Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 57
Observability of Discrete Systems

Dear students, welcome to this lecture on the observability of discrete dynamical systems, so in this lecture, we will see the definition of observability and some conditions on the observability of discrete systems, so this is analogous to the continuous systems or the observability and the condition of observability of continuous systems, only thing is the; there will be little changes for; so as to deal with the discreteness of the system, okay. So, the procedure will be almost same like a continuous system with little, there is a small change.

(Refer Slide Time: 01:15)

Consider the system

$$x(k+1) = A(k)x(k) + B(k)u(k) \qquad (1)$$

and the observation

$A(k)$ is $n \times n$
$B(k)$ is $n \times m$.

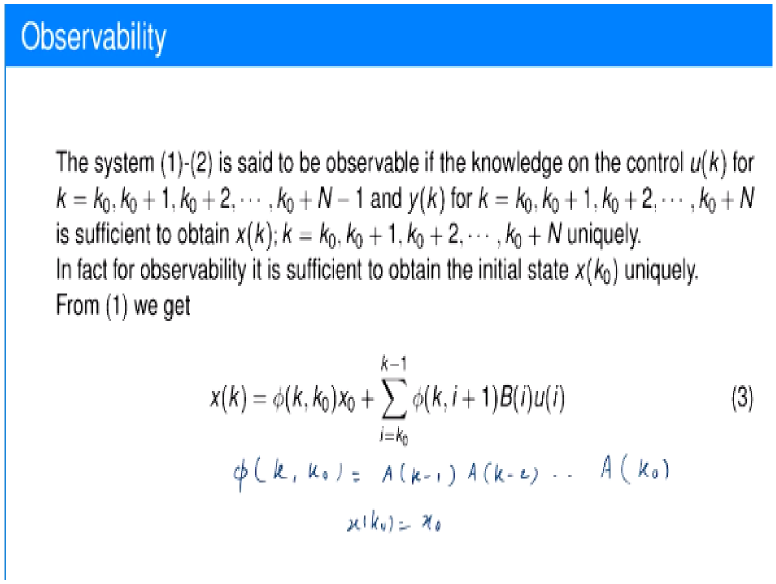
$$y(k) = C(k)x(k) \qquad (2)$$

$x(k) \in R^n$
$u(k) \in R^m$

for $k = k_0, k_0+1, k_0+2, \cdots, k_0+N$.

So, let us consider the discrete control system given by equation 1 that is x of k + 1 = A of k x of k + B + k u of k, so A is n cross n matrix, A of k is n cross n and B of k is n cross m matrix and x of k is the state variable and u of k is the control variable in the spaces, x of k belongs to Rn and u of k belongs to Rm for each value of k, k is ranging from k0, k0 + 1 etc. k0 + n, so these are the time instances in which we considered the dynamical system.

(Refer Slide Time: 02:22)



And the observation of this system is y of k = C k x of k, so now the definition of observability is similar to that of the continuous system except the discreteness of the system, so the system given by 1 and 2 is said to be observable, if the knowledge of the control u of k for these instances, k0, k0 + 1 up to k0 + N - 1 and the knowledge of the observation y of k, which is for k = k0 up to the last stage k0 + N.

So, this knowledge is sufficient to obtain the state of the system x of k for all values of k uniquely, so instead of saying that x of k should be known uniquely for all instances, we can also say that the observability means if the initial state x of k0 is obtained uniquely from the knowledge of u of k and y of k, then also the system is observable because we can see that the solution of the equation 1 is written as x of k = phi of k k0 * x0 + the summation as given here in the equation 3.

Where we have seen in the previous lecture that for phi of k k0 in the general case of the time varying system, it is A of k -1 A of k -2 etc. the products A of k0, so with this notation, we write the solution 3 of the control system 1, so if you know x0 that is x of k0 = x0 uniquely, then from equation 3, we can get x of k for all values of k that is k ranging from k0 to k0 + N, so that is the observability of the system, the definition of observability of the system.

(Refer Slide Time: 04:41)

From (2)

$$y(k) = C(k)\phi(k,k_0)x_0 + \sum_{i=k_0}^{k-1} C(k)\phi(k,i+1)B(i)u(i)$$

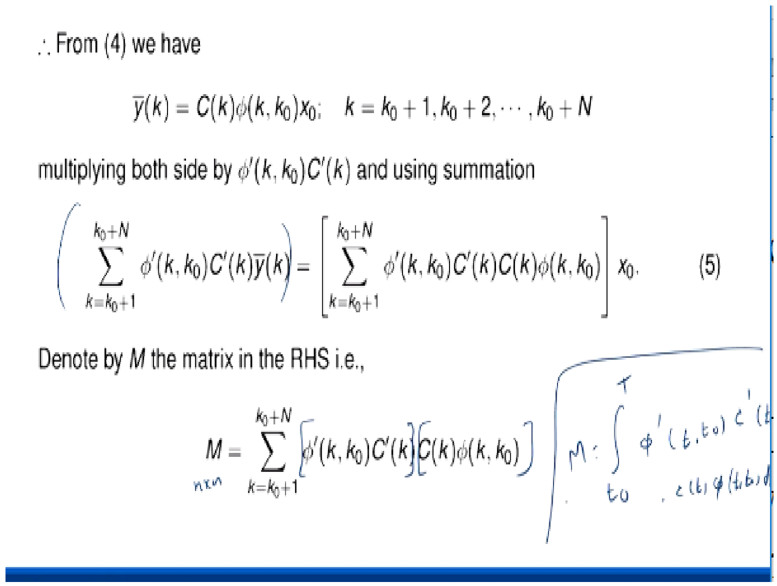
$$\Rightarrow \quad y(k) - \sum_{i=k_0}^{k-1} C(k)\phi(k,i+1)B(i)u(i) = C(k)\phi(k,k_0)x_0 \qquad (4)$$

for $k = k_0 + 1, k_0 + 2, \cdots, k_0 + N$.
As the LFS quantities are known we denote it by $\overline{y}(k)$.

Now, we will see from equation 2, y of k = C of k * x of k, where x of k is given by this expression, so the if you multiply equation 3 both sides with the C of k, we get to the equation 4 that is the observation in terms of this summation and taking this summation term to the left hand side, we will get y of k - summation = C of k phi of k k0 * x0, so this is for k = k0 +1 etc. k0 + N.

Now, we can write; give the notation y bar of k for the left hand side of the equation 4 because all these quantities are known to us, y of k is the observation, u of k is the control and for all the instances, the matrices B, C all are known to us, so we can say that the left hand side is known and we denoted by y bar of k for these instances k = k0 +1 etc. k0 + N.

(Refer Slide Time: 06:02)

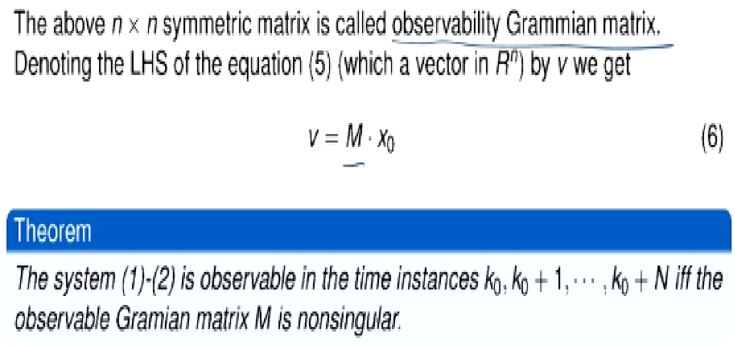


So, from that equation, we get y bar of k = C of k phi of k k0 x0 from the previous slide, so multiply both sides by this matrix, it is phi transpose k k0 * C transpose k and then using the summation from k0 + 1 to k0 + N, we get to this equation 5 here, okay, multiplying by this matrix and then using the summation both sides, we get this expression here, x0 is a constant vector, so we can write it outside the summation as given in the equation 5.

Now, let us call this square bracket summation as the matrix M, so let M be summation k ranging from k0 + 1 to k0 + N of phi dashed C dashed C * phi of k k0 this matrix, so we can note that the size of the matrix, it is a square matrix, it is a n cross n matrix because A is a n cross n matrix, so the state transition matrix is n cross n and the product will give finally n cross n matrix and it is a symmetric matrix.

Because we can see that phi dashed C dashed is 1 matrix and its transpose, so they are multiplied and we are taking the summation over a symmetric matrix, so we get M as a symmetric matrix here.

(Refer Slide Time: 07:58)



The above n cross n symmetric matrix is called the observability Grammian matrix, the matrix M and we can write from the equation, if you denote the equation 5, the left hand side if you denote it by vector v, then we can write the equation 5 as M * x0 I = the vector v, so this can be done for all systems with irrespective of what the matrix A, B, C etc. of the dynamical system but the system will be observable in the time instances k0, k0 +1 etc. if and only if the observability Grammian matrix M is non-singular.

So, for observability, we need the condition that the matrix M is non-singular, so this is exactly similar to the continuous case, the continuous case what we have earlier seen is; if the initial time is t0 on the final time is capital T, then the observability Grammian for the continuous case is given by phi dashed t t0 * C dashed of t * C of t * phi of t t0 dt, so this matrix is M matrix for observability Grammian matrix for the continuous case.

So, if M is non-singular, then the system is observable for the continuous case now, for the discrete case we have a similar result.

(Refer Slide Time: 09:53)

Proof:

Sufficiency: Sufficiency is clear from (6) since M is invertible then x_0 is obtained uniquely as

$$x_0 = M^{-1}v$$

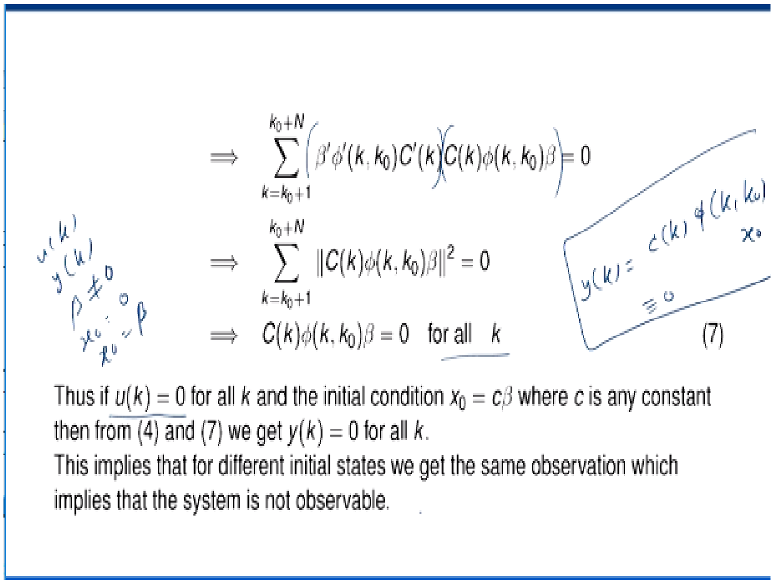
Then from eqn (3) $x(k)$ is obtained for all k uniquely.

Necessity: If the system is observable then M is nonsingular. If M is singular then $\exists \beta \in R^n$ such that $\beta \neq 0$ and $M\beta = 0$. Hence $\beta' M \beta = 0$

And we can prove it in a similar fashion, so now let us consider; see the sufficient see of the condition, if M is non-singular, it is invertible, so directly we can see from equation 6, we will get x0 uniquely, the initial condition x0 is obtained uniquely by taking the inverse, M inverse v is x0, so according to the definition of observability if the initial condition is obtained uniquely, then we call the system observable.

So, sufficient condition is proved directly like this one, now the necessary condition; if the system is observable, then we have to show that the matrix M is non-singular, so let us assume that M is singular, so in that case there will be a vector beta in the space Rn and beta is non-zero such that M * beta = 0 because M is singular, so then we will get beta transpose M * beta is also = 0 from that expression.

(Refer Slide Time: 11:14)



So that implies if you substitute the value of the matrix M, we will get summation k = k0 + 1 to k0 + N, the definition of M as we have seen here, this expression, so we multiply the left with beta dashed and in the right with beta, so we get this expression, summation k0 + 1 to k0 + N beta dashed phi dash C dash C phi * beta = 0. Now, we observe that this 3 terms, beta dashed phi dashed C dashed and C phi beta, so these are transposed to each other.

So, it means that this vector, it is a row vector, it is a column vector and the product will give the norm of this vector, so norm square of C of k phi of k k0 * beta norm square summation = 0 and inside the summation, all the terms are positive therefore, each term separately should be = 0, so we get C of k phi of k k0 * the vector B beta = 0 for all values of k from k0 k0 +1 etc. up to the last stage.

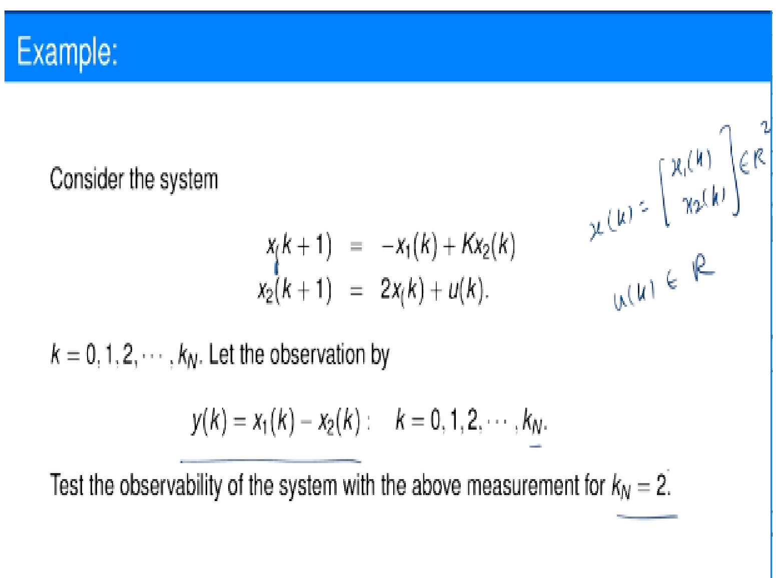
So, now the system observability definition is using the knowledge of the control u of k and the observation y of k, if you are able to get the initial condition uniquely, then the system is observable, so let us assume that the control u of k is 0 for all k, so for using the observability definition let us say u of k is 0 for all k and the initial condition x0 is C times beta, where C is any constant.

Because we know that beta is non-zero according to the assumption in the here, we assume that M is singular therefore, there exist a non-zero beta, so beta is given to us and we assume that initial condition x0 is C times beta for some value of C, then from this last equation 7, we will get that C of k phi of k k0 * C times beta is also = 0 for all value of k, so this implies that if x0 is 0 that is if x0 is 0, then C of k phi of k k0 * 0 is 0.

And for nonzero value of beta that is also 0, so for various initial condition, we get the same observation this y of k = C of k phi of k k0 * x0 because u is 0, we get the observation y to be like this and for let us say, 2 different initial condition, 0 and beta we get the same value y of; this is identically = 0 for all values of k, so this is a contradiction because the initial condition should be obtained uniquely for a given observation, so that is violated here.

So, it means that the beta cannot be non- zero, it has to be 0, if the system is observable, so based on that we say that the matrix M has to be non-singular otherwise, we will get a contradiction, so this proves the result that the necessary and sufficient condition for the control; observability of the system 1 and 2. So, now we will see a simple example to illustrate the observability.

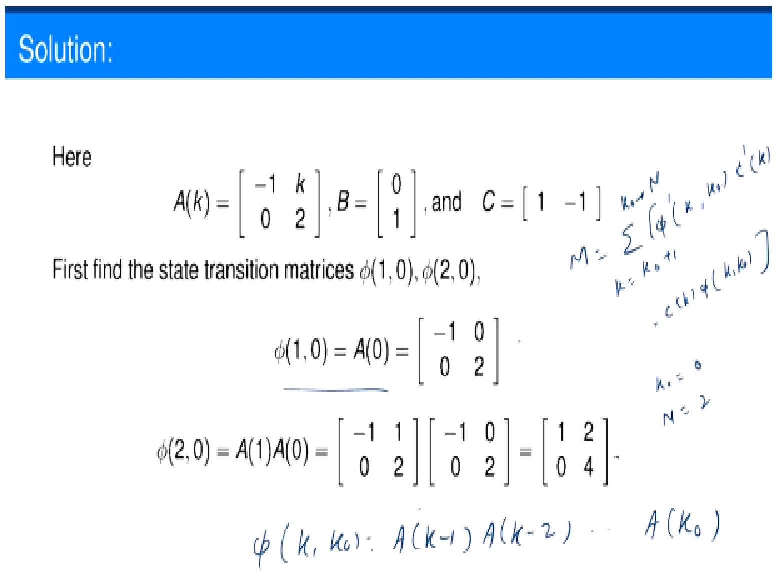
(Refer Slide Time: 15:38)



So, consider the system, x1 of k + 1 = - x1 of k + k times x2 of k, here x1, and x2 of k + 1 = 2 times x of k + u of k, here u of k is the control variable and x1, x2 are the state variable, x of k is x1 of k x2 of k belongs to R2; R2 is a state space and u of k is a single function, so it belongs to R for each value of k, so the control space is R, state spaces R2 here and the observation y of k is given by x1 of k – x2 of k, this gives the observation for the values k = 0, 1, 2, 3 up to the last instance k suffix N.

Now, to test the control observability of the system, we can utilise the observability Grammian matrix, so let us take the simple case where the final instant is 2, initial time is 0 and final time is 2, so for this we test the observability of the system.

(Refer Slide Time: 17:11)



So, here in this case, A of k is given by -1 k 0 2 and B matrix is 0 1, C matrix is 1 -1 and so the state transition matrices phi 1, 0 and phi 2, 0 to be obtained because for the M matrix, we need the formula that k ranges from k0, so if you see the M matrix here, k varies from k0 + 1 to k0 + N; k0 + N of phi k, k0 dashed and C dashed of k * C of k phi of k k0, so this we have to calculate, so in this case k0 is 0 and N is 2.

So, we have to find from 1 to 2 here, that is k starting from 1 and 2, there are only 2 terms we have to calculate, so we have to obtain time this terms, here when we be put k = 1 and k0 = 0, we will get first term is phi of 1, 0 that is phi dashed that is A of 0 that is phi of 1 0, according to the definition of phi k k0 is A of k -1 A of k -2 etc. A of k0, so in this case, k is 1 and k0 is 0, so we will get only A of 0 that is only term for this one.

Similarly, phi of 2, 0 according to this is A of 1 * A of 0, so from here when we put k = 0 and 1, we will get this 2 matrices as the state transition matrix part of that.

(Refer Slide Time: 19:33)

Then the observability Gramian matrix is

$$M = \sum_{k=1}^{2} \phi'(k,0)C'C\phi(k,0).$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \quad \text{nonsingular.}$$

$\Longrightarrow$ the system is observable.

Then M is calculated by this expression and by substituting the appropriate matrices, we get the M to be non-singular because the determinant is non- zero, so the system is observable.

(Refer Slide Time: 19:51)

Let

$$\phi(k,k_0) = A(k-1)A(k-2)\cdots A(k_0) \quad (8)$$

$$\text{then} \quad \phi(k_0,k) = A^{-1}(k_0)A^{-1}(k_0+1)\cdots A^{-1}(k-1)$$

$$\text{and} \quad (\phi(k_0,k))' = \left(A^{-1}(k-1)\right)'\left(A^{-1}(k-2)\right)'\cdots\left(A^{-1}(k_0)\right)' \quad (9)$$

$$= \psi(k,k_0)(\text{say}) \quad (10)$$

Now consider the systems

$$x(k+1) = A(k)x(k) \quad (11)$$

$$\text{and} \quad y(k+1) = \left(A^{-1}(k)\right)' y(k);\ k = k_0, k_0+1, \cdots k_0+N \quad (12)$$

$\psi(k,k_0)$ is State Transition matrix of (12)

$\psi(k,k_0) \to (\phi(k_0,k))'$

Now, let us consider the following, the state transition matrix is given by phi of k k0 is A k -1 etc. A k0 according to the definition, then the transpose or when we write phi of k0 k in the previous lecture, we have seen the definition of phi of k0 k is the inverse of phi of k k0, so it is given by A inverse k0 A inverse k0 +1 etc. A inverse k -1, the inverse of the previous step, so and if you take the transpose of this, we get A inverse k -1 and its transpose.

Then A inverse k -2 transpose etc. A inverse k0 transpose, so this we denote it by psi of k k0, the notation for the right hand side of 9 is denoted by psi of k k 0, so now let us consider the 2 systems; x of k +1 = A of x x of k is one system and another is y of k + 1 is A inverse of k transpose * y of k, k is ranging from k0, k 0+1 etc. and maybe the last is k0 + N, so we see that the state transition matrix of equation 11 is given by equation 8 here.

Because of the matrix A, similarly, the state transition matrix of equation 12 because of this matrix, the first term should start with k - A inverse k -1 transpose and etc. the last term should be A inverse * k is A inverse of k0 transpose, so the psi k k0 is the state transition matrix of the equation 12 here and from this equation, we can easily see that psi of k k0 is nothing but phi of k0 k transpose that is seen from equation 9 and 10.

So, we can see that the relation between the state transition matrix of 11 and 12 are given by this expression, psi of k k0 is phi of k0 k transpose.
(Refer Slide Time: 23:15)

It can be easily seen from the from the definition of the state transition matrix that $\phi(k, k_0)$ and $\psi(k, k_0)$ are state transition matrices of (11) and (12) respectively. Moreover

$$\psi(k, k_0) = \phi'(k_0, k) \qquad (13)$$

So, this we can utilise in the following way.
(Refer Slide Time: 23:17)

Now consider the system

$$\left.\begin{array}{rcl} x(k+1) & = & A(k)x(k) + B(k)u(k) \\ \text{with observation} \quad y(k) & = & C(k)x(k) \end{array}\right\} \qquad (14)$$

and

$$\left.\begin{array}{rcl} \bar{x}(k+1) & = & \left(A^{-1}(k)\right)' \bar{x}(k) + \bar{C}(k)v(k) \\ \bar{y}(k) & = & \bar{B}(k)\bar{x}(k) \end{array}\right\} \qquad (15)$$

where $\bar{C}(k) = C'(k)$ and $\bar{B}(k) = B'(k+1)$.

The system (14) and (15) are called dual systems to (15) and (14) respectively.

Controllability of (14) $\Longleftrightarrow$ observability of (15)

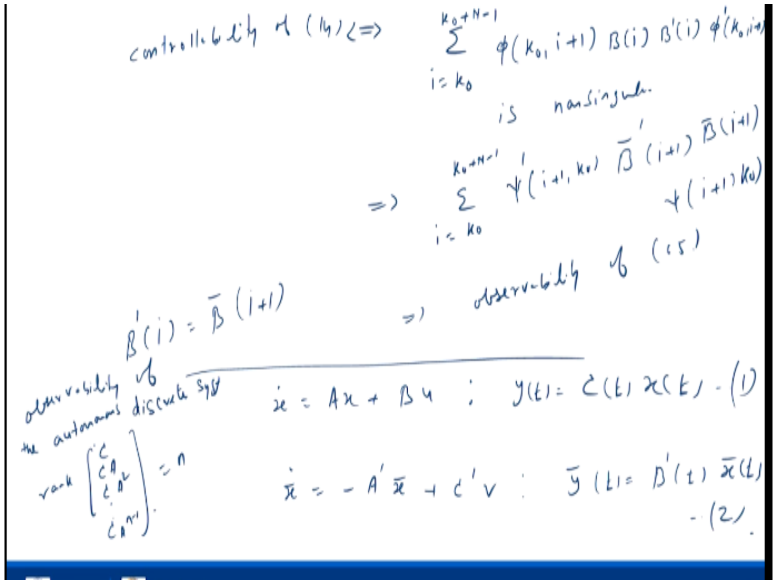
Observability of (14) $\Longleftrightarrow$ Controllability of (15)

Now, let us consider the 2 systems, x of k +1 is A of k x of k + B of k * u of k and the observation y of k is Ck xk that the usual system which we have consider earlier now, we will consider another system x bar of k +1 = A inverse k transpose * x bar of k + C bar of k * v of k and the observation for this system is y bar of k = B bar of k * x bar of k, so here this x and x bar directly they do not have any relation.

So, it is a simply a notation for a different system and similarly, this y and y bar, they do not have direct relation but these are 2 different systems observations, y is observation of system 14 and y bar is observation of system 15. Now, let us select this C bar to be = C dashed of in the equation 15, we select the matrix C bar as C bar of k = C dashed k +1 and B bar of k is B dashed of k – 1.

Because we know the matrices A, B and C so, in terms of A, B, C we are writing the matrices this B bar, C bar etc. so now we can easily verify that the controllability condition of system 14 is same as the observability condition for equation 15.

(Refer Slide Time: 25:23)



Because we can see that the controllability of equation 14, so that implies and implied by the matrix summation i = k0 to k0 + N – 1 of phi of k0 i +1 B of i * B dashed of i * phi of k0 i +1 phi dashed, so this expression it is non-singular, this is called the controllability Grammian matrix and if this matrix is non-singular, this system is controllable and if this is controllable, this is non-singular.

So, we have seen it in the previous lecture, so this condition is same as so, now this implies that this = summation i = k0 to k0 + N - 1 and this gives psi of i + 1 k0 dashed, so earlier we have seen that phi and psi have the relation like this, phi, psi of AB is phi dashed of BA, so we get this relation and B of i so, according to this one, we have B dash of i is B bar of i+1, so we get here B of i is nothing but the B bar dashed of i + 1.

And B dashed of i is B bar of i + 1 and phi dashed k0 i + 1 that is nothing but psi of i +1, k0, so this expression we can easily see that this is nothing but the observability of the equation 15, by seeing the observability Grammian matrix, we can see that it is the same, so we get the result, the controllability of 14 implies observability of 15 similarly, observability of 14 implies the controllability of 15 and vice versa, both sides.

So, this 2 systems are called the dual systems, the 14 and 15 are called the dual system and this result is called the duality theorem, okay this one is; so this is similar to the case of the continuous time systems. So, in the continuous case we have seen that if we take x dot = Ax + Bu and y of t = C of t * x of t, this is the control system and the observation, then the system x bar dot = - A transpose x bar + C dashed of some control v.

And y bar of t = D transpose of t * x bar of t, so the first system and second system are dual to each other, the controllability of one, this equation 1 implies the observability of this equation 2 etc. vice versa, so this duality theorem is similar to the duality theorem for the discrete system here and so, using this we can derive similar conditions as in the case of the continuous systems. So, using this result we can also observe that the observability of the autonomous system; autonomous discrete system.

If A, B, C all of them are constant matrices, so we can conclude that the rank of C, CA, C square A, C, sorry, okay, so the condition, the rank C, CA, CA square C A n – 1, if it = N, then the system is observable or if the system is observable, then the rank of this matrix has to be N, so this can be proved using the duality theorem for the discrete system, so because we know that the controllability of the equation 15 is given by the rank of B, AB, A power n – 1 B = n.

When AB are constant matrix, this is a condition, so using the controllability and observability relation of the dual system, we can prove that the observability of equation 4 will give the condition for the controllability of equation 15 and similarly, the controllability of equation 15 will give the observability condition for the equation 14, so that will turn out to be this condition, so with this we close the lecture on the observability and the relation between observability and controllability of the dual systems, so thank you.