Dynamical Systems and Control

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Lecture – 54
Optimal Control for Discrete Systems-I

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Let $x(k) \in R^n$ for $k = k_0, k_0 + 1, \cdots, k_0 + N, \cdots$ and $F : R^n \times R^n \times R^n \to R$ be a functional. Let $J(k_0)$ be a functional defined as

$$J(k_0) = \sum_{k=k_0}^{k_0+N-1} F(x(k), x(k+1), k) \qquad (1)$$

To find the extremum value of the functional $J(k_0)$ and the corresponding optimal solution (sequence) $x(k) : k = k_0 + 1, k_0 + 2, \cdots, k_0 + N$, the necessary condition is that the first variation of J is zero.

$$\delta J(k_0) = 0.$$

Hello viewers, welcome to this lecture on optimal control for discrete systems, in this lecture before going to the optimal control problems, we will find the procedure for optimising the functional J of k0 given by this expression, summation k ranges from k0 to k0 + capital N - 1 of the function capital F of xk, xk + 1, k, so this function capital F is a function from Rn cross Rn cross R * R here, so this is to be cancelled.

So, it is a real valued function of vectors from Rn and xk is a sequence of vectors where k ranges from k0, k0 + 1 etc. So, we have to find a sequence of vectors xk which minimises the expression J of k0, in some problem it may maximise, so in this lecture we will find this necessary condition for finding the optimum value of this expression. So, the necessary condition may provide the either maximum or minimum value.

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First, we discuss the result for scalar function $x(k)$:

Theorem

The Euler-Lagrange equation for the minimization of (1) is given by

$$\frac{\partial F(x(k), x(k+1), k)}{\partial x(k)} + \frac{\partial F(x(k-1), x(k), k)}{\partial x(k)} = 0 \qquad (2)$$

with boundary condition

$$\left.\frac{\partial F(x(k-1), x(k), k-1)}{\partial x(k)} \delta x(k)\right|_{k_0}^{k_0+N} = 0 \qquad (3)$$

Or it may not provide any of this also because it is only a necessary condition, so for finding the optimum value of J, the necessary condition is the variation in J should be = 0, del J of this expression = 0, so to calculate this del J, we will obtain the necessary condition as follows, given in this theorem the Euler Lagrange equation for the minimisation of the expression 1 is given by this expression.

The partial derivative of F of xk xk – k with respect to xk + the partial derivative of the function, F of xk – 1 xk k with respect to del xk that should be = 0, this is a necessary condition, the boundary conditions are given by this expression, the partial derivative of F, F of xk – 1 F of x of k k – 1/ del xk, this has to be equal to 0. Now, we can observe that the boundary condition for example, if the initial and final conditions are fixed, initial instant is k0; final instant is k0 + N.

So, if the 2 conditions are already given in the problem, then the variation will be 0 at these two points, so this condition is not required, in case that the, these are variables, the initial position x of k0 and final position x of k0 + N are variables, then the variation will be arbitrary, in that case we will get a boundary condition given by this, del F/ del x at these 2 points should be = 0, so that we will see during the example.

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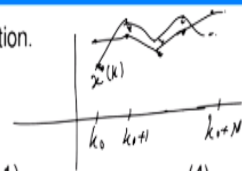
Proof:

Let $x^*(k) : k = k_0, k_0+1, \cdots, k_0+N$ be the optimal solution. A variation of $x^*(k)$ and $x^*(k+1)$ is written as



$$\begin{aligned} x(k) &= x^*(k) + \delta x(k) \\ x(k+1) &= x^*(k+1) + \delta x(k+1) \end{aligned} \qquad (4)$$

Now let

$$J = \sum_{k=k_0}^{k_0+N-1} F(x^*(k), x^*(k+1), k) \qquad (5)$$

$$\text{and} \quad J^{\delta} = \sum_{k=k_0}^{k_0+N-1} F(x^*(k)+\delta x(k), x^*(k+1)+\delta x(k+1), k) \qquad (6)$$

So, let us see how to derive this necessary condition, so if x star k is an optimal sequence in between the instant k0 to k0 + N, then the variation is given by this expression, so we note that first we derived this theorem for the case, where x of k are not vectors, they are scalar functions, so the same expression can be derived later for the vector functions, so that can be easily generalised.

So, right now we are assuming that F is a function of this real valued function, xk, xk + 1 and k is a real number and so now, x of k; x star k is a optimal sequence, so the variation in x star k is given by del x k, so it gives a new sequence, xx of k and xk+ 1 is the sequence starting from x star k + 1 + del x of k + 1, so see if you assume that for the real case, k0, k0 +1 etc. k0 + N, so the optimal value at k0 x of k0 is a number, x of k0 + 1, x of k0 + 2.

So, if you have some sequence like this, this is our optimal solution now, if you give a variation in the first value, it may be a small variation here, positive variation and there may be a negative variation like this, so we may get a new function which may be given by this expression, so it is a; it is like x of k, if this we are assuming to be x star of k for various values, a small increment will give you a new function x of k, this sequence.

And similarly, if you start from the next value for example, if you start from k + 1 that is another sequence, which is given by x of k + 1, now the J value which we want to minimise is given by this expression starting from k0 to k0 + N - 1 of this function and after giving the increment in x star, we get the function, J delta.

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and

$$\delta J = J^{\delta} - J. \tag{7}$$

Using the Taylor series expansion of (6) along with (5), we have

$$\delta J = \sum_{k=k_0}^{k_0+N-1} \left[\frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k)} \delta x(k) + \frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k+1)} \delta x(k+1) \right] \tag{8}$$

Now in order to express the coefficient $\delta x(k+1)$ also in terms of $\delta x(k)$, consider the second expression in (8)

So, the first variation in J is; delta J is J delta – capital J and which can be written like this by directly substituting the values from here, J delta is this, J is this, subtract these 2, we get this expression. Now, if you write the Taylor series expansion of; so the first variation in J is given by J delta - J and if you observe from the previous slide, J delta is the expression 6, now it is having F of x star + del x and x star k + 1 + del x of k+ 1, k.

Now, expanding this expression in the Taylor series and taking only the first variation in the Taylor series, so we will see that the first term of the Taylor series is same as J and J delta – J, in that the first term will get cancelled, the second term is the first variation given from the equation 6, which is nothing but del F/ del x of k * del x of k. Similarly, the derivative; partial derivative del F/ del x of k + 1 multiplied by the first variation x of k+ 1.

So that is what we will get here, the del F/ del x star k and del F/ del xk + 1 term, only first variation we are collecting and omitting all the second variation, we get del J to be like this and this should be = 0 is the necessary condition but from here, we may not get any useful result.

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$$
\begin{aligned}
\sum_{k=k_0}^{k_0+N-1} & \frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k+1)} \delta x(k+1) \\
= & \frac{\partial F(x^*(k_0), x^*(k_0+1), k_0)}{\partial x^*(k_0+1)} \delta x(k_0+1) \\
& + \frac{\partial F(x^*(k_0+1), x^*(k_0+2), k_0+1)}{\partial x^*(k_0+2)} \delta x(k_0+2) \\
& + \cdots\cdots\cdots \\
& + \frac{\partial F(x^*(k_0+N-2), x^*(k_0+N-1), k_0+N-2)}{\partial x^*(k_0+N-1)} \delta x(k_0+N-1) \\
& + \frac{\partial F(x^*(k_0+N-1), x^*(k_0+N), k_0+N-1)}{\partial x^*(k_0+N)} \delta x(k_0+N)
\end{aligned}
$$

So, we can further proceed like this, we take the second term, the second term is del F/ del xk + 1 and we write the entire the summation, summation k = k0 +; k0 to k0 + N -for the second term, so if you expand it term by term, we get this expression, for k = k0 is the first term, k0 + 1 is the second term etc. the last term is for k0 + N.

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$$
\begin{aligned}
& + \frac{\partial F(x^*(k_0-1), x^*(k_0), k_0-1)}{\partial x^*(k_0)} \delta x(k_0) \\
& - \frac{\partial F(x^*(k_0-1), x^*(k_0), k_0-1)}{\partial x^*(k_0)} \delta x(k_0) \qquad (9)
\end{aligned}
$$

And then we add this term and subtract this term that for convenience, del F of x star k0 – 1 x star of k0 k0 -1 term we add and then subtract the same thing now, we can combine this term except the starting from the first term to this term, we can write it in the particular form, so k0 to k0 + N -1 del F of x star of k -1 x star of k k -1/ del x star of k etc. and while writing like this we are omitting this term that is the last term.

Because you are writing it in the form of k -1 appearing first here, we observe that the first values are F of x; x star of k0 but when we write it in the form of k0 -1 as the first term.

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where the last two terms in (9) are added without affecting the rest of the equation.

$$= \sum_{k=k_0}^{k_0+N-1} \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k)$$
$$+ \frac{\partial F(x^*(k_0+N-1), x^*(k_0+N), k_0+N-1)}{\partial x^*(k_0+N)} \delta x(k_0+N)$$
$$- \frac{\partial F(x^*(k_0-1), x^*(k_0), k_0-1)}{\partial x^*(k_0)} \delta x(k_0)$$

So, we have to note that while doing that way, we will be omitting the last term in that previous series and this term we have already introduced artificially in the last step, so this 2 terms are omitted and this summation takes care of all the other terms. See, this last term is not there inside the summation and this last term is also not there in the summation, so we take this term as well as this minus term and combining this 2 here separately, we get the entire expression which we want here, okay.

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$$= \sum_{k=k_0}^{k_0+N-1} \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k)$$

$$+ \left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k) \right] \Bigg|_{k=k_0}^{k=k_0+N}. \quad (10)$$

Substituting (10) in (8) and noting that the first variation should be zero, we have

$$\sum_{k=k_0}^{k_0+N-1} \left[\frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k)} \delta x(k) + \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k) \right] \delta x(k)$$

$$+ \left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} \delta x(k) \right] \Bigg|_{k=k_0}^{k=k_0+N}. (11)$$

So, the second term was written in that particular form, the first term is as it is, so combining the first term of the equation 8 and second term of 8 together, we will write the equation to be like this, these are the expression we get, del F of x star k -1 x star k k -1/ del x star of k * the variation x of k and the last two terms, extra terms which we have taken is written in this particular form.

So, combine, this is only the second term of the necessary condition, the first of term is here as it is so, combining the first term and the second term together, we get this and the boundary condition term is given in this way. So, now the variation del xk is arbitrary, this entire bracket has to be = 0 here and for the boundary condition, we have to; because once the bracket becomes 0, the remaining gives the boundary condition for the solution of the problem.

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For arbitrary variations $\delta x(k)$, we have the condition that the coefficient of $\delta x(k)$ in the first term in (11) be zero. we have

$$\boxed{\frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x^*(k)} + \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)} = 0.} \qquad (12)$$

This equation is called as the discrete-time version of the Euler-Lagrange equation. The boundary condition is obtained be setting the second term in (11) equal to zero. That is

$$\boxed{\left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)}\right] \delta x(k) \Bigg|_{k=k_0}^{k=k_0+N} = 0.} \qquad (13)$$

$x(k_0)$
$x(k_0+N)$

So, we get the necessary condition to be like this, del F of xk xk + 1 k/ del x star of k + del F xk – 1 xk and k – 1/ del xk again = 0 is a necessary condition and the boundary condition is given by this expression, this bracket.

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Now, we discuss two important cases:

- **Case 1:** For fixed end conditions, we have the boundary condition $x(k_0)$ and $x(k_0+N)$ fixed and hence $\delta x(k_0) = \delta x(k_0+N) = 0$.
- **Case 2:** If $x(k_0)$ is given and $x(k_0+N)$ is free then $\delta x(k_0) = 0$ and

$$\left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x^*(k)}\right]\Bigg|_{k=k_0+N} = 0 \qquad (14)$$

So, now depending on the case, if the boundary conditions both of them are fixed values, then the variation at this 2 boundary will be 0, here, the variation, del xk; then it should be del xk also, so this correction also we have to note, okay, equation 13, so this del xk, if the initial time k at k0 and k0 + N, if x of k0 is given, x of k0 + N, if both are given, then the variation at this 2 points will be 0, so this condition is not required only the equation 12 to be solved under this boundary conditions but in case x of k0 is given but the final position is not given, final time is given, k0 + N.

But x of k0 + N is not given and it is free then, we need the boundary condition from here because the variation del x of k0 + N will be arbitrary in that case so, we get a condition del F/ del x star evaluated at k0 + N should be = 0 will turn out to be the boundary condition for such problem. So, for the case 1 we have to solve only the equation 12 with the given boundary condition.

And in the case 2 where x of k0 is given and the other one is free, then we get this boundary condition.

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Now, we consider the conditions for the case in which $x(k)$ is vector function. Thus, consider the functional which is the vector version of the scalar functional (1) as

$$J(k_0) = \sum_{k=k_0}^{k_0+N-1} F(\mathbf{x}(k), \mathbf{x}(k-1), k) \qquad (15)$$

$$\boxed{\frac{\partial F(x^*(k), x^*(k+1), k)}{\partial x_i^*(k)} + \frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x_i^*(k)} = 0.} \qquad (16)$$

$$\boxed{\left[\frac{\partial F(x^*(k-1), x^*(k), k-1)}{\partial x_i^*(k)}\right]\Bigg|_{k=k_0}^{k=k_0+N} = 0} \qquad (17)$$

$i = 1, 2, \cdots, n.$

So, far what we have seen is the derivation of the equation; this 2 equation 12 and 13 where x of k is a real number, the sequence x of k is a real sequence but in case, x of k is a vector sequence, each x of k belongs to Rn, so if x of k belongs to Rn for each k value, then also the procedure is same, only thing is x of k is written as x1 of k x2 of k etc. xn of k, it is a vector, so at every stage we have to replace the real value with the vector.

And then the partial derivative of this function with respect to each value should be considered, so for i = 1, 2, 3 up to n, we can; we get this 2 formulas, instead of 12 and 13 formula, we get 16 and 17 for each variable i, each variable xi, okay, the same formula, so the procedure will be the same exactly, so now we will consider a simple problem of finding the optimum value, optimum sequence.

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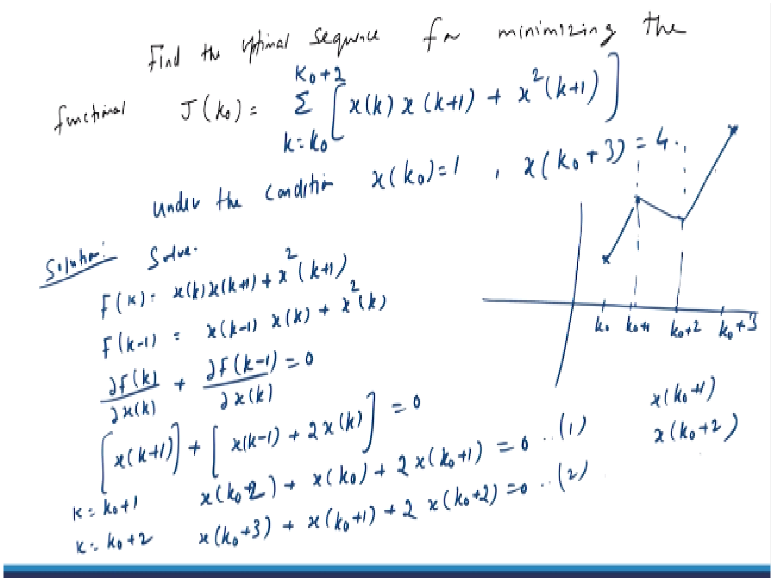

So, find the optimal sequence for minimising the functional J k0, k0 indicates that that is the initial instant, so if it is given by this expression x; x square k0 + 3 + summation, so it is summation k = k0 +2 of; so, let us consider the function J, yeah, so in this problem, we consider the functional J of k0 is given by summation k starting from k0 to k0 + 2 0 and the expression is x of k and x of k +1+ x square of k + 1.

So, we have to minimise this expression and so, it means under the condition x of k0 is 1 and x of k0 +3 is some value, this 2 values are given, so we have to find k0 k0 + 1 k0 + 2 K0 + 3, so 4 instant of time is given, x of k0 is fixed, it this 1 and k0 +3 is also given, it is the value 4, now we have find the values for this k0 + 1 k0 + 2, 2 values we have to find, so the optimum solution it may lie anywhere in this line, the values can be anywhere here, only initial and final are given.

So, if you solve the equation del; the solution, we have to solve the 2 equation which we have mentioned in the previous slide, so that is del F/ del x, when F is starting with x of k and del F/ del xk, if F is starting with k -1, so here F, we can write, if it is starting with k, it is the given expression, x of k x of k + 1 + x square of k+1 and when it the initial thing is starting with k -1, it is x of k – 1 x of k + x square of k.

So, the necessary condition is del F k/ del x of k + del F of k – 1/ del of k = 0, so that is the condition, okay as given here and we do not require the boundary condition 17 because both boundaries are fixed here, so we simply write del F/ del k, derivative with respect to k, it is x of k +1 and del F of k -1/ del xk, if you differentiate this with respect to k, so this is the first one, the second one is with respect to xk, we get x of k -1+2 times x of k.

So, this = 0, now we put k = k0 +1 in this equation, we will get x of k0 +1 + 1 k0 +2 + x of k0 and + 2 times x of k0 + 1 that is = 0 and when we put k = k0 +2 in this equation, we get x of k0 +3 + x of k0 +1+2 times x of k0 + 2 = 0, so we get this 2 equation and it is already given, x of k0 is 1, so we have to solve this 2 equation, from equation 1, we will get the relation between x of k0 + 2 and k0 + 1 = 0.

And k0 + 3, x of k0 + 3 is given as 4, so we substitute it in the equation 2, so that gives you 4 + x of k0 + 1 2 times x k0 +2 = 0, so we get 2 equations with 2 unknowns, the 2 unknowns are x of k0 + 1 and x of k0 + 2 in this equation, so we get the solution, solving this 2 algebraic equation, we get the optimal value of this thing of the 4 values are given, so we will get some value for x of k0 +1 and some value for x of k0 +2, already x of k0 +3 is given.

So, this sequence is the optimal sequence which will optimise the expression J of k0 under this condition, okay. Now, if the final condition is not given let us assume that the initial condition x of k0 is given but final condition is free, so that we can solve by this expression.

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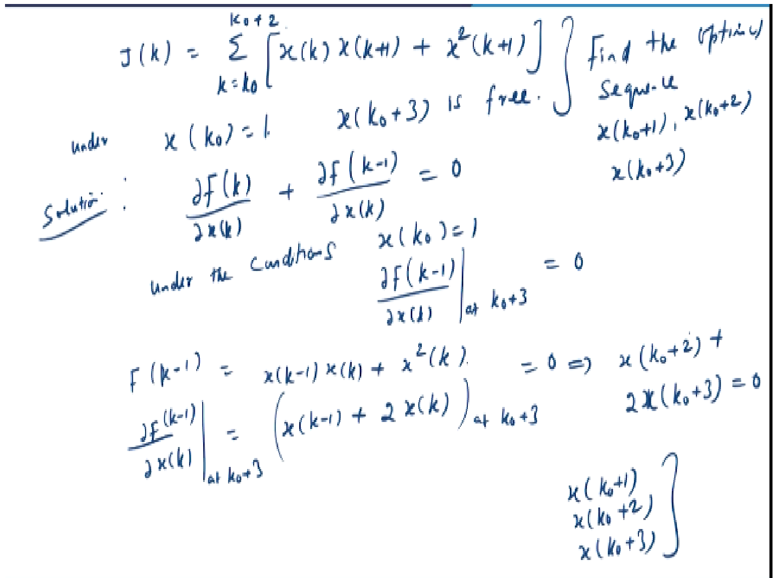

So, now a second example we will see, the example in which the boundary condition is variable, so here we consider k varies from k0 to k0 +2 of x of k x of k +1+ x square k + 1, so you have to minimise this expression under the condition x of k0 is some given value and x of k0 +3 is free, so which is not given, so in that case we have to solve, so for this problem, we have to find the optimal sequence.

We have to find x of; x of k0 is already given, we have to find k0 + 1 x of k0 + 2 x of k0 +3, this 3 values we have to find to minimise or maximise this expression, so we have; the solution is; we have to solve the equation del F, if it is starting with xk and differentiate with respect to this and del F, if we start with k -1 and we differentiate this = 0 under the boundary condition, x of k0 is already 1, it is given.

And because the second boundary condition is free from the expression here, we will see that the del F starting with k -1 derivative with respect to x of k at the endpoint that should be = 0, so we have the boundary condition, del F starting with k -1 term of del of xk evaluated at k0 +3 that should be = 0, so this is the problem, so if you write the; the equation will be the same because the problem expression is the same as in the previous slide here.

So, this 2 equation will be the same, only thing is the boundary condition we have to substitute this expression, so del F x of; if you write K -1 that term is nothing but x of k -1 and x of k + x square of x, we differentiate with respect to del x of k, so that gives x of k -1+2 times x of k, so if you evaluate at k0 +3, so this evaluated at k0 +3 should be = 0, so this implies x of k0 +3-1, so it is k0 +2 and + 2 times x of k0 +3 here that is = 0.

So, we get a relation between k0 +2 and k0 +3 and already we have the 2 equation; equation number 1 and 2 here, so we substitute the relation the, this relation between k; x of k0 +2 and k0 +3, we can write one in terms of the other, so that will reduce one of the variables and we can solve the system using the initial condition by substituting x of k0 = 1 in this 2 system of equation; equation 1 and 2.

And making use of this boundary condition at k0 + 3, we can solve the algebraic equation easily and obtained the optimal sequence for the 3 values that is x of k0 is already given, we want to find x of k0 + 1 k0 + 2 k0 + 3, so this sequence is obtained by solving the simple algebraic system, so in this lecture we have seen the optimisation of a discrete expression, discrete time expression and the optimum sequence, how to find the optimum sequence to minimise J has been shown that is a necessary condition has been shown here.

And in the next lecture, we will consider the optimal control problems that is along with minimisation of J, the constraint is a discrete control system and we will find the optimal control for the discrete time systems, thank you.