Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 53
Optimal Control - IV

Dear students, welcome to this fourth lecture on the optimal control and we continue the lecture from the previous lecture.

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Consider the system

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (1)$$

and the performance index as

$$J_1(u(t)) = S(x(T), T) + \int_{t_0}^{T} F(x(t), u(t), t)dt \qquad (2)$$

So, in the previous lecture we have considered the problem of finding the optimal control for the equation 1 that is dx/ dt is a function of x of t u of t, t, x is the state variable and u is the control variable and the performance index to be minimised is the function J given by S of x, capital T, T + integral t0 to capital T and capital F of xt ut t dt for suitable functions S, F and small f etc.

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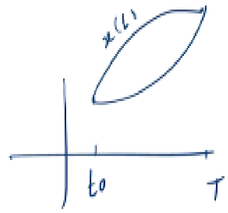
where

Define:

$$H = F(x(t), u(t), \lambda(t), t) + (\lambda(t))' f(x(t), u(t), t)$$

Then the Euler-Lagrange equations in terms of H, for optimal control are given by



$$\left(\frac{\partial L}{\partial u}\right) = 0 = \left(\frac{\partial H}{\partial u}\right) \quad (3)$$

$$\left(\frac{\partial H}{\partial t}\right) = -\dot{\lambda}(t) \quad (4)$$

$$\left(\frac{\partial H}{\partial \lambda}\right) = \dot{x}^*(t) \quad (5)$$

$$\left(H + \frac{\partial S}{\partial t}\right)_T \delta T + \left(\frac{\partial s}{\partial x} - \lambda^*(t)\right)_T \delta x_T = 0 \quad (6)$$

$x_T = T^2$, $\delta x_T = 2T \cdot \delta T$

where x_T is the final position $x(T)$.

So, now we define the Hamiltonian, in the previous lecture we have already defined, so we just recall the result, H = capital F + lambda dashed * small f where lambda is the vector; lambda 1, lambda 2, lambda n, so and lambda transpose is the row vector and f is the column vector given in the equation 1 then, Euler Lagrangian equation to be solved for finding optimal control is given by this equations 3, 4, 5 and 6, where 6 is the boundary conditions.

Now, we have seen one case in the last lecture that is if the end point, if the initial point is fixed and final point is also fixed, so if you consider t0 and capital T and at t0, the value is x of t0 and at capital T, the value is x of capital T, so we consider functions of this type for the optimal state, so x of t is the state variable initial condition x of t0 and final condition x of capital T, both of them are fixed, t0 and capital T are also fixed values.

So, in that case we have the del T, there is no variation in capital T because it is fixed, so that is 0 and there is no variation in the value of the final position that is del x of xt that is also fixed here, so 0, so the conditions 6 will not appear, directly all the 4 conditions are given, all the 2 conditions are there, we have to solve the system 3, 4, 5 subject to the given condition that was what we have seen in the last lecture.

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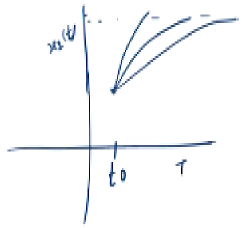
Example 1

Find the optimal control u and optimal state $x(t)$ for

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + 4\,u(t)$$
$$x_1(0) = 1, \quad x_2(0) = 0$$
$$J = \frac{x_1^2(T)}{2} + \int_0^T \left(x_2^2(t) + \frac{u^2(t)}{2}\right) dt$$

where T is free and $x_1(T) = 0$, $x_2(T) = 1$.



Now, we will see the other cases, so for example consider the problem x1 dot is x2, x2 dot = - x1 + u, sorry this is u; u of t and initial condition x1 of 0 is 1 and x2 of 0 is 0, this is fixed and the final time is a variable and final position is fixed here, x1 of capital T is 0, x2 of capital T is 0, so this is the case in which t0 is fixed and the initial condition is fixed here and at final time, t is a variable but this is fixed that is the value is fixed.

For example, it may be a curve of this type, so capital T is not fixed, it can be any of this values, so here we are trying just for one dimensional problem for example, if you take this as x2 of t at t = capital T, the value is 1 and for capital T, the value is 1, so we can get many such functions, out of this functions we have to find the optimal function, similar graphs can be drawn for x1 of t separately.

So, how to find the control u of t, optimal control for this particular problem, for doing that we proceed using this 4 condition; 4 equations; 3, 4, 5 subject to the boundary condition. Now, if you observe the boundary condition, the final position is fixed therefore, del xt is; there is no variation in the final position, so this is 0 but del capital T is arbitrary, there is a variation, so the boundary condition is H + del S/ del t at capital T should be = 0.

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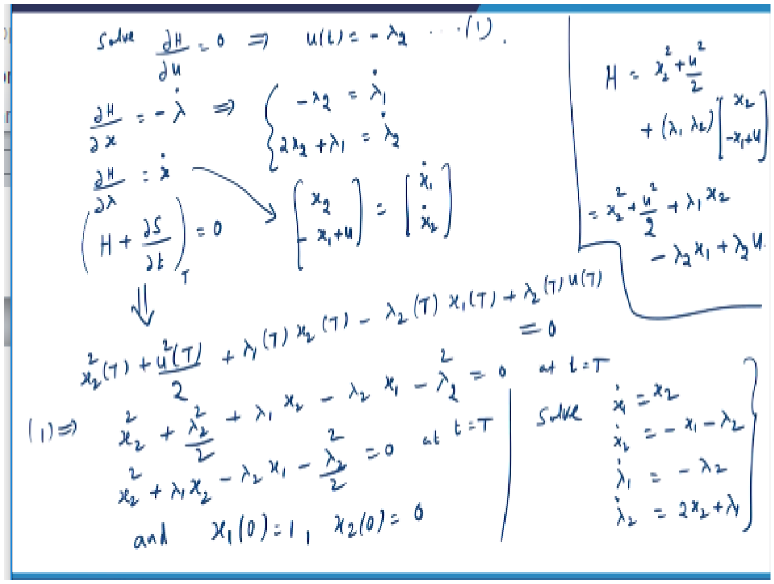

So that is the boundary condition, other conditions are given here, so we have to solve the equation, del H/ del u = 0 is one equation and del H/ del x = - lambda dot d lambda/ dt and the other equation is del H / del lambda = x dot so that turns out to be the given equation itself and the boundary condition is H + del S/ del t evaluated at the final position is 0, okay where capital T is also to be found out unknown function.

So, first del H, we will first write the H value from the given problem, the Hamiltonian is given by the capital F x2 square + u square/ 2 that is capital F + lambda 1 lambda 2 that is zero vector multiplied by small f is the function H1 dot = x2 that is the first value and x2 dot is – x1 + u, so this is the H function which = x2 square + u square/ 2 + lambda 1 x2 – lambda 2 x1 + lambda 2 u, so this is the H function.

Now, del H/ del u, if you calculate from here, it is = u of t and here del H/ del u is + lambda 2, so it is – lambda 2, del H / del u is u + lambda 2 = 0, so therefore u = lambda; - lambda 2 and del H/ del x, so it means, del H/ del x1 first, del H/ del x1 is – lambda 2 that is = lambda 1 dot the first one, the second variable is del H/ del x2 is 2 x2 and + lambda 1, so that is = lambda 2 dot from this equation.

So, this implies these 2 equations and del H/ del lambda if you see that is nothing but the function x itself, it is x2 to – x1 + u that is the left hand side = x1 dot x2 dot that is from the third equation, this equation implies this one. Now, H at capital T, if you substitute capital T for all this thing, so this implies x2 square at capital T + u square at capital T/ 2 + lambda 1 at T, x2 at T – lambda 2 at T and x1 at T + lambda 2 at T and u at T.

So that is = 0, now we substitute from this equation, u =; let us say it is 1, u = -lambda 2, we substitute it in this equation, so x2 square + u = - lambda 2, so it will be lambda 2 square/ 2 here + lambda 1 * x2 at capital T – lambda 2 at T and x1 at T, here again u of T is – lambda 2 if you substitute, it is – lambda 2 square = 0, so here we have lambda 2 square/ 2 etc. so the condition is; this is at T, everywhere we do not write capital T, so we can write like this.

So, we have fixed x2 square + lambda 1 x2 – lambda 2 x1 and this 2 combined, we get lambda 2 square/ 2 = 0 at t = capital T, so this is the condition on capital T, okay so first thing is we have to solve this 4 equation, coupled equation, so this equation and we have to solve the equation lambda; first we will write this one, x1 dot = x2 and x2 dot = - x1 + u, u = - lambda 2, so we substitute this, u is – lambda 2.

And then lambda 1 dot = - lambda 2 and lambda 2 dot is 2 x2 + lambda 1, so it is it coupled equation that is to be solved, so we can solve it like the usual dynamical system problem, it is like x dot = capital A * x, where A is the matrix whose coefficients are given in the right hand side and we can find the solution with these conditions, so we have the condition; initial condition already given in the here, x1 0 = 1, x2 0 = 0.

So, we have this condition also, so after solving this, we will get 4 constants; C1, C2, C3, C4, to find the 4 constant, we have to make use of this condition as well as the boundary condition given in this expression, by substituting in this expression we can solve the problem for the 4 constants. After finding the value of say, lambda 2, we substitute in the equation 1 and that gives the control function u of t is the optimal control.

And the solution x1 and x2 obtained from here is called the optimal state variable for the given problem 1 here, okay.

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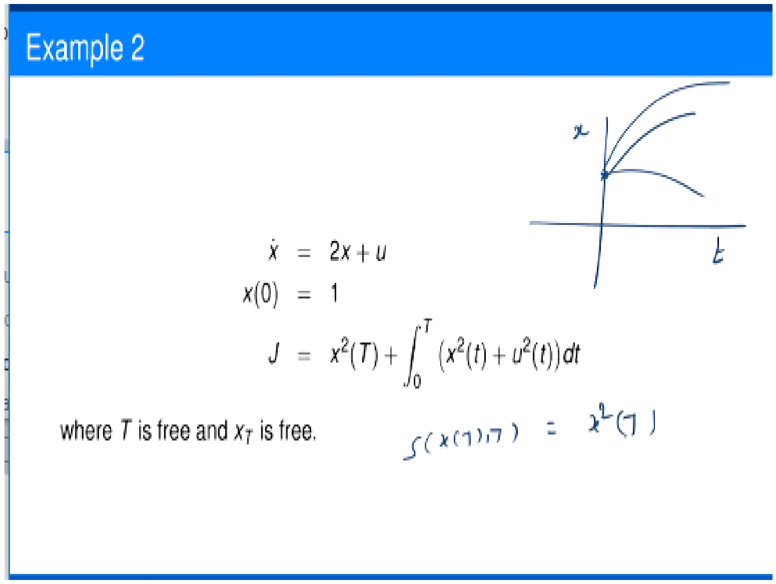

So, now we will consider the example 2, here in this example, we have a simple the example x dot; one variable problem, so here x dot = 2x + u and initial condition is fixed, the initial time is 0 and initial position is 1 that is fixed, we have to minimise the function functional J which is given in this expression and here we have this capital T, final time is free as well as the final position is also free.

So, we have it is a one variable problem, so t is an independent variable and x is the dependent variable, x of 0 = 1 at t = 0, it is 1 and capital T is not fixed, it can be anything and final time is also anything, so the optimal control problem is; it can be any of these curves and because final time, final position both of them can be arbitrary here but it should minimise the functional J, so to solve this problem we make use of the usual equations 3, 4, 5 subject to the boundary condition here 6.

So, we have to; you see here H; so this expression, so here the initial time is fixed and the final time is arbitrary, similarly initial position is fixed and final position is arbitrary, now we will make use of the equations 3, 4, 5 subject to the boundary condition 6, so let us see because final time is free, so del t is free, del t is arbitrary. Similarly, final position is also free, del x of xt is also arbitrary, so it is; these are the 2 arbitrary functions; del t and del xt.

So that implies that this 2 brackets are to be 0 separately that is H + del S/ del t at capital T should be 0 and del S/ del x – lambda at capital T should be 0, so these are the boundary condition, other conditions are the same in both the problems. So, let us write this equations, so H = x square + u square + lambda times the right hand side 2x + u.

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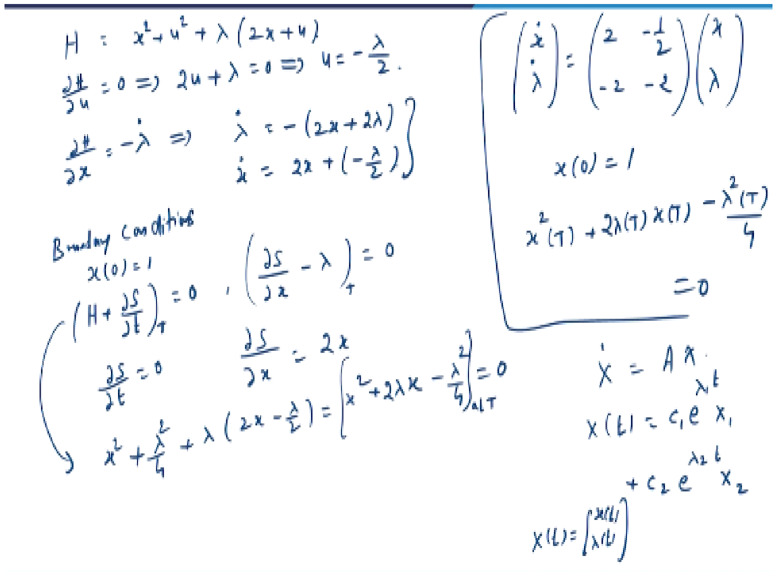

So, we will write here in this problem, the H function is x square + u square + lambda times 2x + u and the conditions are del h / del u = 0, so that implies 2u + lambda = 0, so that implies u = - lambda / 2 that is equation, the second equation is del H/ del x = - lambda dashed, so that implies lambda dot; sorry, lambda dot, so lambda dot is - del H/ del x, if you differentiate H with respect to x, it is – 2x and + 2 lambda.

So that is lambda dot = - del H/ this and already, the problem given is x dot = 2x + u, where u is – lambda/ 2, so – lambda / 2 and the boundary conditions are; so these are the 2 coupled equation, so boundary condition or the first boundary is x of 0 = 1 that is given, other conditions are H + del S/ del t at capital T = 0 and del S/ del x – lambda at capital T; because both capital T and the final position both are free, this 2 should be = 0.

Now, we can evaluate, S is given in the problem that is x square of capital T is the function, S of capital T, so here S of x of T, T that is given by x square of capital T, so del S / del t is 0 because S is not a explicit function of T, so del S/ del t = 0 but del S/ del x = 2x because S function; S is nothing but x square function, if you to differentiate, it is 2x and we have to evaluate all this at capital T.

So, when we substitute this thing, so this implies x square at capital T + u square means, it is lambda square/ 4 + lambda * 2x + u; u is again – lambda/ 2 that is = x square + 2 lambda x and here lambda square/ 4 lambda square/ 2, so we get – lambda square /4, so that = 0, this is at capital T, this, wherever this small t should be replaced by capital T that is 0, so this equation gives the coupled equation with these boundary conditions, we to solve, so we get this expression.

We can write x dot lambda dot that is = 2 and – 1/2 and –lambda dot is – 2x and – 2 lambda, so, this is the coupled equation with the condition x of = 1 and this condition x square T + 2 lambda T x of T – lambda square T/4 = 0, so this to be solved, so first we solve x and lambda, so we get 2 constants C1 and C2 and using x of 0 = 1, we can find a relation between C1 and C2, so C2 can be written in terms of C1.

Then substituting in this expression, the values of x and lambda at capital T, we will get an equation in capital T, nonlinear equation in capital T and we can find the roots of that nonlinear equation in capital T, so that values will give the final position or the optimal position for; the optimal final position; final time capital T and which can be substituted and we can find the optimal control and optimal state of the system.

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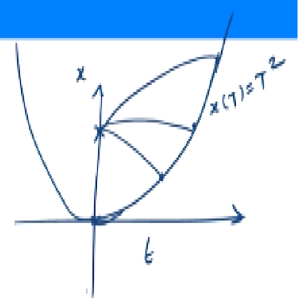
Example 3



$$\dot{x} = 2x + u$$
$$x(0) = 1$$
$$J = x^2(T) + \int_0^T (x^2(t) + u^2(t))dt$$

where T is free and $x_T = x(T) = T^2$.

So, this is for both final time is free and final position is free, okay, now we consider a similar problem, x dot = 2x + u and initial position is fixed, so here this is function of t at t = 0, x of 1 and final position; final time is free, t is free and x of t is also free but it is an little bit depending on capital T, so if you write this equation, x = T square, this is the x axis, this is t axis, so x = T square curve is like this.

So, the final position x of capital T should lie on this curve, so it is not like the previous one, it can be arbitrary, it is arbitrary but it should lie on this curve, the end point should be any of this say, on this particular curve, so x of capital T = T square is given, so again we can make use of these same equations, first of 3, 4, 5 equations are the same we can make use and in the sixth equation, the boundary condition delta T is; T is free.

So, delta T is an arbitrary variation and del xt is not arbitrary but it depends on delta T because we have x suffix T, it is T square this term in that particular problem, the final position is depending on the final time, so it like the square of the final time, so if you calculate the variation, del x suffix T, it is the derivative of this, it is 2T * del capital T, so we have to substitute in the place of del x suffix T, we will substitute this.

So, this will also be in terms of del capital T, we can combine this equation, so finally what we will get is the following.

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$$H = x^2 + u^2 + \lambda(2x + u) \quad \cdots (1)$$

$$\left(H + \frac{\partial S}{\partial t}\right)_T \delta T + \left(\frac{\partial S}{\partial x} - \lambda\right)_T \delta x_T = 0 \quad \cdots (2)$$

Given $x_T = T^2$

$$\delta x_T = 2T \cdot \delta T \quad \cdots (3)$$

Substitute (3) in (2)

$$\left[\left(H + \frac{\partial S}{\partial t}\right)_T + \left(\frac{\partial S}{\partial x} - \lambda\right)_T 2T\right] \delta T = 0$$

δT is arbitrary

$$\Rightarrow \left(H + \frac{\partial S}{\partial t}\right)_T + \left(\frac{\partial S}{\partial x} - \lambda\right)_T \cdot 2T = 0$$

$$S(T) = x^2(T) \qquad \frac{\partial S}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial S}{\partial t} = 0$$

So, here also H function is x square + u square that is one thing + lambda times 2x + u and the boundary conditions are H + del S/ del t at capital T * del t + del S/ del x – lambda evaluated at T and the variation of the final position x suffix T, so this should be = 0 but given that x suffix T or x of capital T that is T square, so the variation in the final position is the derivative into the variation into T.

So, this we have to substitute here, if you call this equation as 1 and this equation as 2 and this as 3, so substituting 3 in 2, we get H + del S/ del t evaluated at capital T + del S/ del x - lambda evaluated at capital T, here del xt is 2t, so we have to multiply here 2t, so whole multiplied by del T can be taken out, now this is arbitrary, del T, the variation in final time arbitrary, therefore this has to be = 0.

So that implies, so this = 0 and we have the initial condition, so the equation will be the same because expressions are the same whatever we derived in the last problem, the equation is the same, x dot * lambda dot is given by the dynamical system this one, the matrix A is here, so we can solve the equation x dot = Ax as x of t = C1 e to the the power lambda 1 t * x1 + C2 e to the power lambda 2t * x2, where lambda 1 lambda 2 are the eigenvalues of capital A, x1 x2 are the or the corresponding eigenvectors.

So, the solution can be written like this, where capital X means, small x and lambda, so this lambda 1, lambda 2 are the eigenvalues, they are constants but here lambda t means, it is the Lagrange multiplier that is totally different thing, okay, there is no connection between this lambda 1, lambda 2 and this lambda and so the solution can be written like this, so from this expression capital X of t, we can find small x of t and lambda t separately in terms of C1 and C2.

Then substitute the condition x of 0 = 1, we will get one relation between C1 and C2 and the remaining relation we have to find from here because we know all this expression, H is given and S function is x square t that is also given, so S of T is given as x square T, so del S/ del t = 0 and del S/del x is 2x and evaluated at capital T is 2x of capital T, okay, so substituting these values here, we will get another relation for the final time T.

So, those equation to be solved and we can find simultaneously capital T as well as the other constant C2, substituting those equations in this one, u; we have already obtained, u = - lambda 2, after finding lambda value, we substitute lambda 2 of T which is the optimal control and then we can find, we also found the x of T after finding the constants C1 and C2, okay, so we get to the optimal control as well as the optimal state of the system.

So, this covers all the cases, in the previous lecture we have seen the constant boundary condition and in this lecture, we have seen variable boundary conditions and so this completes the procedure for finding the optimal control problems in the continuous system, so this type of systems we call it as continuous system because the derivatives are there. In the next lecture, we will consider the discrete time systems and how to find the optimal control for the discrete systems, thank you.