Dynamical Systems and Control
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Lecture - 55
Optimal Control for Discrete Systems - II

Dear students. Welcome to this lecture on optimal control for discrete systems.

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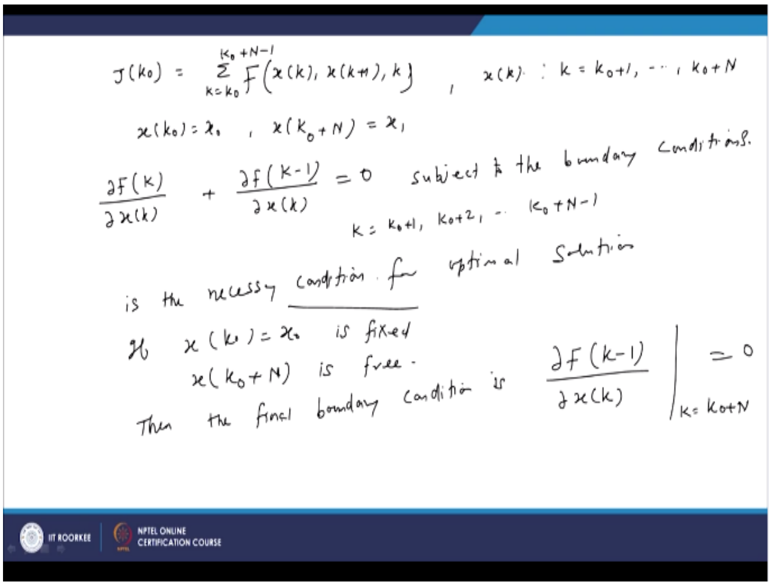

So in the previous lecture, if you recall that we have studied the necessary condition for finding the optimal control of discrete performance index which is given by a functional J of k0 summation k is=k0, k0+N-1 of a function of x of k, x of k+1 and k. So we want to minimize this functional J such that the sequence x of k, k is=k0+1, etc k0+N. So this finite sequence minimizes this expression.

So if you are given the initial value of x of k as x0 and the final position x of k0+N as x1 at initial instant k0 and the final instant k0+N if the value of the vector function x is given as x0 and x1, then how to minimize the functional J of k0 given in this expression so where F is a suitable smooth function. So we have seen that the necessary condition for solving this problem is the following.

The del F of k/del x of k+del f k-1/del x of k. Here F of k means this one F of x k+1 and k and F of k-1 it denotes wherever k is there in this expression we will put k-1 that is F of x of k-1, x of k, k-1 that expression is the second one. So this is=0 and the boundary conditions are given subject to the condition which are already given at k0 and k0+N the conditions are given, so this is solved for various values of k, k is=k0+1 and k0+2 etc k0+N-1.

So if you substitute k value all these values we will get a set of equations algebraic equation and we can solve it using these two conditions. So this gives the necessary condition for optimal solution for this problem. It is also called the Euler's equation. Now we will see if the boundary is also free if x of k0=x0 is fixed as in the previous case and the final value x of k0+N is free which is not specified.

In that case, we have seen that the condition is then the final boundary condition, it is del F k-1/del x of k evaluated at k is=k0+N should be=0. So this derivation we have seen in the last lecture for the final boundary condition and the equation is the same. The equation with initial boundary condition is already given, final boundary condition is given by this expression so we can solve the system of equation and get the optimal solution.

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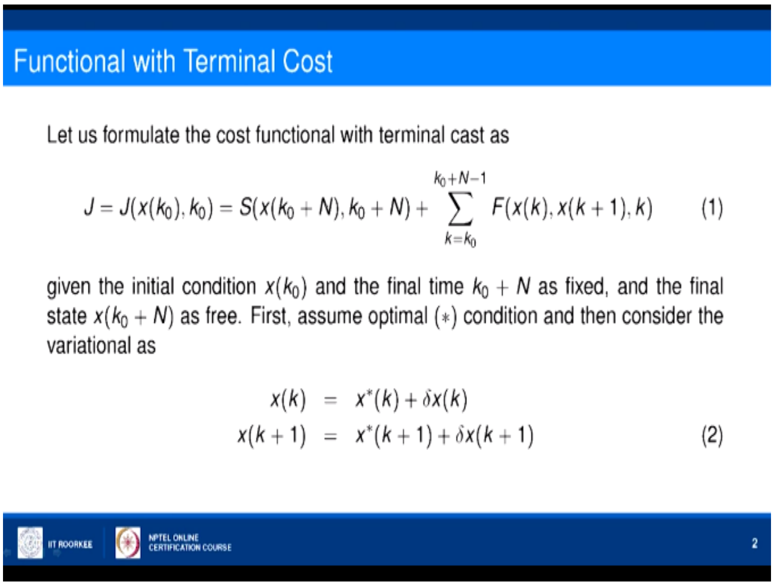

Now we will see that functional with terminal cast. So if the expression instead of the functional J only with the summation if you have an additional function, function of the final position. So if the final position is fixed, there is, it does not have any meaning because if x of k0+N is a fixed value, then the function S of that expression will also be a fixed one. So minimizing J without the first expression S or with the first expression, both will give the same optimal solution.

But in case the x of k0+N is a free boundary condition. So then the minimization of this expression has a meaning. So in this case, if you follow the similar procedure as we did in the previous lecture that is the optimal solution for this is x star of k and the increment in x star of k is x of k that is x star of k+a variation in the function x of k and when we put k=k+1 the next position x of k+1 is the increment from x star of k+1 which is given by this.

And when we substitute this to x of k, x of k+1 in the expression of J and find the first variation of J.

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Then, the corresponding functionals J and J^{δ} become

$$J = S(x^*(k_0+N), k_0+N) \sum_{k=k_0}^{k_0+N-1} F(x^*(k), x^*(k+1), k) \qquad (3)$$

$$\text{and } J^{\delta} = S(x^*(k_0+N) + \delta x(k_0+N), k_0+N) + \sum_{k=k_0}^{k_0+N-1} F(x^*(k) + \delta x(k), x^*(k+1) + \delta x(k+1), k). \qquad (4)$$

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So when we substitute x of k that is the first one when we substitute x of k+1 that is second one and if you take the difference between these two, we will get the variation in J.

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Following the same procedure as given previously for a functional without terminal cost, we get the first variation as

$$\delta J = \sum_{k=k_0}^{k_0+N-1} \left[\frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k)} + \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \right]' \delta x(k) + \left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k) \right] \Bigg|_{k=k_0}^{k=k_0+N} + \frac{\partial S(x^*(k_0+N), k_0+N)}{\partial x^*(k_0+N)} \delta x(k_0+N). \qquad (5)$$

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And using the Taylor series and neglecting the higher order terms, we get this expression. So this can be easily seen from the previous lecture in the similar manner. So we get this expression for the first variation and equating the first variation to 0 and observing that the variation del xk is arbitrary, so we have to get this bracket should be=0 and the boundary conditions are given by this expression.

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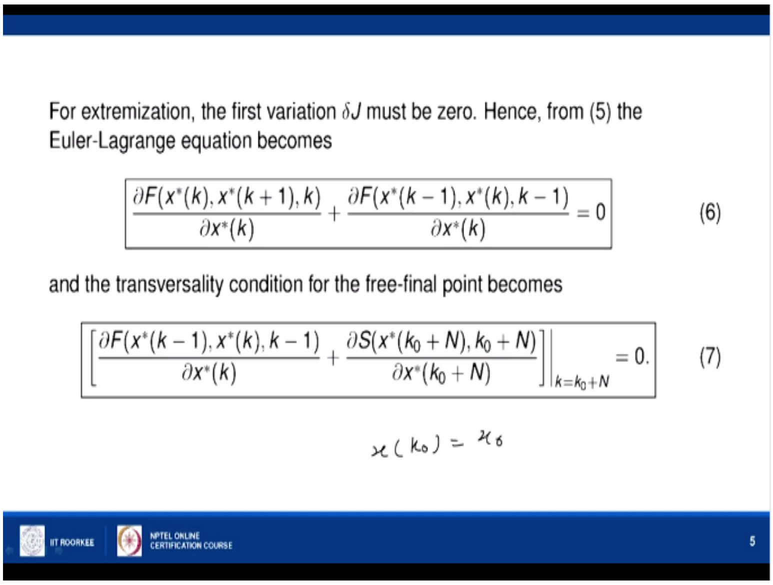

So equation 6 is nothing but the square bracket within the summation sign that should be=0 because the variation is arbitrary and then this 2 expression should be=0. Now because at k-k0, the x of k0 is given so the variation at the initial position is 0 and x of k0+N is free, so the variation at k0+N is arbitrary. Therefore, the expression will give del F of k-1/del x of k at k0+N+this expression should be=0.

So we get the boundary condition to be like this at the point at the final instant k0+N, the condition is this. So we have to solve 6 and 7 together along with the initial condition x of k0=x0, so we will get the solution of this thing optimal control provided it exist because these conditions are only necessary conditions and if the optimal control exist for this problem it can be solved through these equations. For example, we will consider the following.

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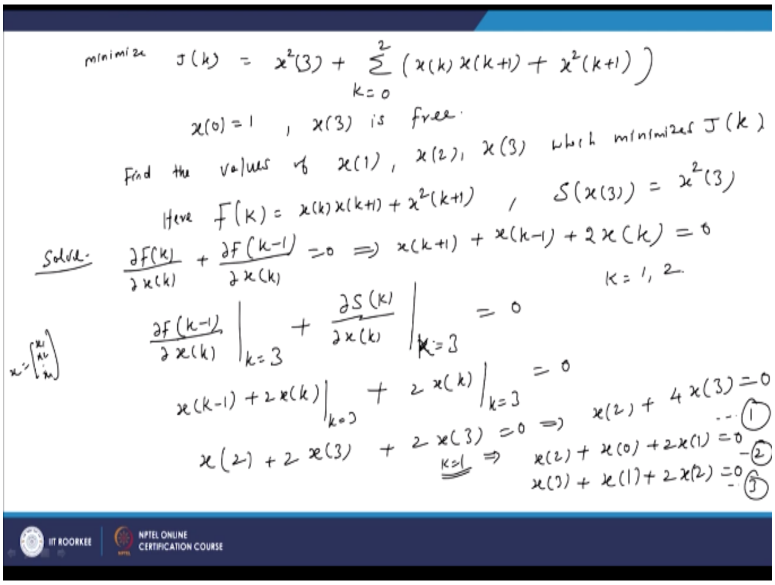

So example so we will say minimize J of k which is given by x square 3+summation k varies from 0 to 2 of x of k*x of k+1+x square of k+1. So here x of 0 is 1 and x of 3 is not specified. It is a free expression, free value here. So we have to find the sequence, find the values, the finite sequence. We want to find, x of 0 is already given, we want to find x of 1, x of 2 and x of 3 which minimizes the expression J of k whatever is given in the summation.

So for this we can adopt the procedure. The function capital F in the previous slide is here capital F of k is this expression. It is x of k, x of k+1+x square of k+1 and S function S of x of 3, it is=x square of 3. So that if you compare the expression in the previous J, we want to minimize J in this, so S is a function of the final position and final time. So in that particular example, it is x square of 3 where 3 is k0+N.

So in that problem k0 is 0 and N is capital N is 3, so we have the function S to be x square of 3. So now if you apply the condition, two conditions, so we have to solve the problem del F k/del x of k+del F k-1/del x of k=0. So this is the necessary condition. So del F/del x of k that gives from this expression, it is x of k+1 and if you put k=k-1 in this we get x of k-1, x of k and here x square of k.

So when we differentiate with respect to x of k, we will get + this will be x of k-1 and here + 2 times x of k from when we differentiate this. So this should be=0 for the values k is=1, 2. So we will get the expression and the boundary condition here is del F k-1/del x of k that is at k is=3 final condition+del S/del x of k that is also at k is=3, this should be=0. So that is this boundary condition.

Del F when you put k-1 in the expression/del x of k+del S/del x at the final position that should be=0. So if you take this expression F of k-1 when we differentiate with respect to xk, we will get x of k-1 first and here +2 times x of k at k=3+del S/del xk that is 2 times x of k at k is=3 that is=0. So this condition will give when we put k=3 we get x of 2+2 times x of 3 and here also we will get 2 times x of 3 that is=0.

So this implies x of 2+4 times x of 3=0. So this is one equation we get, other equations are from here. When we put k is=1 in this expression, we will get x of 2 the first one, first k=1 this will imply x of 2+x of 0+2 times x of 1 that is=0. This is one equation. When we put k=2 here x of 3+x of 1+2 times x of 2=0. This is equation 3. So if you solve this, now the unknowns are we want to find x1, x2, x3 and we have the 3 equations with 3 unknowns x2, x3, etc.

But x of 0 is known to us, x of 0 is 1 here, so here the second equation we get x of 2+2 times x of 3=-1 in that case. So we have 3 equations with 3 unknowns. By solving, we will get the sequence. So similarly, we can solve for more general expression of the summation. Here we note that this function x of k, it is a real-valued function but it can be a vector-valued function also. The x of 1, x of 2, all these functions may be a vector.

In that case, the derivatives will be with respect to the components. So if x is=so if we have the component x is x1, x2, xn the vector, in that case the equation will be in the form, this equations we will have if x is a vector del F of k/del x suffix i where i is=1, 2, 3, up to N. So in this wherever x is there in the denominator, we have to write the x suffix i here and then we will get so many number of equations and it has to be solved in that corresponding manner.

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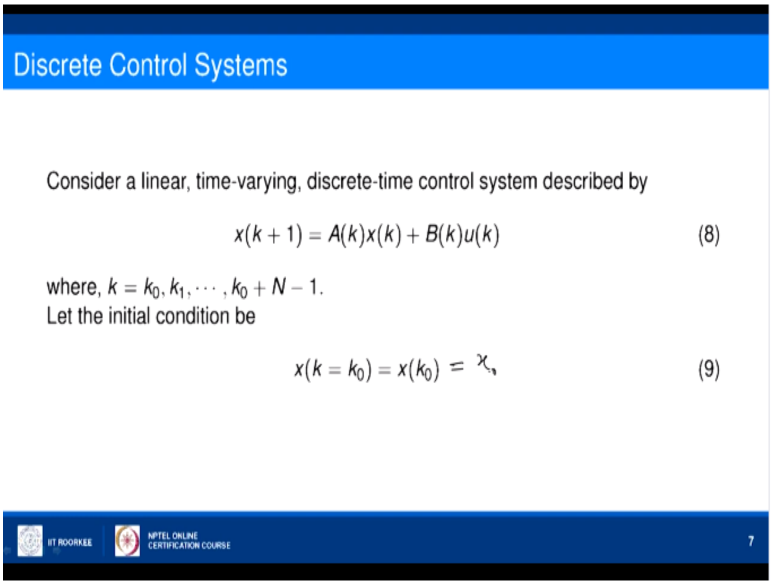

So now we will consider the optimal control problem. Earlier, we have seen only the optimizing a cast function J but we want to optimize a cast function J under the constraint. So if we have the constraint as the control system, discrete control system like this x of k+1 is A of k*x of k+B of k*u of k where u is the control function and x of k is the state of the system and the time instants are given as k is=k0, k0+1, etc k0+N-1.

So these are the time instant at which we evaluate this state of the system x of k and the initial condition is given as x at k0=x suffix 0, we can say like this.

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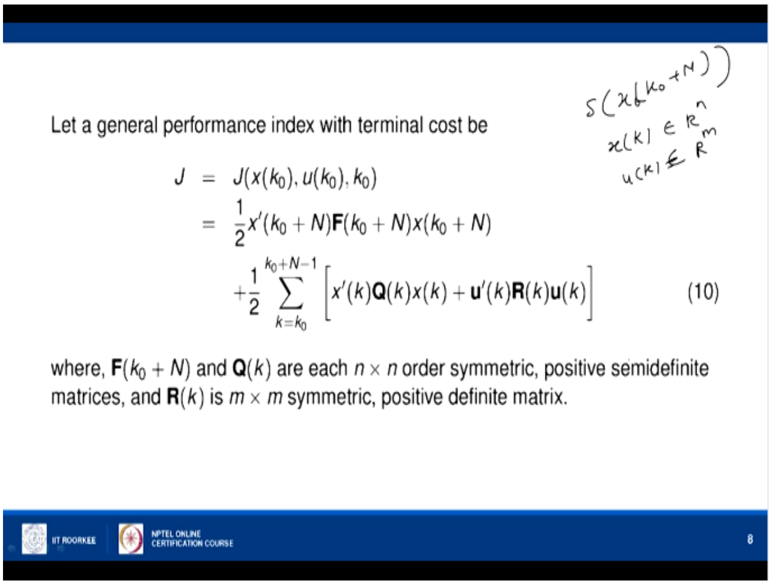

So with this constraint if you want to minimize the expression J, the performance index J which is given by this expression, so what we observe here is earlier we have written a general function S x of k0+N a function of the final position. So say instead of a general function here we are considering a linear expression of the function S. So S is given by x transpose at the final time*F of x0+N*x of sorry F of k0+N*x of k0+N, so k0+N is the final instant.

So here F is a n x n matrix and x is the column vector and x dashed is the row vector. So when we multiply all this three we will get a scalar expression at the final instant k0+N this one+1/2 times summation k varies from k0 to k0+N-1 of this expression. Here also Q is a n x n matrix and x is the vector in Rn and u is a vector in Rn. We have already introduced that x of k is a vector in Rn, it has N components.

And the control u of k it belongs to Rm, so the matrix R here is m x m matrix and we assume that it is positive definite matrix and Q is a positive semi definite matrix, F is also a positive semi definite matrix, symmetric matrix. So these conditions are required in some practical situations, practical problems. Otherwise, we can replace them with a general function also as we have seen in the earlier expression.

Here we are seeing that there is no restriction on the function S and F, only thing is they have to be differentiable functions and we solve the necessary condition this Euler-Lagrange condition, so same thing applies to these equations also. Here it is a specific expression, we are taking the particular expression for the linear discrete control systems and we can solve in the similar manner.

So S function is replaced by the first expression given in this equation 10 and the capital F function in the previous problem is replaced by the linear expressions here, x dashed Qx+u dashed Ru is a quadratic expression it is called. So for example if we replace Q with identity matrix, this is nothing but the norm of the vector x square and similarly R is replaced with identity we will get this as u dashed u is nothing but the norm of u square u of k square.

So it is a quadratic expression. So generally it is called a quadratic optimization problem for the discrete system.

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The methodology for linear quadratic optimal control problem.

Augmented Performance Index: The augmented cost functional using Lagrange multiplier $\lambda(k+1)$ is

$$J_a = \frac{1}{2}x'(k_0+N)\mathbf{F}(k_0+N)x(k_0+N) + \frac{1}{2}\sum_{k=k_0}^{k_0+N-1}\left[x'(k)\mathbf{Q}(k)x(k) + \mathbf{u}'(k)\mathbf{R}(k)\mathbf{u}(k)\right] + \lambda(k+1)\left[A(k)x(k) + B(k)u(k) - x(k+1)\right] \quad (11)$$

Minimization of the augmented cost functional (11) is the same as that of the original cost functional (10), since $J = J_a$.

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So if we adopt the same procedure because only the functions, expressions are written as a quadratic and some linear expression, other things are same. We can adopt the same procedure, only difference here is instead of just finding the optimum value of J, we also have the constraint given by the control system. So when there is a constraint, we adopt the Lagrange's multiplier method.

So along with the J function, we add plus a Lagrange's multiplier lambda at k+1*the constraint that is x of k+1=A*x+B*u because that is 0 we are writing in any order, we are writing it as A*x+B*u-x of k+1. So when the optimal function x of k is minimizing the J as well as satisfying the constraint then the final term this last bracket is anyway going to be 0 because it will satisfy the constraint.

So the optimum value of Ja, the augmented function is same as the optimum value of the performance index J without this last term. So it is first we will optimize Ja and whatever solution we obtain that is the solution for the original problem of optimizing J along with this constraint.

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Lagrangian:

$$\mathcal{L}(x(k), u(k), x(k+1), \lambda(k+1))$$

$$= \frac{1}{2}x'(k)\mathbf{Q}(k)x(k) + \frac{1}{2}u'(k)\mathbf{R}(k)u(k) + \lambda'(k+1)\Big[A(k)x(k) + B(k)u(k) - x(k+1)\Big]. \quad (12)$$

So the Lagrangian is defined as this expression. The Lagrangian is the expression given here. Except the first term, this term and the next term the second and third term constitute the Lagrangian of the problem.

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Euler-Lagrange Equations: We now apply the Euler-Lagrange equation (6) to this new function $\mathcal{L}$ with respect to the variable $x(k)$, $u(k)$, and $\lambda(k+1)$. Thus, we get

$$\frac{\partial \mathcal{L}(x^*(k), x^*(k+1), u^*(k), \lambda^*(k+1))}{\partial x^*(k)} + \frac{\partial \mathcal{L}(x^*(k-1), x^*(k), u^*(k-1), \lambda^*(k))}{\partial x^*(k)} = 0 \quad (13)$$

$$\frac{\partial \mathcal{L}(x^*(k), x^*(k+1), u^*(k), \lambda^*(k+1))}{\partial u^*(k)} + \frac{\partial \mathcal{L}(x^*(k-1), x^*(k), u^*(k-1), \lambda^*(k))}{\partial u^*(k)} = 0 \quad (14)$$

And the necessary condition for the optimum value is del L at k/del xk+del L at k-1/del xk=0. It is exactly similar to what we have seen earlier. Only thing is the L is replaced by capital F in the previous case because we have S function here+inside summation we have the capital F function if you compare it with the previous expression. So the necessary condition is in terms of the capital F function which comes inside the summation.

In the place of the capital F function, we are defining the Lagrangian capital L. So the condition is written in terms of L here, del L at k/del xk+del L at k-1/del xk=0. Similarly, the boundary condition is here instead of one variable in the previous problem now we have 3 variables. L is a function of not only, so L is not only a function of x alone like the previous one, it is also a function of u of k and it is also a function of lambda of k+1.

So there are 3 different functions involved here. So the procedure adopted in the previous one can be extended for all these 3 variables. So L is differentiated with respect to x of k. Similarly, L is differentiated with respect to u of k these two terms and L should be differentiated partially with respect to lambda of k also, is not it?

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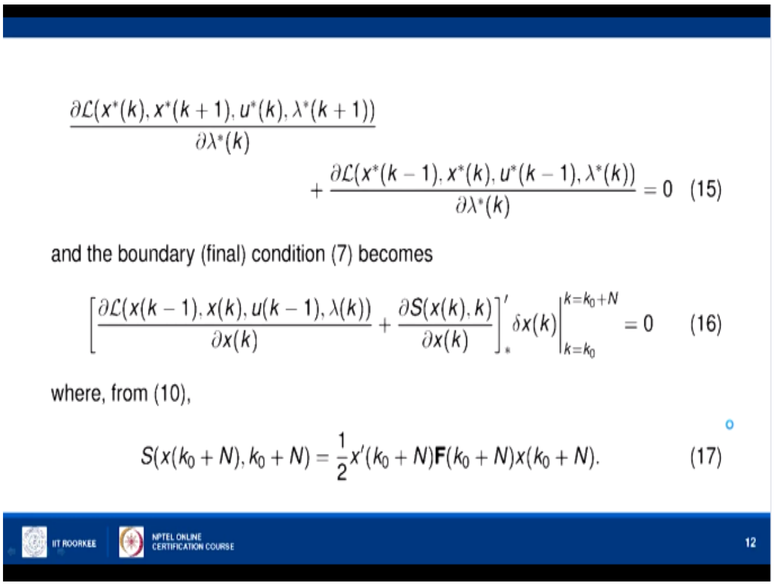

So with these 3 necessary conditions, we get the boundary condition in the following way. Exactly similar to the previous one, L at k-1 is partially differentiated with respect to del xk and the S function del S/del x of k at the final position. So this will give the boundary condition for because S is only a function of x of k not u of k or lambda of k, etc.

So only the boundary condition involves the derivative with respect to x of k so where the function S is given by the expression the linear expression which we have seen. So these equations if you see the equation 13, 14 and 15, 16, so they constitute the necessary condition for getting the optimal control problem. So a simpler way or simplifying the expression we can write or introduce the Hamiltonian function.

Hamiltonian function is defined by this expression. It is nothing but the Lagrangian function if you observe except the k+1 x of k+1 terms all the remaining terms are called the

Hamiltonian and so the Lagrangian is nothing but the Hamiltonian function-lambda dashed k+1*x of k+1.

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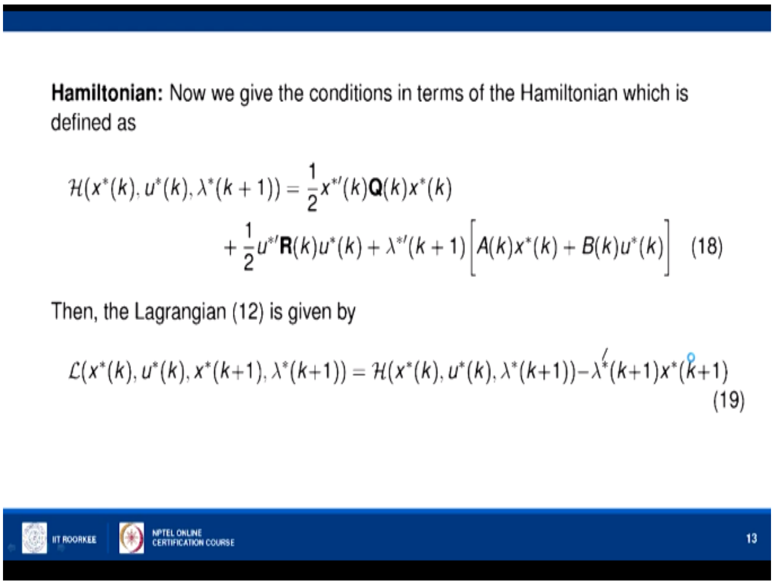

So we define the Hamiltonian in this manner. So Lagrangian L is written as Hamiltonian-lambda dashed*x of k+1. So here the star everywhere it denotes the optimal solution. So these equations will be satisfied at the optimal values of x, optimal values of u and optimal value of lambda, etc are substituted. So but when we solve the problems, we generally differentiate it with respect to the variables and then solve.

So the meaning of putting star in each equation is that at the optimal values all these equations are satisfied okay. So now L is written in terms of H in this way and converting the equations 13, 14, 15, 16 in terms of H, directly we can verify del L/del u for example it means del H/del u because the next term is not there. So the Lagrangian L is written in terms of H as in the equation 19.

Now converting these equations 13, 14, 15, 16 in terms of the Hamiltonian, so we can easily see that for example we get del L/del u if you want to calculate. So that is in this equation 19 del L/del u is nothing but del H/del u because other term there is no u involved in the second term. So similarly, all the terms in terms of all the derivatives of L can be replaced by the derivatives of H from the equation 19.

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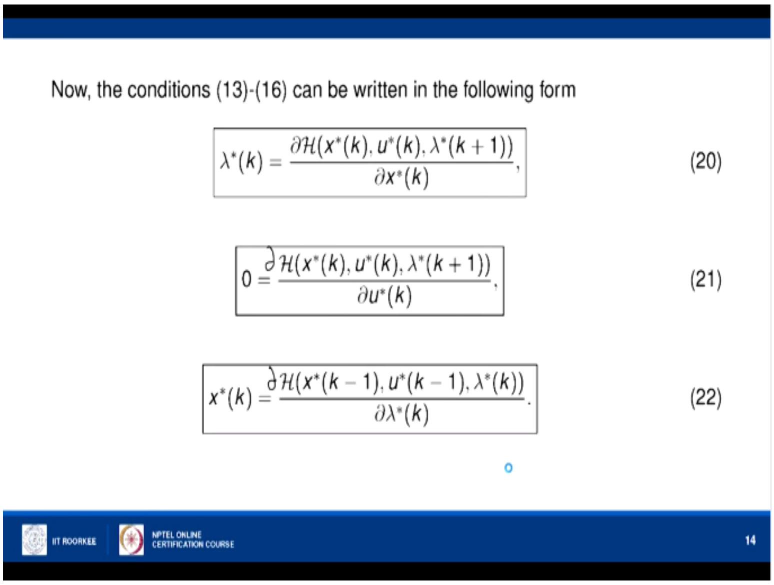

So we will get the system of equation in this way in terms of H. So del H/del xk=lambda of k and del H here del H/del uk=0 and similarly del H at k-1 here/del lambda k that is x of k. So these are the 3 necessary conditions we get and the boundary condition written from this expression. Boundary condition is given in the equation 16.

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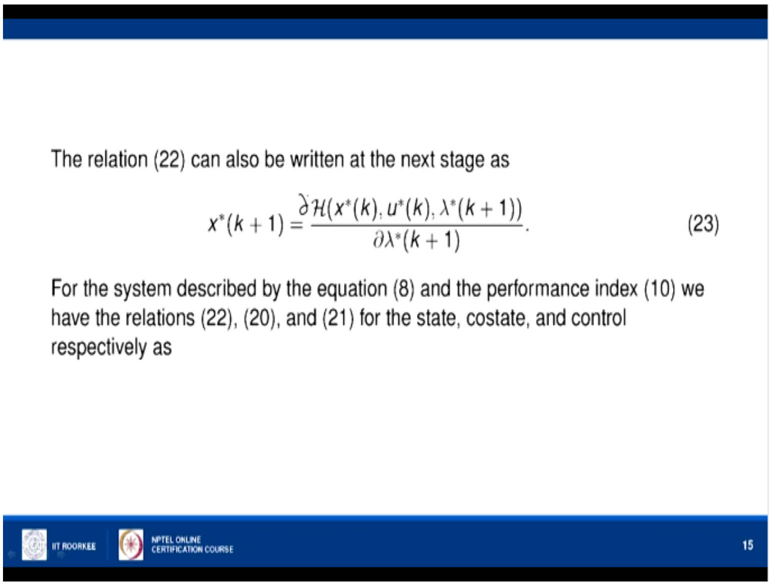

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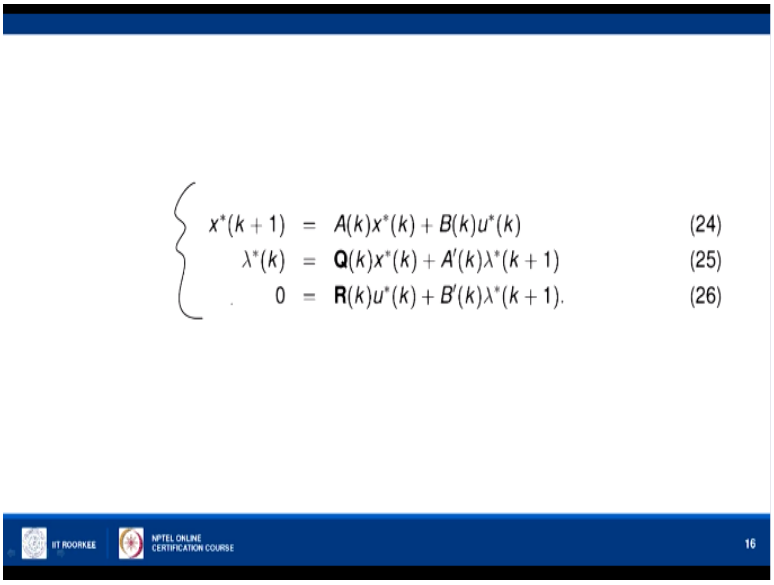

Now if you solve this system of equations with the boundary condition, we get the optimal control value here okay. So now converting actually substituting the expression of H, S, etc, so when we see that H is involving the matrices Q, R, A, B, etc. So converting actually substituting H and then differentiating with respect to those variables, we will get the system of equation to be like this.

We have to solve these 3 equations along with the boundary condition so that we will get the optimal control for the system.

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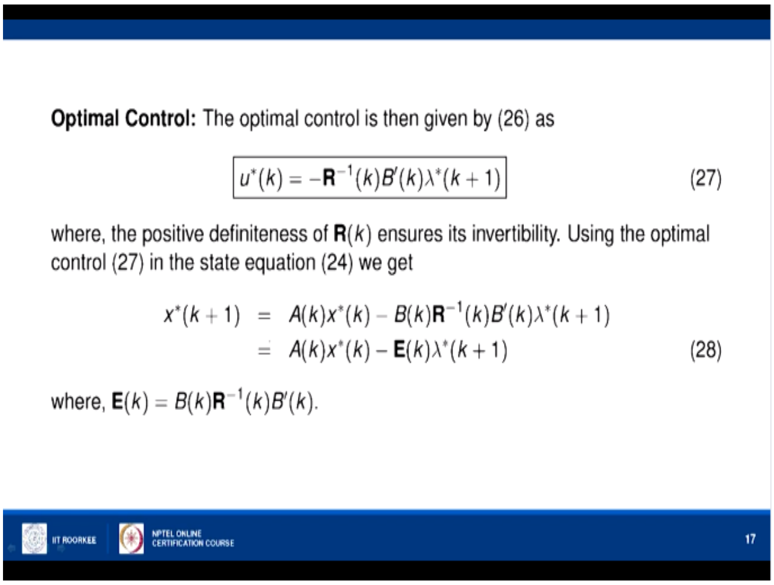

So the optimal control from the equation we can see for example the equation 26 we can directly get the control optimal control u by taking the remaining terms to other side and writing u star of k in this manner. So we get the optimal control expression in terms of lambda. Then, substituting this u expression in equation 24, then we will get a coupled equation in x and lambda.

So we will get the equation like this substituting u value from 27, we will get the equations like this.

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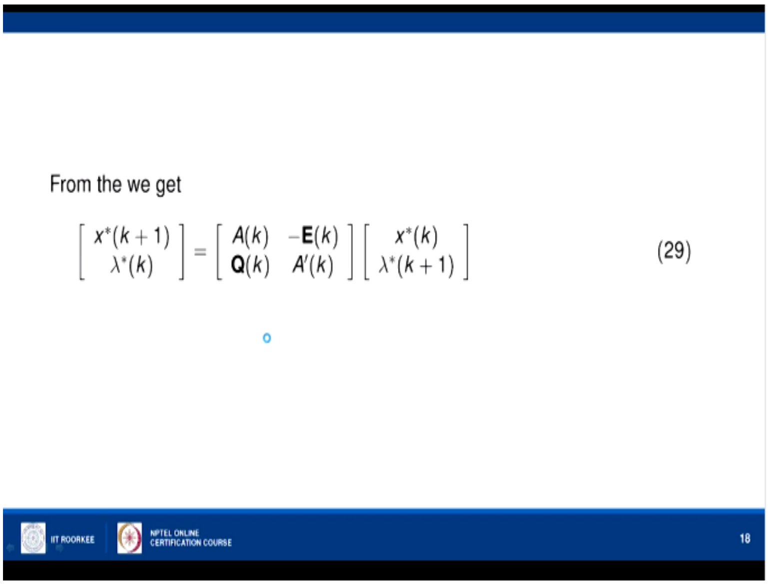

And finally we will get a coupled equation of this type and we can solve this equation along with the boundary condition we get the optimal control solution.

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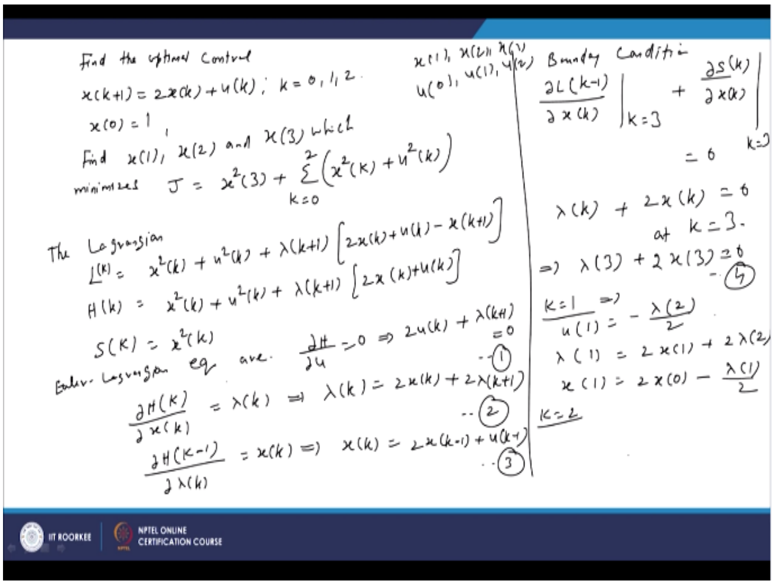

So this can be demonstrated by a simple example for the equation x of k+1=2 times x of k+u of k, here k is=0, 1, 2 and we take the similar to the previous problem x of 0=1 and we want to find x of 1, x of 2 and x of 3 which minimizes the expression J which is given by x square

3+summation k is ranging from 0 to 2 of x square of k+u square of k. So the performance index is given by this expression and the initial condition.

We have to find the finite sequence x1, x2, x3 for minimizing this under the constraint. So we can formulate the Lagrangian directly. The function L of k at the point k is so that is given by x square of k+u square of k this expression+lambda k+1*2 x of k+u of k-x of k+1. So this is Lagrangian expression we have. Then, we can write it in the form of the Hamiltonian. Hamiltonian at k, this is Lagrangian when we write k first.

So Hamiltonian is given by x square of k, u square of k+lambda k+1*2 x of k*u of k except the x of k+1 term others are Lagrangian. So now the function S, S function is given by this thing. S of k we can write it as x square of k that first term here. So the equations are given by so the necessary condition the Euler-Lagrange equation. So they are given by del H/del u=0 that is the first equation.

So del H/del u this implies 2 uk and here sorry this is + here differentiate with respect to u, u of k we will get lambda k+1=0. So this is one equation. We can get u of k from in terms of lambda k+1. Then, the second equation is del H of k/del x of k is=lambda of k. If we recall that is the other equation. So this implies lambda of k is del H/del xk so that is nothing but differentiate with respect to xk, it is 2 x of k+here also it is 2 lambda k+1 from the derivative.

So this is the second equation and the other equation will automatically give the same equation. We have del H k-1/del lambda k that is=x of k. So this is the third equation. This implies x of k=so when we write H k-1 everywhere, so we will get when we write k-1 we will get here lambda*k and the remaining things are 2 x k-1+u k-1. When we differentiate that with respect to lambda k, we will get 2 x of k-1+u k-1.

So it is nothing but the same equation whatever constraint already given, the control system, it is just one step before. Instead of k+1 we have k and the right hand side k is replaced by k-1. So it is the same equation, equation 3 and the boundary conditions can be written as given in the previous slides. So these are the equations which we wrote just now and the boundary condition we can have from this expression.

Del L at k-1/del xk+del S/del xk, so when in the L function we replace it with k-1. So first term is x square k-1, then u square k-1, etc and we have to differentiate with respect to xk, so that will come only here. So the boundary condition is del L k-1/del x of k at k is=3 that is the final time similarly del S/del xk at k=3 that will be=0. So del L k-1/del xk that is nothing but lambda k, only this term.

And del S/del xk that is 2 times x of k=0 at k=3, this is only at k=3. So this implies we get lambda of 3+2 times x of 3=0, this is the fourth equation. So now we can solve this equation 1, 2, 3, 4 and we will get the values of them, so we will write one by one. If you write k is=1 in all these equations, we will get a set like this. Similarly, so for example if you put k is=1 in all these equations that will give u.

So wherever u comes we can replace it with the lambda value okay. So we will get u of 1=-lambda 2/2 so that can be substituted in the next equation and then lambda of 1 that is given by 2 times x of 1+2 times lambda of 2 and the next equation is x of 1=2 times x of 0 and here u of 0. So u of 0 is nothing but -lambda of 1/2 from the equation 1. So we get a set like this. Similarly, we can write at k=2 similar expression.

So by writing system of equation, we will get exactly 6 unknowns that is we want to find x1, x2, x3 and u1, u2 these 5 unknowns we want to find x1, x2, x3 and we want to find the control for this expression is u0, u1, u2. These are the control because we do not want to find u3 because u2 the control given at the instant 2 will take the system to the step 3 from the given system of equations.

So we need to find u0. At initial instant, the control will take the system to the first instant and control at first instant will take it to the second and second instant will take it to the third. So we want to find this and we want to find the state values x of 1, x of 2 and x of 3 which will minimize the expression J. So we have 6 unknowns and we will get 6 equations which can be solved in a usual manner and then get the optimal control for the problem okay. Thank you for listening.