, we have seen that if there exist a circle of radius r in which this conditions are satisfied.

Then, the system is asymptotically stable, so this illustrate the asymptotically stability of this the Lyapunov theory, okay, so with this we conclude the stability concept for the time invariant system for the discrete dynamical system, okay, thank you.