Dynamical Systems and Control
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Lecture - 56
Controllability of Discrete Systems

Hello viewers. Welcome to this lecture on the controllability of discrete time varying systems.

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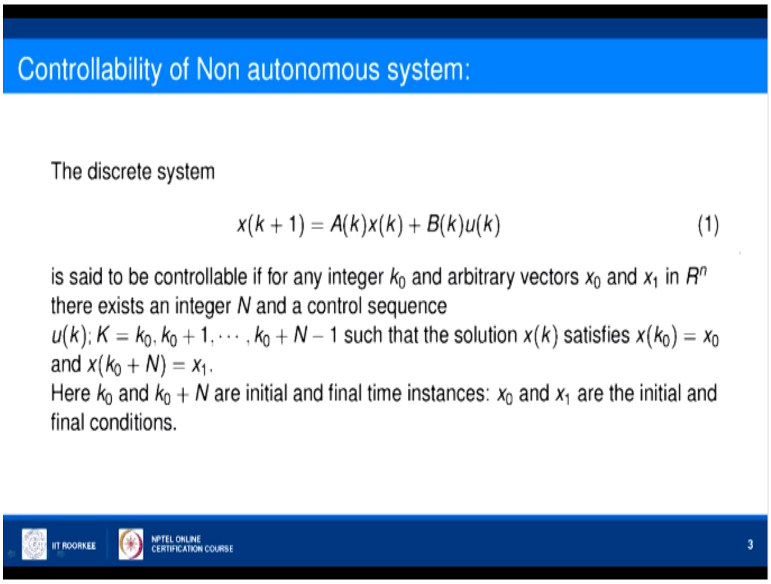

So let us consider the time varying system x of k+1=A of k x of k+B of k u of k as given in equation 1. So the system is said to be controllable if for any initial integer k0 and arbitrary vectors x0 and x1 in Rn the state space there exists an integer N that is final instant and a control sequence u of k0, u of k0+1, etc u of k0+n-1 such that the solution of 1 satisfies x of k0 is the initial condition given x0 and x of k0+N is x1 the final condition.

So if this is the case then we say that the system is controllable in the time instances k0, k0+1 up to k0+n.

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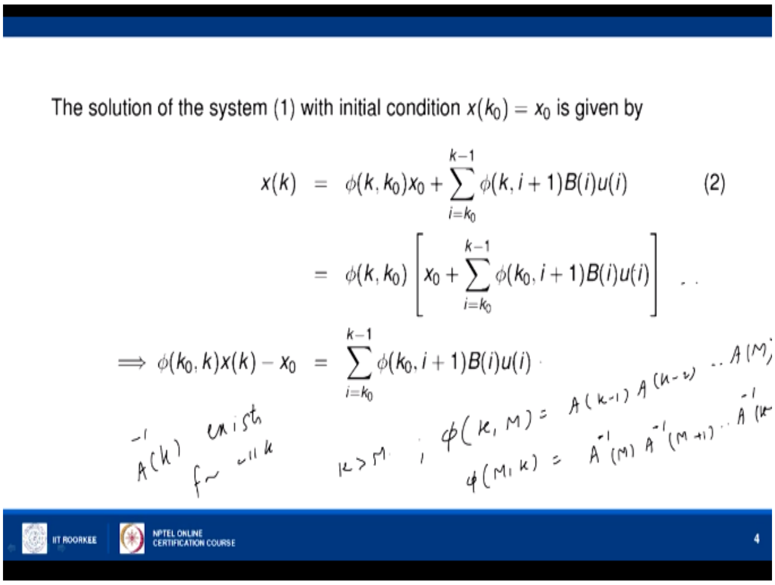

So now we will see that this controllability of the time invariant or time variant system were analogous to the continuous systems. So in the previous lecture, we have seen the Kalman condition for the controllability. So similarly in this lecture, we will see analogous result using the controllability Gramian matrix. So before that let us write the solution of the equation 1 in the following way.

X of k is phi of k, k0 x0+summation i ranging from k0 to k-1 phi of k, i+1*B of i u of i. So we recall that the definition of the state transition matrix is phi of k, any integer M where k is strictly>M if we have then this is=A of k-1, A of k-2 up to A of M. So this is the definition of the state transition matrix and phi of M, k it means the inverse of this one that is A inverse of M, A inverse of M+1, etc A inverse k-1 is the notation for phi of M, k.

So using this state transition matrix and their properties, we write the solution of the system 1 as given in the equation 2 and then it can be using the property phi of k, i+1 can be written as phi of k, k0*phi of k0, i+1. So we can write equation 2 in the form of the next equation as given here. Now because phi is invertible, we are assuming that all these matrices A of k are invertible, this inverse always exists for all k.

So with this assumption we can take the inverse of the matrix phi k, k0 which is=phi of k0, k. So taking phi k, k0 to the left hand side we get phi k0, k*x of k and taking this x0 also to the left side we get –x0=this summation i is=k0 to k-1 this expression.

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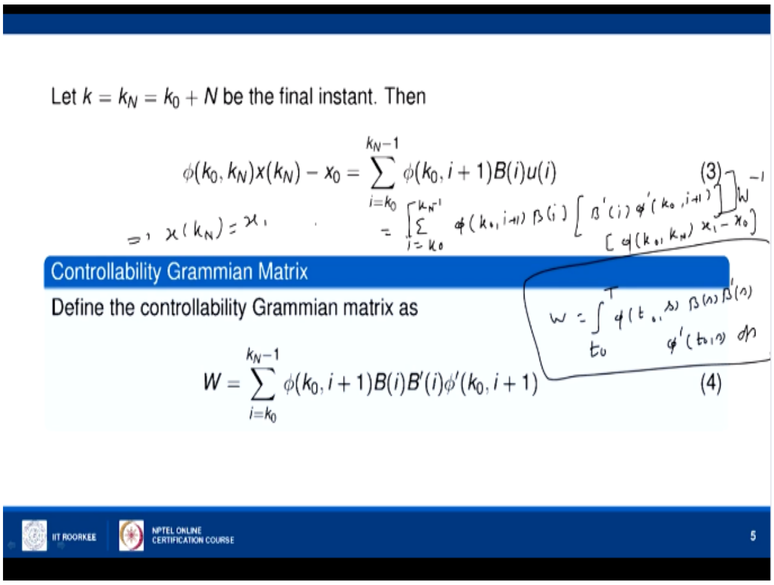

So now we get the equation 3 by substituting the final time instant that is when k is=k0+N which is denoted by k suffix N we get from the place of k the equation 2 we put k is=k suffix N here in this expression, we get this phi of k0, k suffix N*x of k suffix N-x0 is given by the right hand side in 3. So now we define the controllability Gramian matrix. So if you recall for the continuous case, we say W is=t0 to the initial time to final time of phi t0, s B of s B dashed of s phi dashed t0, s ds.

So this says the definition of the controllability Gramian matrix in the continuous case where dash denotes the transpose. So the same similar definition replacing integral to summation and using the time instances like this, so W is defined as summation i is=k0 and k N-1 of phi of k0, i+1 B of i B transpose i phi transpose k0, i+1. So this is the controllability Gramian matrix.

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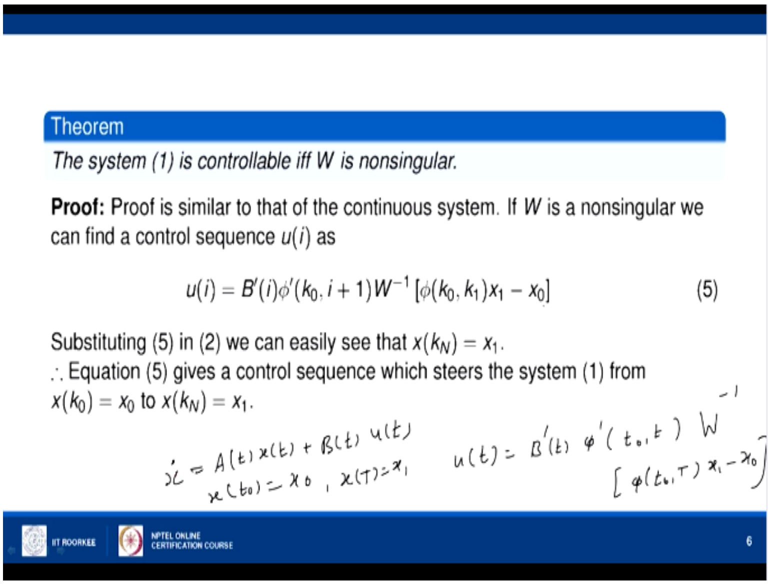

So the theorem analogous to the continuous case is, the system 1 is controllable if and only if the controllability Gramian matrix W is nonsingular. So the proof is similar again to the continuous case. Earlier, we defined the case, in the continuous case we define u of t to be=B dashed of t phi dashed t0, t*the Gramian matrix for continuous case, its inverse*phi of t0 and final time*x1-x0 the initial condition of the system.

So this is the one such control for the continuous case which steers the system is x dot=A of t x of t+B of t u of t. For this, we define the control if you substitute this x of t0 is x0 and x of t is x1. So u of t steers the system from x0 to x1. So similar to this for the discrete system, we define the control given in the equation 5 which will steer the system from the initial condition x0 to final condition x1.

So this proof is direct because if you substitute this control in the equation 5 sorry in the equation 3 u of i we substitute as given in equation 5 here. So if you see this, so if you substitute this will be equal to summation i is=k0 kN-1 phi k0, i+1 B of i. Now in the place of u of i, we substitute the definition B dashed of i phi dashed of k0, i+1*W inverse*phi of k0, kN*x1-x0*this expression okay.

So if you substitute this it can be easily seen that the bracket from here to this position, this indicates or by the definition of W, we can say that this is W and then W inverse becomes identity and the remaining portion is phi of k0, kN x1-x0 which is same as the left hand side. So this implies that x at kN in that place you have x1, so the final condition is given by x suffix 1.

So it means that the control given in equation 5 it steers the system from initial condition x0 to final condition x1. So that is the proof for the sufficient condition. Sufficient condition means it is the theorem is if W is nonsingular then the system is controllable. If W is nonsingular, we can use W inverse in this expression. So by substituting this control, we get the controllability of the system.

Similarly, the converse part, if the system is controllable then W has to be nonsingular. So the proof is again similar to the continuous case exactly same wherever the integral is there, they are replaced by summation, so that can be easily verified. So it is necessary and sufficient that the controllability Gramian matrix is nonsingular for the controllability.

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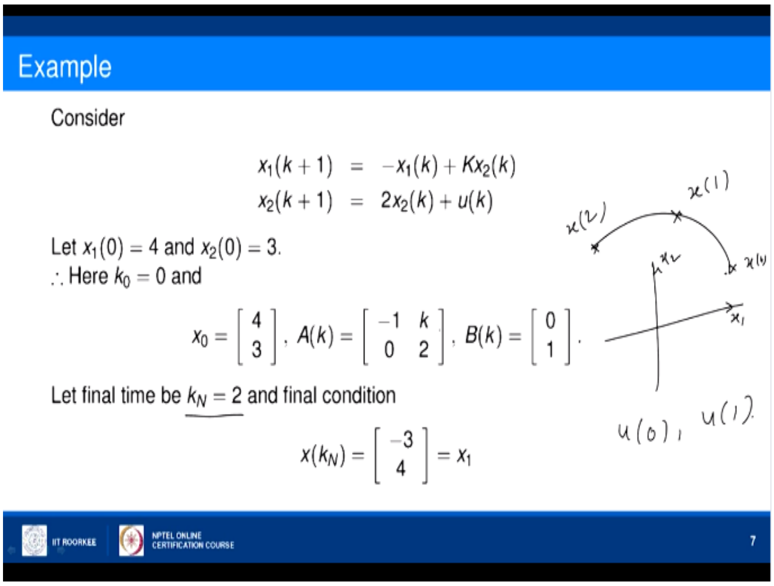

So now we consider this following example, x1 of k+1 is -x1+k times x2 of k and x2 of k+1 is 2 x2+u of k, u of k is the control here. So in this case, the matrix A is time varying which is given by -1 k and 0 2 and the matrix B is given by 0 and 1, the control is u of k. Initial condition at time t=0 is 4 and 3, this vector 4 3 is there, initial time is k0 is=0. Now if you see final if you take final time to be k suffix N which is=2, then we will try and the final condition is given by -3, 4 for example.

So we want to steer the system from the initial condition 4, 3 to -3 and 4. So we want a control which will steer the system from this to this. There may be several controls which will perform this solution but we can find one such control here. So this is the x1 axis and x2 axis but we note that it may not be a continuous solution, it will be a discrete solution and it will find the solution in 2 steps.

Because at k0=0 the point is here, at kN=2 it should be at the end position, so in between you have only one point. So we need one point as the solution that is x at 1, this is x at 0 is the vector 4 3 and x at 2 is the vector -3 4 and we need this thing and we also want to find the control u of 0 and u of 1, so that this control steers the system to this final position.

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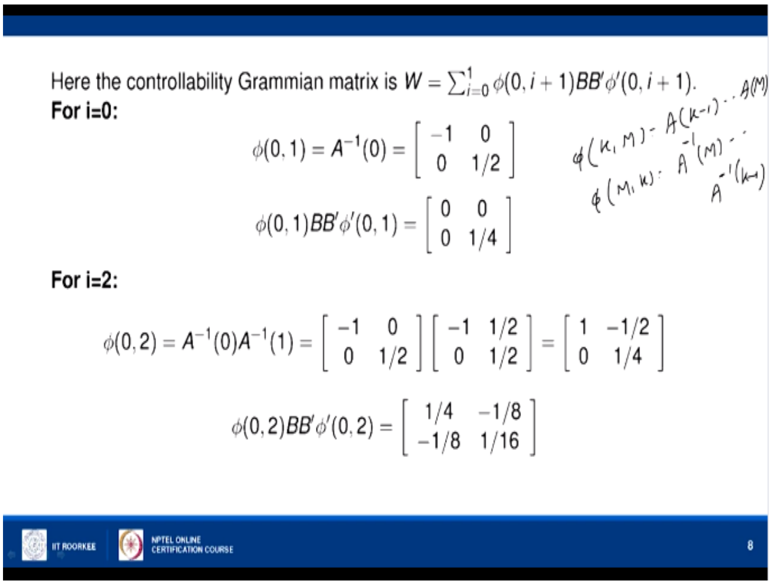

So if you directly apply the formula, the controllability Gramian is summation i ranges from 0 to k N-1 so that is 2-1, it starts from 0 and end with 1 phi of k0, i+1 BB dashed phi dashed 0, i+1 that is k0 is 0. Now we can calculate by recalling phi of k, M it means A of k-1 up to A of M. So when we take and similarly phi of M, k we have seen that it is A inverse of M, etc up to A inverse of k-1.

So this is phi of 0, 1 means it is A inverse of 0 and A of 0 is given here, A of 0 is -1 0 0 2, so we can find the inverse of that and A of 1 is -1 1 0 2. So substituting those values similarly phi of 0, 2 is A inverse 0, A inverse 1 that gives this one and so for calculating W we need the summation of these two expressions phi of 0, 1 and BB dashed phi dashed 0, 1+phi of 0, 2 BB dashed phi dashed 0, 2.

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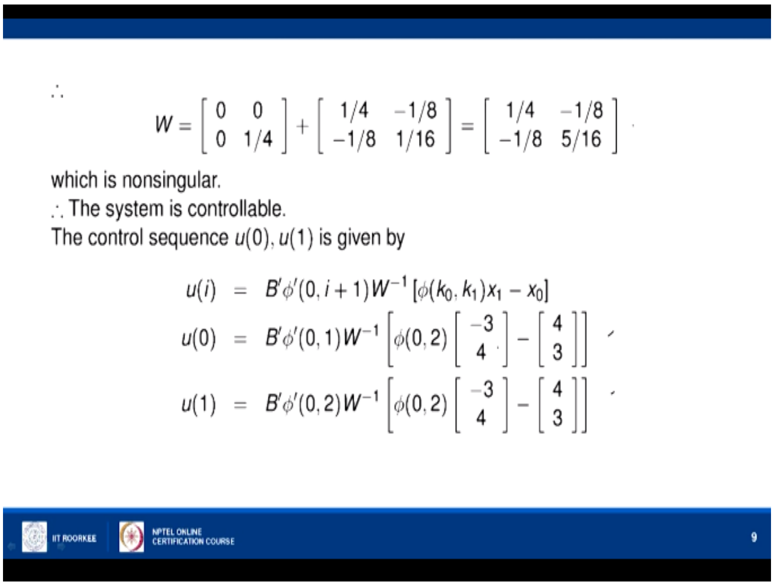

So summing these two we get the controllability Gramian matrix given by this expression and can be easily verified that it is nonsingular. So according to the theorem, the system is controllable. So the control variable which we want to find is given by this expression. We have seen the formula in 5 is one control which will steer the system from initial x0 to the final condition x1.

So applying this formula, we get the variables u of 0 is given by this expression and u of 1 is given by this. So it will steer the system from initial condition 4 3 to final condition -3 4. So this illustrates the method of finding a particular control for steering the system. So with this, I conclude this lecture. Thank you.