Dynamical Systems and Control

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Lecture – 52
Optimal Control-III

(Refer Slide Time: 00:32)

Consider the system

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (1)$$

and the performance index as

$$J_1(u(t)) = S(x(T), T) + \int_{t_0}^{T} F(x(t), u(t), t)dt \qquad (2)$$

Then a necessary condition for optimal control is given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad (3)$$

$$\frac{\partial L}{\partial u} = 0 \qquad (4)$$

$$\delta J = L|_{\text{at}T}\delta T + \left[\left(\frac{\partial L}{\partial \dot{x}}\right)' \delta x(t)\right]_T = 0 \qquad (5)$$

Dear students, welcome to the third lecture on the optimal control, so in the previous lecture, we have seen the problem of finding the optimal control of 1 and 2, so we considered the system dx/dt = f of x of t u of t t; u of t where x is the state variable and u is the control variable and the performance index to be minimised or maximised is given by the expression J1, which is = the function x of capital T, t + integral t0 to capital T f of xu t dt.

And here, capital T is the final time and t0 is the initial time and our aim is to minimise or maximise the expression J under the constraint given in the equation 1, so last time we have seen the necessary condition which is Euler Lagrange equation is given by del L/ del x – d/ dt of del L/ del x dot = 0 and del L /d el u = 0, this is the fourth equation and the boundary condition to solve this equation is given by the del J should = 0 which is the necessary condition for the minimisation or maximisation problem where del denote the variation.

The variation; first variation of J = 0 is the necessary condition so here, del J = 0 implies the value of the L at the final time t into the variation of capital T + del L/ del x dot transpose * del x dot evaluated at final time t should be equal to 0.

(Refer Slide Time: 02:36)

where

$$
\begin{aligned}
L &= L(x^*(t), \dot{x}^*(t), u*(t), \lambda(t), t) \\
&= F(x^*(t), u^*(t), t) + \left(\frac{\partial S}{\partial x}\right)' [\dot{x}^*(t)] + \left(\frac{\partial S}{\partial t}\right) \\
&+ (\lambda(t))' \{ f(x^*(t), u^*(t), t) - \dot{x}^*(t) \},
\end{aligned}
$$

$$H = F(x(t), u(t), \lambda(t), t) + (\lambda(t))' f(x(t), u(t), t)$$

$$\therefore L = H + \left(\frac{\partial S}{\partial x}\right)' [\dot{x}^*(t)] + \left(\frac{\partial S}{\partial t}\right) - (\lambda(t))' \dot{x}^*(t)$$

Here, L denotes the Lagrangian function given by this expression, L is a function of x x dot u and the Langrange multiplier lambda and t and which is given by this expression, capital F function + del S/ del x transpose * x dot + del S/d el t + lambda transpose, the Langrange multiplier is a vector function and so lambda is the column vector and lambda transpose is the row vector, the Langrange multiplier and it is multiplied by small f of the xu – x dot function, okay.

All these are evaluated at the optimum value x star u star etc. so in general, L is a function of all these variable; x, x dot u lambda etc. and the Hamiltonian H is defined by capital F function + lambda dash times small f function therefore, the Lagrangian L in terms of the Hamiltonian H is given by this expression which can be easily seen from the previous step.

(Refer Slide Time: 04:17)

Then the Euler-Lagrange equations in terms of H, for optimal control are given by

$$\left(\frac{\partial L}{\partial u}\right) = 0 = \left(\frac{\partial H}{\partial u}\right) \tag{6}$$

$$\left(\frac{\partial H}{\partial t}\right) = -\dot{\lambda}(t) \tag{7}$$

$$\left(\frac{\partial H}{\partial \lambda}\right) = \dot{x}^*(t) \tag{8}$$

$$\left(H + \frac{\partial S}{\partial t}\right)_T \delta T + \left(\frac{\partial s}{\partial x} - \lambda^*(t)\right)_T \delta x_T = 0 \tag{9}$$

where x_T is the final position $x(T)$.

Now, the condition which was the Euler Lagrange equation 3 and 4, if you convert it in terms of Hamiltonian H is given by this expression, del L/ del u = 0 same as del H/ del u from the previous expression we can easily see this one from here and del H/ del x, it should be x here, = - lambda dot that is the seventh equation, del H/ del lambda = x dot is the equation which is already given, del x/; del H/ del lambda, we can see from here.

It is nothing but this function small f, so the equation 8 is nothing but the equation given, x dot = small f which is the equation 1 itself okay, so already, so this 3 expressions along with the boundary condition, H + del S/del t evaluated at capital T * the variation of capital T + del S/ del x - lambda evaluated at capital T * variation of the final position x suffix t, so this we have not proved in the last lecture.

We have to prove that the boundary condition given by 5 in terms of capital L, if you convert it into by del u = 0 is the fourth equation and the boundary condition to solve this equation is given by the del J should be equal to 0 which is the necessary condition for the minimisation or maximisation problem where del denote the variation. The variation; first variation of J = 0 is the necessary condition.

So, here del J = 0 in place the value of L at the final time t into the variation of capital T + del L/ del x dot transpose * del x dot evaluated at final time t should be = 0, here L denotes the

Lagrangian function given by this expression, L is a function of the x x dot u and the Lagrange multiplier lambda and t and which is given by this expression, capital F function + del S/ del x transpose * x dot + del S/ del t + lambda transpose, the Lagrange multiplier is a vector function.

And so, lambda is the column vector and lambda transpose is the row vector, the Lagrange multiplier and it is multiplied by small f of the x u – x dot function, okay all these are evaluated at the optimum value x star u star etc. So, in general L is a function of all this variable; x, x dot, u, lambda etc. and the Hamiltonian H is defined by capital F function + lambda dash times small f function.

Therefore, the Lagrangian L in terms of the Hamiltonian H is given by this expression which can be easily seen from the previous step. Now, in the terms of capital H is given by 9, so this we have to prove it and we will prove it as a theorem.

(Refer Slide Time: 08:15)

Theorem

$$L(J)\delta J + \left[\left(\frac{\partial L}{\partial \dot{x}} \right)' \delta \dot{x}(t) \right]_T = 0 \qquad (10)$$

we can observe that

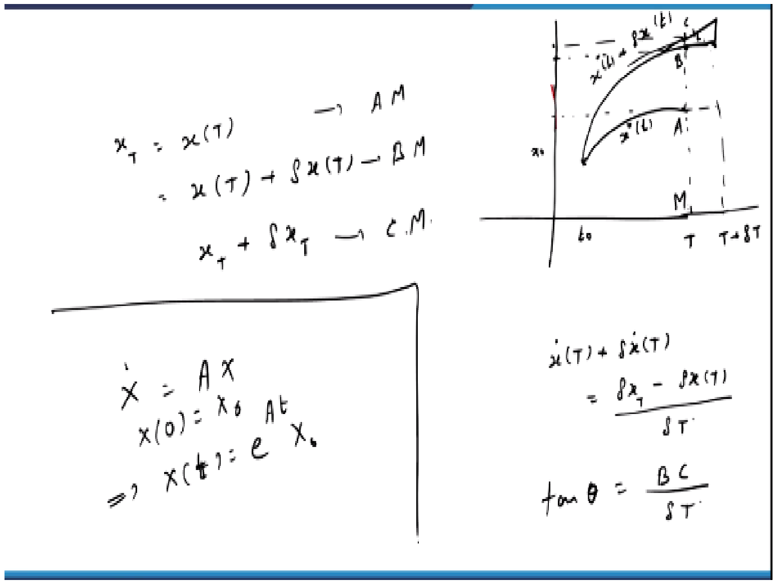
$$\delta x_T = \delta x(T) + (\dot{x}^*(T) + \delta \dot{x}(T))\delta T$$

$$\Rightarrow \delta x(T) \simeq \delta x_T - \dot{x}^*(T)\delta T \quad \text{(by omitting second variation)}$$

$$\therefore L(T)\delta T + \left(\frac{\partial L}{\partial \dot{x}} \right)' \delta x(T) = \left[L - \left(\frac{\partial L}{\partial \dot{x}} \right)' \dot{x}(T) \right] \delta T + \left(\frac{\partial L}{\partial \dot{x}} \right)'_T \delta x_T = 0 \qquad (11)$$

So, first the given condition is this one, L evaluated at capital T, this is a correction, it should be capital T * del of capital T + del L/ del x dot transpose * del x dot at evaluated at capital T that is our condition 5, so this condition 5 we want to show it to be equal to the condition 9, so let us take the condition; the boundary condition in terms of L as given here. Now, we can easily see this expression, del x suffix t = del x of capital T + x dot of capital T + del x dot of capital T * del t.

(Refer Slide Time: 09:16)



So, this we can see in the following way, so if we consider the initial condition is; let us say t0 is this one and x0 is this expression, so that is initial condition and let us say this is our optimal solution, x star of t and this is the variation of this thing, so it is x star of t + the variation in the function x; x of t. now, let us say this is the time t and this is the variation in t; t + delta t because we are; the t may not be a fixed value, it also has the variation.

The final position is not also a fixed value, it can have a variation, so here this is the final position for x dot and the final position for the variation of x star; x star + del x star is this expression, now we can see here this expression and this particular point, if we take, so this is the variation in x of capital T is this expression, x suffix capital T and the next one is x of capital T + del x of capital T.

The variation in this expression that is different, this expression is this particular point and there is another thing which is x suffix T + del x suffix T that is the final position of x star is x suffix T and x suffix T + del x suffix T is the actual final position of the incremented function x star + del x star, so we have 3 values here and from here, we can easily see this expression, you can see that del x suffix T = del x of T, the variation in x of T + the point A is the final position of x star.

And the point B is the incremented position of the x star at capital T that is x of capital T is this position and the incremented value is x + del x of capital T and this x suffix T is the final position and the final position of the x + delta x function is here, so C, this MC or CM is nothing but x suffix T and its increment, del x x suffix T is given by this one. Now, we can see that approximately, the tan of this angle = BC is this height.

And delta t is from here to here, the increment in capital T, so BC/ delta T is the tan theta; tan of this angle but that is nothing but the slope of this curve at the capital T position that is x of T + del x of T, its derivative evaluated at capital T is the slope of this tangent and that is = tan of; tan theta which is BC height/ delta capital T, so that is the expression, so tan theta is nothing but x dot at capital T + del x dot at capital T.

And BC is nothing but del xt, from here we can see, del s suffix T - del x of capital T and divided by del capital T is given, so from this expression keeping del x suffix T in one side, we get this expression, del x suffix T; capital T is del x of capital T + x dot of capital T + del x dot capital T multiplied by del capital T. Now, by neglecting the second variation del x dot of T * del T, the product of 2 first variation we omit.

And keeping only the first variation, we get bill del x capital T = del x suffix T and bringing it to the left hand side, we get the x dot at capital T * del T, okay approximately equal, by neglecting the second variation, so substituting this del x of capital T in the expression of the boundary condition in the equation 10 and converting it into the H value, we will get this following expression.

So, first L at capital T * del capital T is there + del L/ del x dot * del x dot capital T, this del x capital T we are replacing it with this expression, so will get L - del L / del x dot transpose * x dot of T, from here we will get this expression. See from here, we can see that del x of capital T = x dot of capital T * del T that is substituted here plus the remaining term del L/ del x dot and del x of small t is substituted here, so we get this expression 11.

(Refer Slide Time: 16:32)

$$L = H + \left(\frac{\partial S}{\partial x}\right)'\dot{x} + \frac{\partial S}{\partial t} - \lambda'\dot{x}$$

$$\frac{\partial L}{\partial \dot{x}} = \left(\frac{\partial S}{\partial x}\right)' - (\lambda(t))'$$

$$L(T) - \left(\frac{\partial L}{\partial \dot{x}}\right)_T \dot{x}(T) = (H)_T + \left(\frac{\partial S}{\partial t}\right)_T$$

Therefore the condition (11) can be written as

$$\left(H + \frac{\partial S}{\partial t}\right)_T \delta T + \left[\left(\frac{\partial S}{\partial x}\right) - \lambda(t)\right]_T' \delta x_T = 0 \qquad (12)$$

And now, converting L in in terms of H as shown in the previous slide here, L in terms of H equation is given here and substituting in this equation 11, we get the equation 12 here directly because L = H + del S/ del x transpose etc. and del L/ del x dot from here, we can see that it is nothing but del S/ del x transpose + - lambda dashed here, okay from this expression. Now, L evaluated at capital T is similarly this expression, L – del L/ del x dot * x dot T, if you substitute in terms of H, L evaluated at capital T - del L / del x dot * x dot is in terms of H is given by H at capital T + del S/ del T evaluated at capital T.

So that we can see from the previous equation, so substituting this expression in the boundary condition, we get the equation 12, so this is the proof that the boundary condition 5 is equivalent to the boundary condition 9 in terms of H.

(Refer Slide Time: 17:47)

Example 1

Write the Euler-Lagrangian equation and boundary conditions to find the optimal control for the following problem.

The system equation is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u$$
$$x_1(0) = 1, \quad x_2(0) = 0$$

performance index is

$$J(u(t)) = \frac{1}{2}(x_1(T))^2 + x_2(T) + \int_0^T (x_1^2(t) + u^2(t))dt$$

For the cases:

(*i*) $T = 5$, $x_1(T)$ and $x_2(T)$ are variables.

(*ii*) T is a variable $x_1(T) = 3$, $x_2(T) = 0$.

(*ii*) T is a variable $x_1(T)$, $x_2(T)$ are related as $x_2(T) = x_1^2(T)$.

So, this equations will be very useful in solving various optimal control problems for example, we consider the problem x1 dot = x2, x2 dot = -x1 + u, the boundary condition x1 0f 0 = 1 and x2 0f 0 is 0, so here initial condition is given, okay the final boundary at capital T we will consider different cases, the performance index to be minimised is given by J, it is 1/2 x1 of capital T square + x2 of capital T.

This is the function capital S in the expression of the problem, here considering the problem given in 1 and 2, so S as a function of capital T and inside the integral, we have the capital F function, so in this particular problem, we consider capital F function as x1 square + u square, the boundary conditions; the final boundary condition we consider are the following; case 1; T is fixed that is 5 and the final positions x1 of capital T, x2 of capital T are variables.

And the case 2 and case 3 are different cases, T is a variable and the final positions is fixed in one case, in the second case final position is also a variable.

(Refer Slide Time: 19:28)

$$L(x(t), \dot{x}(t), u(t), t) = F(x(t), u(t), t) + \left(\frac{\partial S}{\partial x}\right)' \dot{x}(t) + \frac{\partial S}{\partial t} + (\lambda(t))' [f(x(t), u(t), t) - \dot{x}(t)]$$

and

$$L = H(x(t), u(t), \lambda(t), t) + \left(\frac{\partial S}{\partial x}\right)' \dot{x}(t) + \frac{\partial S}{\partial t} - (\lambda(t))' \dot{x}(t).$$

where

$$H(x, u, \lambda, t) = F(x, u, t) + (\lambda(t))' f(x, u, t)$$

In case (i).

$$t_0 = 0, \quad T = 5, \quad x_1(0) = 0, \quad x_2(0) = 0$$

$$F(x, u, t) = x_1^2 + u^2, \quad S(x(T), T) = \frac{1}{2}(x_1(T))^2 + x_2(T).$$

So, we will see how to solve this optimal control problem, so the Lagrangian function L according to the definition, it is capital F + this expression and in terms of H, it is the expression given here, so in this problem H is which is capital F + lambda dashed * small f and we substitute capital F is x1 square + u square and capital S function is 1/2 x1 of capital T square + x2 of capital T, initial condition is 0, final time is 5.

And initial boundary condition x1 of 0 is 0, x2 of 0 is 0, so this is the first case as given in the problem.

(Refer Slide Time: 20:29)

$$H(x, u, \lambda, t) = x_1^2(t) + u^2(t) + \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) + u(t) \end{bmatrix}$$

By Euler-Lagrangian equation

$$\frac{\partial H}{\partial u} = 0$$

gives

$$\begin{aligned} 2u + \lambda_2 &= 0 \\ \Rightarrow u(t) &= -\frac{\lambda_2}{2}, \qquad (13) \\ \frac{\partial H}{\partial \lambda} &= \dot{x} \Rightarrow \dot{x} = f \end{aligned}$$

Now, we have to find the Euler Lagrangian equation necessary condition in this following way, so H is given by this expression, capital F + lambda dashed * small f function is this and the Euler Lagrangian equation is del H/ del u = 0, so if you substitute directly del H/ del u, it is 2u + lambda 2 = 0, so the control u is given in terms of the Lagrange multiplier lambda 2, so u = - lambda 2/ 2.

So, the control u is given by – lambda 2/2, now del H/ del lambda = x star is the second condition but del H/ del lambda is nothing but small f from here we can see directly, so the condition gives x dot = f which is the same equation as given in the equation itself in the given statement of the problem.

(Refer Slide Time: 21:37)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}\right\} \qquad (14)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix}$$

$$\left.\begin{aligned} \dot{\lambda}_1 &= -2x_1 + \lambda_2 \\ \dot{\lambda}_2 &= -\lambda_1 \end{aligned}\right\} \qquad (15)$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \frac{\lambda_2}{2} \end{aligned}\right\} \qquad (16)$$

Hence we have to solve the coupled system (15) and (16).

So, x dot = f is; x dot = f means it is x1 dot = x2, x2 dot is –x1 + u the given system of equation, the third Euler Lagrangian equation is lambda dot = -del H/ del x or it is – del H/ del x1 and del H/ del x2, so if you substitute this value using the H expression, we get lambda 1 dot = - 2x1 + x2 + lambda 2, lambda 2 dot = - lambda 1, so we have the expression as given here, if you see the equation 14 in the place of u, we can replace it with the equation 13 that is you u = - lambda 2/ 2.

So, by substituting u value here, we get x1 dot = x2 and x2 dot = - x1 – lambda 2/2, so the system 15 and 16, they are the coupled system of ordinary differential equations, so we have to solve this 4 equations; 4 first-order differential equation.

(Refer Slide Time: 22:58)

Boundary conditions are

$$\left[H + \frac{\partial S}{\partial t}\right]_T \delta T + \left[\frac{\partial S}{\partial x} - \lambda(t)\right]_T^{\prime} \delta x_T = 0$$

Case (i)

$$\delta T = 0$$

$$\Rightarrow \left[\frac{\partial S}{\partial x} - \lambda(t)\right]_T^{\prime} = 0$$

$$\begin{bmatrix} x_1(T) - \lambda_1(T) \\ 1 - \lambda_2(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \lambda_2(T) = 1, \quad \lambda_1(T) = x_1(T)$$

And the boundary conditions to solve this problem or as we have already derived, it is given in the expression, so in case 1 given by this one, T = 5, so it is a fixed value, there is no variation in T, so delta T = 0 and x1 T, x2 T are variables, so the variable boundary condition is only in the second term, delta capital T is 0, therefore the boundary condition is given by del S/ del x – lambda T transpose evaluated at capital T should be = 0.

Because del capital T is already 0 and de x suffix T, the variation in the final condition is a arbitrary, therefore the expression into an arbitrary value = 0 that implies the vector given in this bracket should be = 0, so we get the condition del S / del x - lambda dashed should be = 0 evaluated at capital T, so if you substitute S value and then differentiate with respect to x1 and x2, we get this equation.

(Refer Slide Time: 24:24)

$$L(x(t), \dot{x}(t), u(t), t) = F(x(t), u(t), t) + \left(\frac{\partial S}{\partial x}\right)' \dot{x}(t) + \frac{\partial S}{\partial t} + (\lambda(t))'[f(x(t), u(t), t) - \dot{x}(t)]$$

and

$$L = H(x(t), u(t), \lambda(t), t) + \left(\frac{\partial S}{\partial x}\right)' \dot{x}(t) + \frac{\partial S}{\partial t} - (\lambda(t))'\dot{x}(t).$$

where

$$H(x, u, \lambda, t) = F(x, u, t) + (\lambda(t))' f(x, u, t)$$

In case (i),

$$t_0 = 0, \quad T = 5, \quad x_1(0) = 0, \quad x_2(0) = 0$$
$$F(x, u, t) = x_1^2 + u^2, \quad S(x(T), T) = \frac{1}{2}(x_1(T))^2 + x_2(T).$$

Because S equation is this, so S function is nothing but 1/2 x1 square + x2, so if you differentiate with respect to x1, we will get x1 and differentiate with respect to x2, we will get 1 here, so del S/ del x1 is nothing but x1 – lambda 1, del S/ del x2 is 1 – lambda 2 that is the zero function, so from here we will get lambda 2 = 1, lambda 2 evaluated at capital T is 1 and lambda 1 at capital T is nothing but x1 of T.

(Refer Slide Time: 25:04)

$\therefore$ we need to solve

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \frac{-1}{2} \\ -2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

we get

$$x_1(0) = 1, \quad x_2(0) = 0, \quad \lambda_1(T) = x_1(T), \quad \lambda_2(T) = 1$$

So, the coupled equation; 4 equation is written in the matrix form like this, x1 of x1 dot is x2, x2 dot is – x1 -1/2 lambda 2 etc. as we saw in the previous thing, the 15 and 16 equations are coupled and there they can be written in the matrix form like this and the boundary conditions is given in the problem that is at T = 0, these 2 are already given and this 2 boundary conditions are derived just the previous slide here.

So, along with this 4 boundary conditions, we can solve the system of equation in the usual way that we know that the equation x dot = Ax with some initial condition x of 0 = x0, so the solution is x of t; small t = e to the power At * x0, so using this form, we can solve the equation, we can solve this equation and then substituting the boundary condition which can get the solution of the problem.

So, in solving this equation, we will get the values of x1 of T, x2 of T lambda 1 T lambda 2 T, now after getting the value lambda 2 T, we substitute it in this equation that is equation 13, u of T = - lambda 2/ 2, so lambda 2 of T we are solving from the system of equation, substitute here we get the control u of T that is called the optimal control for this problem and the solution x1 of t and x2 of T after getting the solution with the boundary condition we call that as the optimal state of the system x1 star x2 star are the optimal state that is coming from the solution of the system.

(Refer Slide Time: 27:25)

Example 2

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x_1(0) = 1, \quad x_2(0) = 0.$$

$$J = \frac{x_1^2(T)}{2} + \int_0^T \frac{u^2(t)}{2} dt$$

$$H = \frac{u^2}{2} + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

So, this can be solved like this, so now we consider the second example quickly, this case x1 dot = x2, x2 dot = u and initial condition is x1 0 = 1 and x2 0 = 0, the function performance index to be minimised is J which is given by x1 square at final time + this expression, so S of capital T is nothing but x1 square capital T/ 2 and capital F function is given by u square/ 2, so these are the two things and H = F function + lambda times small f function is given in the right hand side of the equation x2 and u, this is the small f function.

(Refer Slide Time: 28:21)

$$S(T, x(T)) = \frac{x_1^2(T)}{2}$$

$$\frac{\partial H}{\partial u} = 0 = u + \lambda_2 \Rightarrow u = -\lambda_2,$$

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -\frac{\partial H}{\partial x} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

$$\Rightarrow \dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{x}_1 = x_2, \quad \dot{x}_2 = u = -\lambda_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Now, by applying the formula, Euler Lagrangian the necessary condition del h / del u = 0 will give u = - lambda and lambda dot = - del H/ del x that equation gives this one, lambda 1 dot is 0, lambda 2 dot = - lambda 1 by directly differentiating H with respect to x1 and x2, we get this one, so the solution; the equation is lambda 1 dot = 0, lambda 2 dot = - lambda 1 and then the given system of equation is x1 dot is x2 and x2 dot is u.

In terms of lambda because del H/ del lambda is the equation, the 2 system of equation already given and converted in terms of lambda and then lambda 1 dot, lambda 2 dot equation.

(Refer Slide Time: 29:36)

Case 1: If $T = 5$, x_T is free then

$$\left(\frac{\partial S}{\partial x}\right)'_T - \lambda'(T) = 0$$

$$\Rightarrow \begin{bmatrix} x_1(T) \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix}.$$

All the 4 is written in the form of a system like this which can be solved easily, now for the boundary condition, if T is fixed and the final position is free, as we have seen in the previous example, del of capital T is 0 and so, the remaining condition if you substitute we get the boundary condition as x1 at capital T is lambda 1 at capital T and lambda 2 at capital T becomes 0, by differentiating the S function with respect to x1 and x2, we get this value.

And lambda 1T and lambda 2T is given by this, so we get the boundary condition in this form and we have to solve the system of the 4 by 4 matrix.

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$$\begin{aligned}
\lambda_1(t) &= c_1 \\
\lambda_2(t) &= -c_1 t + c_2 \\
\dot{x_2} &= -\lambda_2 = c_1 t - c_2 \\
x_2(t) &= c_1 \frac{t^2}{2} - c_2 t + c_3 \\
\dot{x_1} &= c_1 \frac{t^2}{2} - c_2 t + c_3 \\
x_1 &= c_1 \frac{t^3}{6} - c_2 \frac{t^2}{2} + c_3 t + c_4
\end{aligned}$$

So, the solution can be easily obtained directly here because lambda 1 dot = 0, it implies that lambda 1 is a constant and when you substitute that constant here lambda 2 dot is – constant, so when we integrate, we will get lambda 2 is - constant * T + some other constants C2, so this 2 is solved directly from here easily, so we can substitute directly, lambda 1 is a constant, lambda 2 - that constant T + C2.

And now, substitute this lambda 1, lambda 2 in the equation, x2 dot is – lambda 2 and when we substitute lambda 2 here, we get this expression C 1 T – C2, so integrating this both sides with respect to a T, we get x2 of T is C1 T square/ 2 – C2 T + C3 constant and we are given that x1 dot is x2, so substitute x2 in the right hand side, then integrate, we will get x1 as the solution like this.

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$$
\begin{aligned}
x_1(0) &= 1 = c_4 \\
x_2(0) &= 0 = c_3 \\
x_1 &= c_1\frac{t^3}{6} - c_2\frac{t^2}{2} + 1 \\
x_2(t) &= c_1\frac{t^2}{2} - c_2 t
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2(5) = 0 &\implies 0 = 5c_1 + c_2 \\
\lambda_1(5) = x_1(5) &\implies c_1 = \frac{125}{6}c_1 - \frac{25}{2}c_2 + 1
\end{aligned}
$$

Now, this C1, C2, C3, C4 all this can be found out by directly substituting the boundary conditions, all the conditions can be easily substituted and it can be easily solved, all the 4 constants can be obtained, so after substituting all this constants in the expressions, so we get x1 of t is this, x2 of t is this that is the optimal state variable of the problem and the control is given by this expression that is u of T is – lambda 2 and lambda 2 is obtained by this expression.

Lambda 2 of T is - C1 T + C2, so this is the optimal control for the given problem, so here we have seen 2 examples in which the final time is a fixed value okay, where the variation of final time is 0 and made use of the boundary condition accordingly but in the next lecture, we will see various other cases, the remaining 2 cases we have to discuss, this the cases where final time is variable and the final position are fixed.

And both final time and final position, both are variables, in such problem how to find the optimal control, so these cases we will discuss in the next lecture and solve some examples, okay. Thank you.